Mathematics for Elementary Teachers Topics List

01 Operations with Tallies (K-3)

We define and perform the fundamental arithmetic operations using the oldest mathematical notation - tallies. These definitions lay the foundation for all of our arithmetic algorithms and many problem solving techniques we will use later.

02 Comparing Numbers with Tallies (K-2)

We use one-to-one correspondences to compare sets of tallies. This, combined with a little laziness, leads to the notion of counting and the counting numbers.

03 Base Ten Notation (K-4)

We introduce base ten notation and bundling diagrams. Base ten notation is a compact notation that allows us to extend arithmetic with numbers less than ten to arbitrarily large numbers.

04 Bar Models (K-8)

We introduce bar models as a problem solving tool. Here, we solve a variety of problems, but only with single digit arithmetic.

05 Properties of Addition and Subtraction (K-2)

We use our definitions of addition and subtraction to explain several arithmetic properties of arithmetic and subtraction.

06 Addition Algorithm (1-5)

We use bundling diagrams to derive the standard algorithm for adding base ten numbers.

07 Subtraction Algorithm (1-5)

We use bundling diagrams to derive the standard algorithm for subtracting base ten numbers.

08 Properties of Multiplication and Division (3-5)

We use the definitions of multiplication and division to explain several arithmetic properties of multiplication and division, and we introduce array diagrams as a tool for working with multiplication.

09 Multiplication Algorithm (4-5)

We use array diagrams to derive the partial products multiplication algorithm and the standard multiplication algorithm.

10 Division Algorithm (4-5)

We use tallies and the definition of division to derive the standard division algorithm.

11 Number Lines and Negative Numbers (Number Lines K, Negative Numbers 4-7)

We introduce number lines and explain the relationship between the arithmetic operations and the number line. We also use the number line to motivate negative numbers and extend arithmetic to negative numbers.

12 Comparing Numbers (2-6)

We explain the standard algorithm for comparing numbers in base ten and introduce rounding and approximate arithmetic.

13 Order of Operations (4)

We introduce exponents, parentheses, and order of operations to simplify notation.

14 Fractions (3-6)

We define fractions and solve fraction word problems without any fraction arithmetic.

15 Forms of Fractions (4-5)

We use box and bar models to explain equivalent fractions and mixed numbers, and we convert between mixed numbers and improper fractions.

16 Operations with Fractions (3-6)

We compare fractions using equivalent fractions, and we use bar models and boxes to explain fraction arithmetic and arithmetic with mixed numbers.

17 Ratios and Proportions (6-8)

We solve ratio and proportion problems with bar models, tables, and unit rates, and we identify proportional and inversely proportional relationships.

18 Decimals (4-6)

We define decimal notation and learn how to perform arithmetic with and compare numbers in decimal notation.

19 Scientific Notation (5-8)

We convert between standard notation and scientific notation and perform arithmetic with scientific notation.

20 Percents (6-7)

We define percentages and solve percent problems with tables, multiplication, division, fractions, and exponents.

21 Expressions (5-8)

We introduce numerical and algebraic expressions and use bar models to write expressions related to word problems geometric diagrams.

22 Equations (Equations K-8, Solving Algebra Word Problems 4-8)

We solve linear equations, and we write and solve equations for bar models derived from word problems.

23 Sequences (K-8)

We define sequences, and we identify and fix patterns in sequences. We learn to identify and find expressions for arithmetic and geometric sequences and solve problems with sequences.

24 Functions (8)

We describe functions with words, tables, ordered pairs, and equations. We plot points to graph functions, and we interpret graphs of functions.

25 Linear Functions (8)

We learn to recognize linear functions, and we find equations for linear functions from points and tables. We then solve problems using linear equations.

26 Odd and Even Numbers (2)

We explore possible definitions of the words even and odd and use these definitions to explain arithmetic properties of even and odd numbers.

27 Divisibility (4)

We explore factors and multiples and apply divisibility tests. We also use base ten bundling to explain why certain divisibility tests work.

28 Prime Numbers (4)

We define prime and composite numbers and explain the importance of prime numbers through the Fundamental Theorem of Arithmetic. We find prime numbers using the Sieve of Eratosthenes and use factor trees to find prime factorizations.

29 Common Factors and Multiples (6)

We find least common multiples and greatest common factors with brute force, prime factorization, and the slide method, and we solve problems related to factors and multiples.

30 Rational and Irrational Numbers (8)

We convert between decimal and fraction notation. We also identify rational and irrational numbers and draw a Venn diagram for the number systems.

31 Geometry

We explain the axiomatic method and the contributions of Thales and Euclid to mathematics and science. We also begin listing basic primitives and definitions in geometry.

32 Angles (2-8)

We define and classify angles, and we use the Parallel Postulate, vertical angles, and the angles in a triangle to solve problems involving angles and explain why certain basic theorems in geometry are true.

33 Triangles (K-5)

We define and classify types of triangles and draw a Venn diagram to organize them. We also use the sum of the angles in a triangle to solve problems involving triangles.

34 Quadrilaterals and Other Polygons (K-5)

We define and classify quadrilaterals and draw a Venn diagram to organize them. We use properties of angles and the Parallel Postulate to explain some properties of quadrilaterals. We also define and classify general polygons.

35 Measurement (K-5)

We explain the concept of measurement and explore one, two, and three dimensional features of objects. We also discuss various units of measurement and measure lengths with a ruler.

36 Unit Conversions (2-6)

We use conversion maps and unit rates to convert between different units.

37 Area (3-7)

We explain why basic area formulas are correct and use them to calculate the areas of composite shapes. We then approximate the area of irregular shapes.

38 Circles (K, 7)

We use circles as tools for locating regions. We derive the formulas for the area and circumference of a circle, and we solve problems involving them.

39 Pythagorean Theorem (8)

We state and prove the Pythagorean Theorem, and we solve problems using the Pythagorean Theorem.

40 Polyhedra (K-2)

We define and classify various types of polyhedra and use Euler's Formula. We also describe the five Platonic solids.

41 Nets and Surface Area (6-7)

We draw nets for polyhedra and cylinders and use them to calculate surface area, and we find the surface area of a sphere.

42 Volume (5-8)

We find volumes of prisms, pyramids, and spheres.

43 Transformations and Symmetry (4, 8)

We apply translations, rotations, reflections, and dilations, and we identify symmetries in patterns and construct patterns with symmetry.

44 Congruence and Similarity (7-8)

We identify shapes that are congruent or similar, and we solve similarity problems involving length, area, volume, and weight.

45 Data and Statistics (2-8)

We classify types of data and use sample data to draw conclusions about a population proportion. We also read and draw basic statistical graphs.

46 Summarizing and Comparing Data (5-7)

We calculate various measures of center and the five number summary of a set of data values. We also interpret and explain percentiles.

47 Variation and Relative Standing (6-7)

We calculate various measures of variation and apply Range Rule of Thumb to find a usual range of data values. We also use Z-scores to compare data values.

48 Probability (7)

We approximate probabilities using relative frequency, and we calculate probabilities by listing outcomes of an experiment.

49 Multi-Stage Experiments and Counting (7)

We calculate probabilities by using trees and arrays to list outcomes of multistage experiments and by using the multiplication principle. We also count sequences and outcomes using the Fundamental Counting Principle.

Operations with Tallies

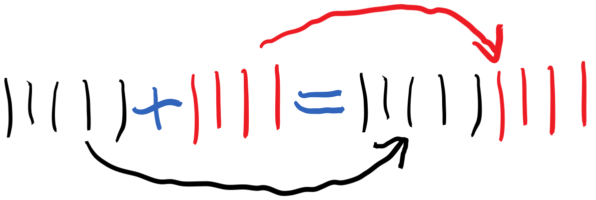
Perhaps one of the earliest and simplest cases of written mathematics is the use of tallies. For example, a young shepherd may have seen ||| foxes and |||||| vultures. Tallies are simple and precise. In this section, we will see that tallies and can be used to demonstrate the four fundamental mathematical operations and some mathematical problems.

Problem: Sam the shepherd had ||||| sheep. He traded some chickens for |||| more sheep. How many sheep did he have then?

Solving this problem with tallies is simple. We just place all of the black sheep and red sheep together to see that Sam has ||||||||| sheep. Counting the number of objects in two combined sets or groups like this is called addition. We could express this arithmetic in this way:

$$|||||+||||=|||||||||$$

The symbol "+" is a plus sign and denotes combining groups or sets. Notice how this arithmetic works. We simply copy tallies:

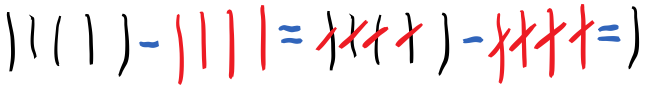


Notice that it does not matter which sheep, the black or the red, we copy first. Therefore, it seems like adding |||| to ||||| is the same as adding ||||| to ||||. When adding, order does not seem to matter.

Problem: Sam the shepherd had ||||| sheep. He then traded |||| sheep for some chickens. How many sheep did he have then?

To solve this problem, we start with the black sheep and cross out one sheep for each red sheep.

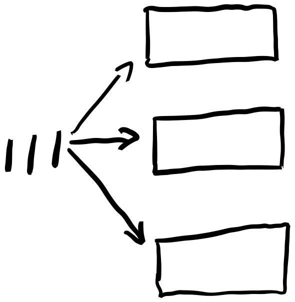
To keep track of the crossing out, we cross out one red tally and then one black tally and repeat until no red tallies are left. We might wright this arithmetic in this way:



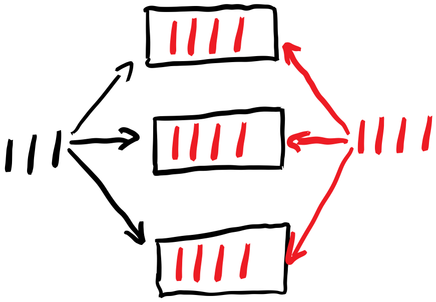
Sam seems to have | sheep left. Counting the number of objects left after removing some objects like this is called subtraction.

Problem: Each of ||| shepherds have |||| sheep. How many sheep do they have together?

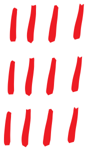
To solve this problem, we will end up drawing a tally for every single sheep. First, we draw a “box” or “container” or “pen” for each shepherd’s sheep:



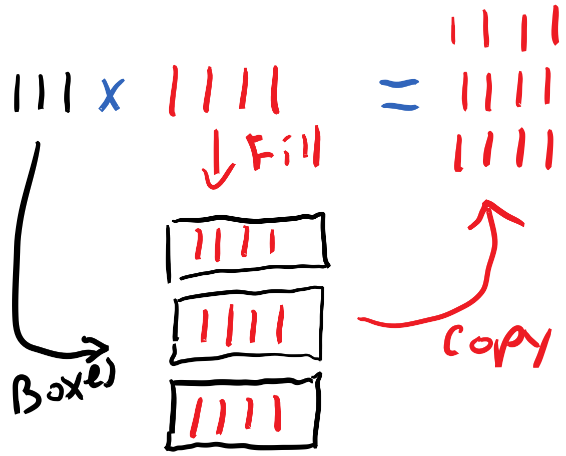
Then we place |||| sheep in each container:



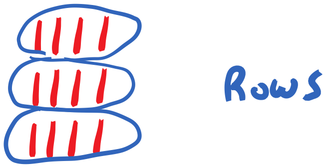
After removing some of the marks used for the computation, it appears as if the shepherds have this many sheep combined:



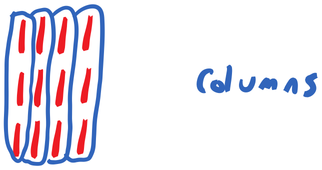
We can summarize these steps this way:



The process we just went through is called multiplication. The final answer we have drawn is called an **array**. We can group the objects in an array in two ways. If we group them horizontally, we call the groups **rows**.

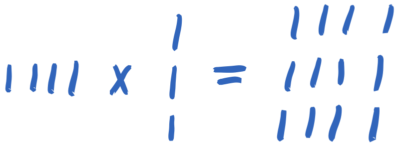


If we group the objects vertically, we call the groups **columns**.



If we focus on rows, we have  groups of  tallies, which we could express this way:

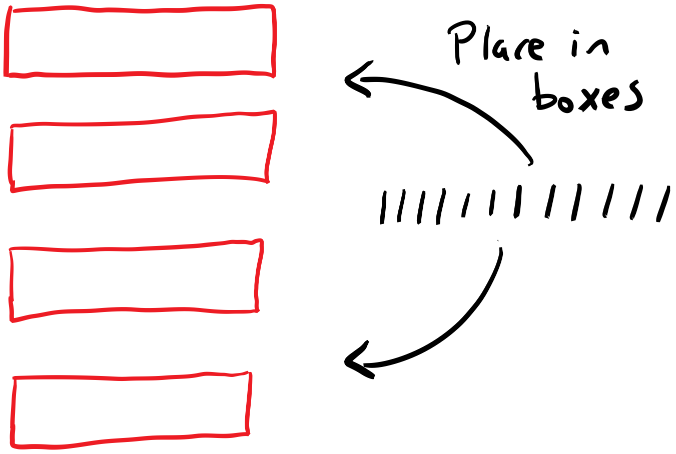
If we focus on columns, we have  groups of  tallies, which we could express this way:



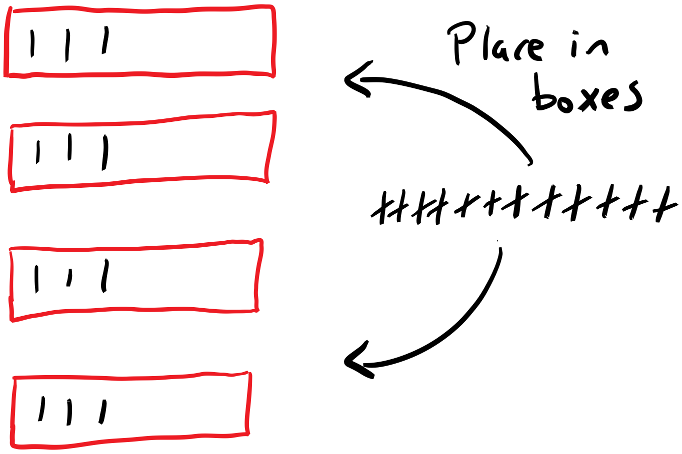
Notice that, just like addition, the order in which we multiply does not seem to matter.

Problem: Suppose that |||||||||||| sheep were placed into |||| pens with the same number of sheep in each pen. How many sheep are in each pen?

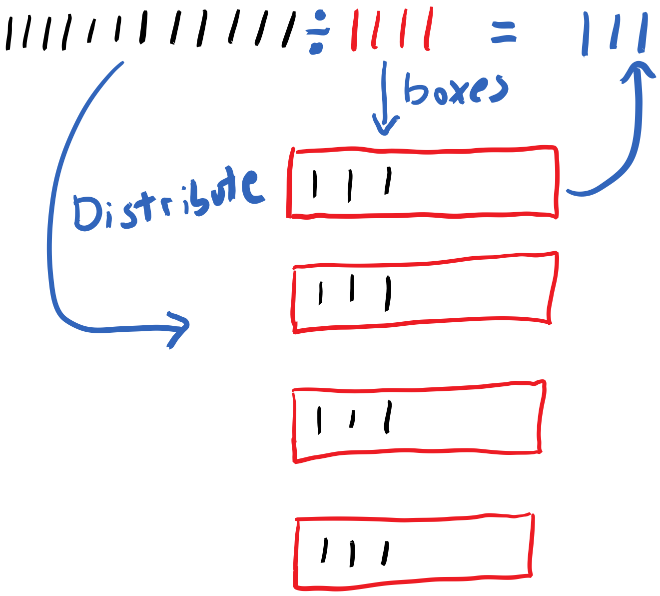
To solve this problem, we will first draw a box for each of the |||| pens. We will then place each of the |||||||||||| sheep into a pen, one at a time, rotating pens. First, we draw boxes for the pens:



Now we place each of the black tallies, one at a time, in separate boxes. We cross out the tallies as we place them in boxes.



It would seem that each pen has ||| sheep in it. This process of distributing objects among boxes or containers is called division. We might represent our arithmetic in this manner:



To solve these four problems about sheep, we have introduced four basic operations:

Operations

Addition: The **sum** of A and B is the number of objects in a group formed by combining a group of A objects with a group of B objects. The sum of A and B is denoted as $A + B$ and is read "A plus B." In the sum $A + B$ the numbers A and B are sometimes called **terms**, **addends**, or **summands**. Calculating a sum is called **addition**.

Subtraction: The **difference** of A and B is the number of objects left over after B objects are removed from a group of A objects. The difference of A and B is denoted $A - B$ and is read "A minus B." Calculating a difference is called **subtraction**.

Multiplication: The **product** of A and B is the number of objects in A groups containing B object each. The product of A and B is denoted as $A \times B$ and is read "A times B." Calculating a product is called **multiplication**. In the product $A \times B$, the numbers A and B are called **factors**.

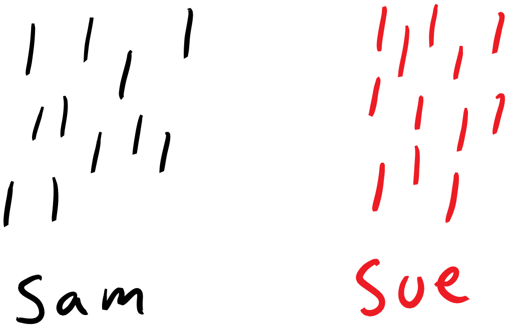
Division: The **quotient** of A and B is the number of objects in each group when A objects are placed into B groups which are all the same size. The **quotient** of A and B is denoted as $A \div B$ and is read "A divided by B." Calculating a quotient is called **division**. In the quotient $A \div B$, A is called the **dividend**, and B is called the **divisor**.

Comparing Numbers with Tallies

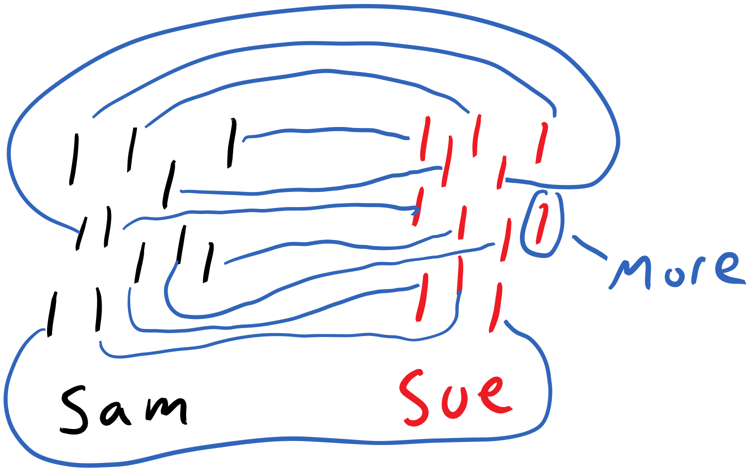
In the previous section, we saw that we can use tallies to describe the four fundamental mathematical operations. However, performing arithmetic with tallies is a bit tedious. Here we will see that simple comparisons with tallies can be even more tedious. Trying to use tallies to compare will lead us naturally to the mathematical notions of pairing and grouping. These notions will lead us to more useful mathematical notation for numbers.

Problem: Sam and Sue used tallies to count their sheep. The results are below. Who has more sheep?

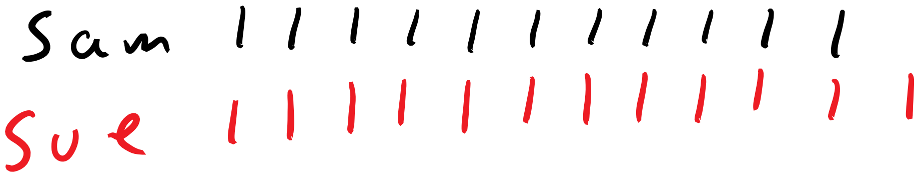
This problem is somewhat overwhelming because there are so many tallies. We will address the same question with fewer tallies.



To compare these tallies, we are going to, one at a time, pair one of Sam's tallies with one of Sue's tallies. If we run out of Sam's tallies before we run out of Sue's tallies, Sue has more. If we run out of Sue's tallies first, Sam has more. If we run out at the same time, they have the same number of sheep. With the tallies in place, this process looks like this:

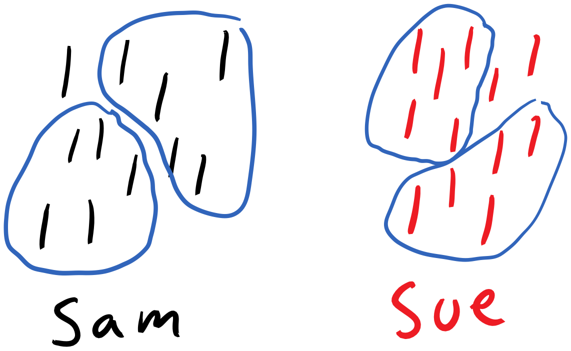


It appears as if Sue has more sheep than Sam. This pairing process looks more reasonable if we rearrange the tallies into rows that we can place side by side.



If we pair each of Sam's tallies with Sue's tally directly beneath it, we see that Sue has more tallies. We could also see that Sue has more tallies here by noting that her row is longer.

Another approach at determining who has more sheep is grouping. We can place the tallies into groups so that it is easier to see who has more tallies. If we (arbitrarily) use groups with size ||||| then this looks like so:

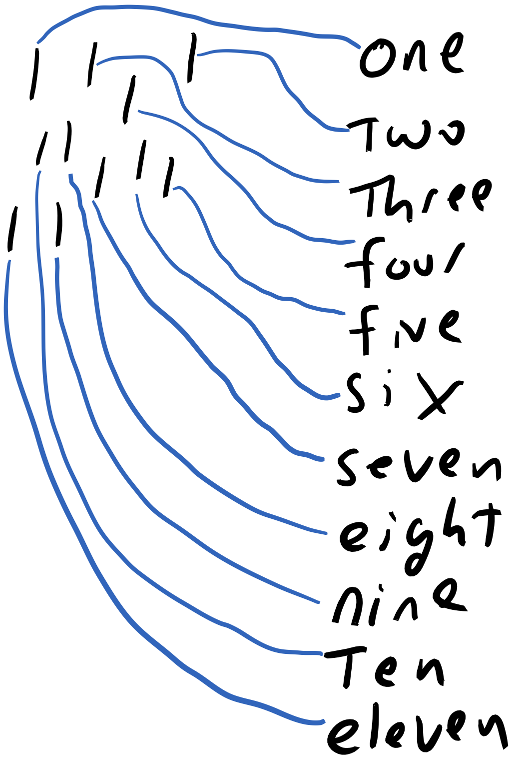


We see that Sam has || groups of ||||| with | left over. Sue also has || groups of ||||| but she has || left over. Thus Sue has more tallies.

We can make the process of pairing objects in groups more useful for comparing the sizes of groups. We first name the numbers of objects that might be in a group. The names, initially, are somewhat arbitrary.

|	one	|||||||	seven
||	two	||||||||	eight
|||	Three	|||||||||	nine
||||	four	||||||||||	Ten
|||||	five	|||||||||||	eleven
||||||	six	||||||||||||	Twelve

We call this set of names of sizes of sets **counting numbers**. Notice that the counting numbers come with a natural order. Here, we pair Sam's tallies with some of the counting number, taking the counting numbers in order starting at one.



Since we can pair Sam's tallies with the counting numbers one through eleven, we say that Sam has eleven tallies. This process of pairing a group with the counting numbers in order starting at one is **counting**. If we do the same thing with Sue's tallies, we see that she has twelve tallies. Since eleven comes before twelve in the order of the counting numbers, Sue still has more sheep.

It will be convenient to have a name for the number of objects in an empty group, for example the number of sheep in an empty sheep pen. If a group has no objects in it, we say the group has **zero** objects. Zero has traditionally not been included as a counting number. The collection of all counting numbers along with zero is often called the **whole numbers**.

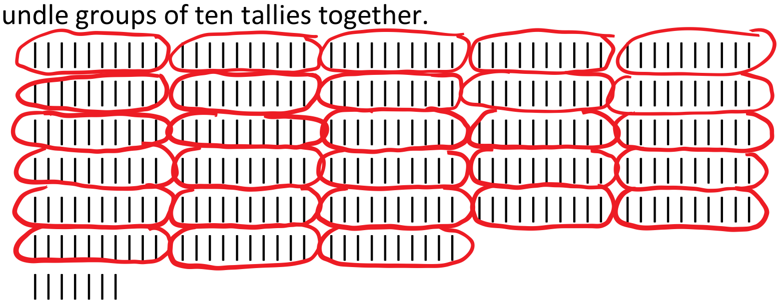
Summary up to this point: The counting numbers are an ordered set of names of number of objects that can be in a group. Pairing the objects in a group with the counting numbers in order is called counting. The last counting number paired with an object in a group when counting is the number of objects in a group. This number of objects can be used to compare the sizes of groups. If a group is empty and contains no objects, we say that the group has zero objects.

Base Ten Notation

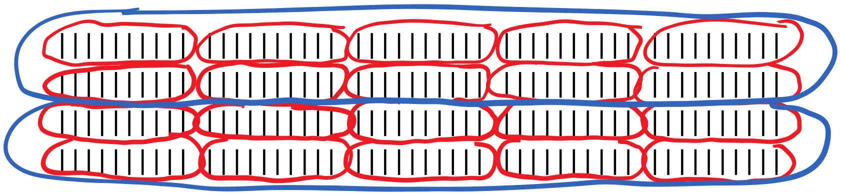
At this point, we have an ordered set called the counting numbers that we use to count the number of objects in a group. The counting numbers are names of possible numbers of objects in nonempty groups. The number of objects in an empty group is zero. So far, we have only named these counting numbers: one, two, three, four, five, six, seven, eight, nine, ten, eleven, and twelve. These names are somewhat arbitrary and somewhat limited. For counting to be useful, we need names of every possible number of objects in a group. However, there are infinitely many such numbers, so naming them all is unreasonable. We need a simple system that allows us to name arbitrarily large numbers by using only a few symbols. The modern system used to name numbers is the Hindu-Arabic or Indo-Arabic or base ten system which was first conceived in India prior to the fourth century, adopted by Arabia by the ninth century, and introduced to the western world in the thirteenth century.

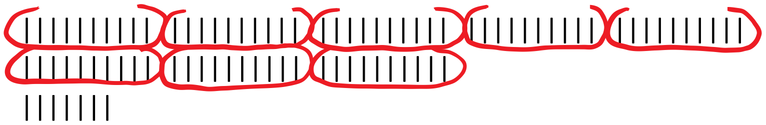
The base ten number system is based on bundling objects into groups of ten repeatedly and then using place value to describe how many ungrouped objects there are, how many bundles of ten there are, how many bundles of ten bundles of ten, and so forth. For this to work, we need symbols for the numbers zero through nine, and we need names for certain size bundles. We use the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 to represent the numbers zero, one, two, three, four, five, six, seven, eight, and nine. We call these symbols **digits**. The number of objects in ten groups of ten is one **hundred**. That is, one hundred is ten times ten. The number of objects in ten groups of one hundred is one **thousand**. That is, one thousand is ten times one hundred. We have names for larger sized groups, but we will hold off on them for now. Consider this group of tallies:

We can first bundle groups of ten tallies together.



Notice that there are 7 (seven) tallies left over at the bottom ungrouped. There are still so many groups of ten here that it is difficult to tell how many tallies there are, so now we bundle ten groups of ten together.





The blue outlined groups each contain ten groups of ten, or ten times ten, or one hundred tallies. The red groups each contain ten tallies, and there are seven ungrouped tallies at the bottom. Therefore, there are two groups of one hundred, eight groups of ten, and seven single tallies. The base ten system expresses this number as 287. We could read this as “two hundreds + eight tens + seven,” but we shorten it to “two hundred eighty seven.” Each location or place of a digit in a base ten number carries a value with it. The right-most digit is the number of ones (tallies). The second digit from the right is the number of tens. The third from the right is the number of hundreds. The fourth from the right is the number of thousands. The base ten number

5302

represents

5 thousands + 3 hundreds + no tens + 2.

Frequently (but not always) commas are put in base ten numbers after every third digit from the right to make place value easier to see. For example, 1234567 could be written as 1,234,567. Note now that we can write ten as 10, one hundred as 100, and one thousand as 1,000. Here are some names of common place values in base 10.

1	one	1,000,000	million
10	ten	10,000,000	ten million
100	hundred	100,000,000	hundred million
1,000	thousand	1,000,000,000	billion
10,000	ten thousand	1,000,000,000,000	trillion
100,000	hundred thousand	1,000,000,000,000,000	quadrillion

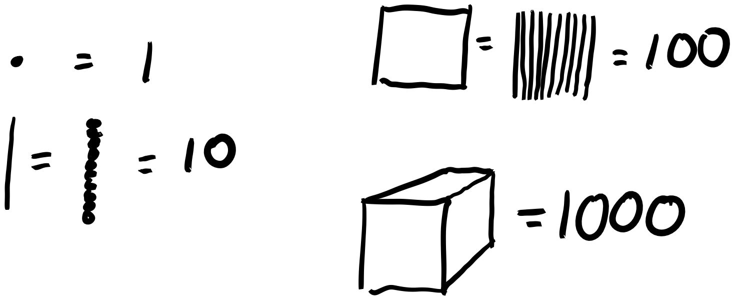
We can now write a base ten number such as 7532 in several **expanded forms**:

7 thousands + 5 hundreds + 3 tens + 2

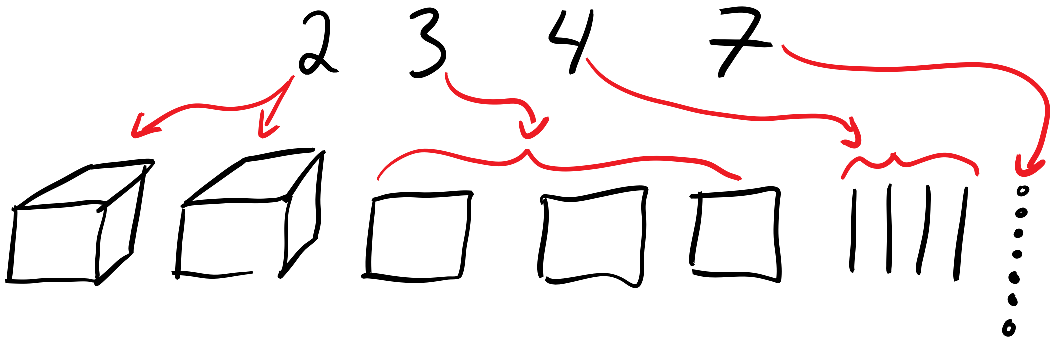
7000 + 500 + 30 + 2

$$(7 \times 1000) + (5 \times 100) + (3 \times 10) + 2$$

We will also frequently draw **bundling diagrams** of base ten numbers. These are diagrams which are based on groups of ten. We use a single dot for one, a line segment for ten (imagine ten dots glued together to form a line segment), a square for one hundred (imagine ten lines glued together), and a box for one thousand (imagine ten squares stacked). Here are the basic shapes:



And here is a bundling diagram for 2347:



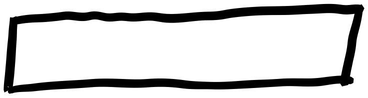
Bundling diagrams will be useful to us when we start discussion algorithms for adding and subtracting later.

The beauty of the base ten system is that it allows us to express infinitely many numbers with only ten symbols (0, 1, 2, 3, 4, 5, 6, 7, 8, 9), and it provides easy algorithms to extend arithmetic with single digits to arithmetic with arbitrarily large numbers.

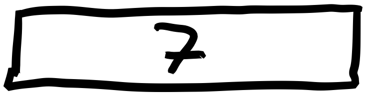
Bar Models

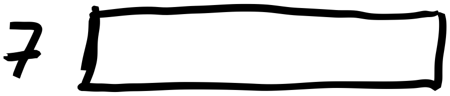
In this section we introduce a problem solving technique which has become popular in recent years in math classes starting from kindergarten on. The technique involves drawing diagrams (called bar models) of problems. Students solve problems by labeling and revising their diagrams and discovering what values parts of their diagrams represent. The technique is simple enough that it can be taught to kindergarten and first graders who can then solve problems (with single digit arithmetic) that generally would have required algebra in more traditional curriculum. As student progress, and as problems become more complex, bar models transition smoothly into basic algebra.

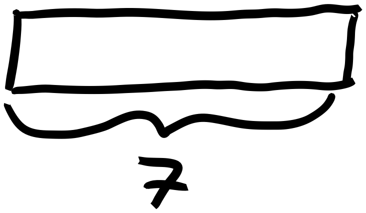
The fundamental tool here is the bar.



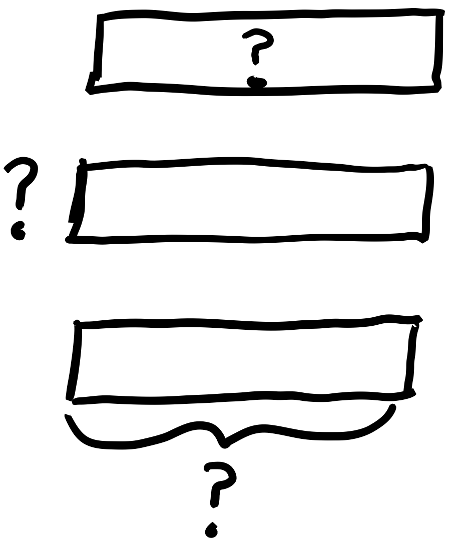
The bar represents a number. When students first encounter bar models, the bar might be an abstraction of a bar made of snap-together blocks. Students might first use bar model techniques with concrete blocks. A bar might represent a known number. Each of these bars represent the quantity 7.



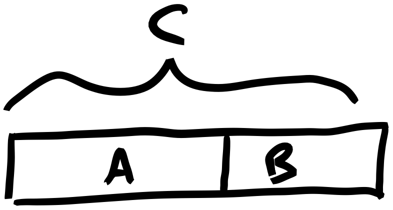




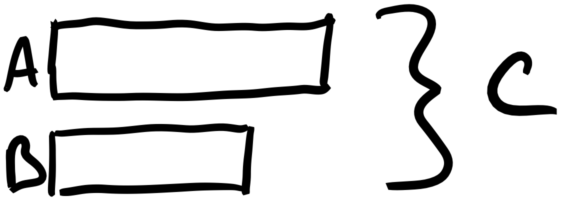
A bar might represent an unknown number. Each of these bars represents an unknown number.



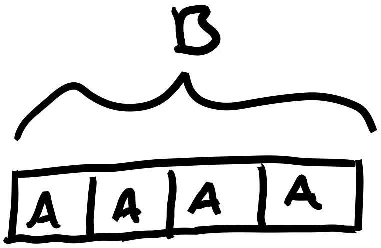
The length of bars roughly represents their value on the level of order. If you know that one bar represents a value larger than a different bar, then you should draw the bar for the larger value longer. However, if you accidentally draw a bar longer than another, do not assume that it represents a larger value. Bars are sometime divided into parts like so:

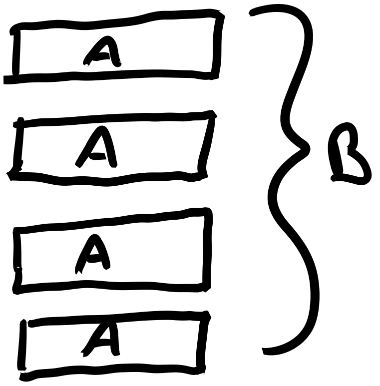


In this diagram, A and B represents part of the whole bar C. A has been drawn longer than B, but that is accidental. The two parts are draw different sizes to emphasize they may not be equal. In a diagram such as this, if the parts A and B are known, then we can add to get C. If C is known and if one of the parts is known we can subtract to find the other part. Sometimes, we might also want to consider combining separate groups or bars rather than separate parts of one group or bar. This might look like:

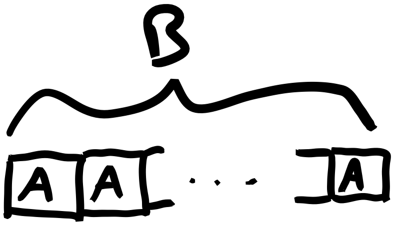


Sometimes, a single bar might be divided into several parts that are all the same size, or several bars that are all the same size are considered together:





In both of these diagrams, it takes 4 copies of A to make B. If we know the number of parts (4 here) and the size of each part, we can multiply to get B. On the other hand, if we know B and the number of parts, we can divide to get the size of each part. If we want to draw a diagram of a situation where we do not know how many parts there are (of if we are lazy and do not want to draw all of the parts) we can draw a diagram such as this one:

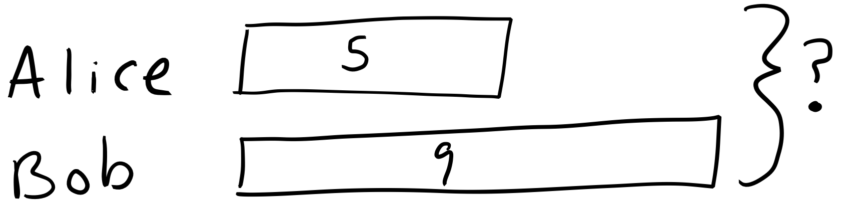


Here, if we know B and the size of each part, we can divide to discover the number of parts.

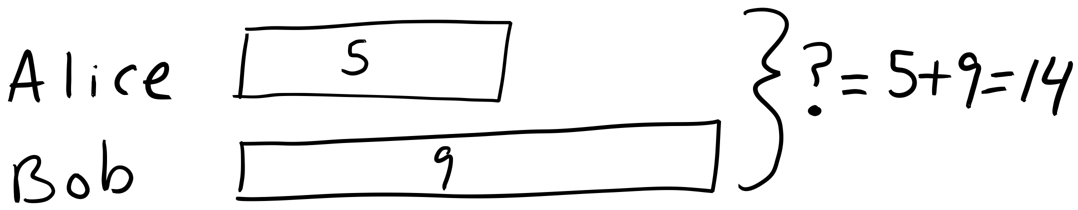
We illustrate here the use of bar models to solve problems involving arithmetic based on single digit arithmetic.

Problem: Alice has 5 blocks. Bob has 9 blocks. How many blocks do they have together?

To solve this problem, we draw two bars, one for Alice's blocks and one for Bob's. We label each bar with the number of blocks each child has. We also indicate that we are looking for the combined number of blocks.



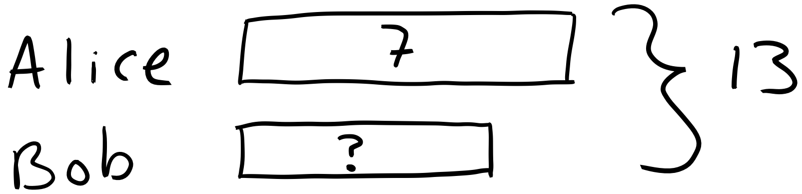
Since we know two parts and are looking for the combined whole, we add to find the total number of blocks.



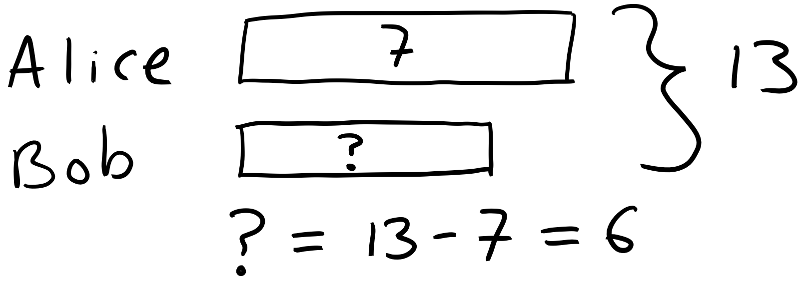
Alice and Bob have 14 blocks together.

Problem: Alice and Bob have 13 blocks together. Alice has 7 blocks. How many blocks does Bob have?

To solve this problem, we first draw a bar for Alice's blocks and a bar for Bob's blocks. We label Alice's bar with 7 and the combined bars with 13. Since we do not know Bob's number of blocks, we label his bar with a question mark. Notice that at this point we do not know whose bar should be longer so any comparison of the lengths of the bars should be avoided.



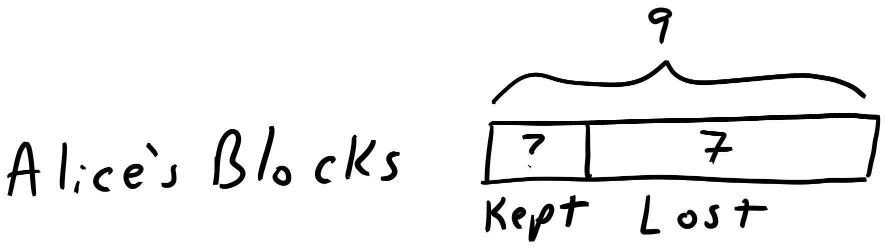
Since we know a whole (13) and we are looking for a part (Bob's number of blocks) we subtract.



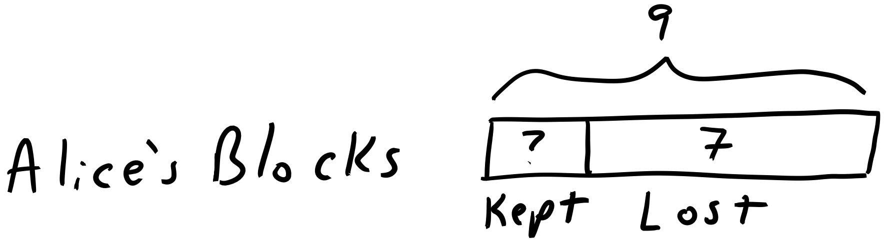
Bob has 6 blocks.

Problem: Alice had 9 blocks, but she lost 7. How many blocks does she have now?

For this problem, we draw one bar for Alice's blocks. We divide the bar into two parts – those that were lost and those that were kept. We label the entire bar with 9 and the lost part with 7. Since we do not know how many blocks Alice has now, we label the kept part with a question mark.



Since we know the whole and a part, we subtract to find the other part.

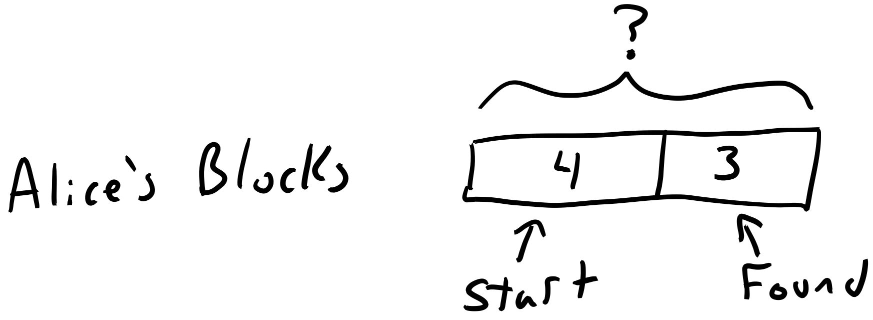


$$? = 9 - 7 = 2$$

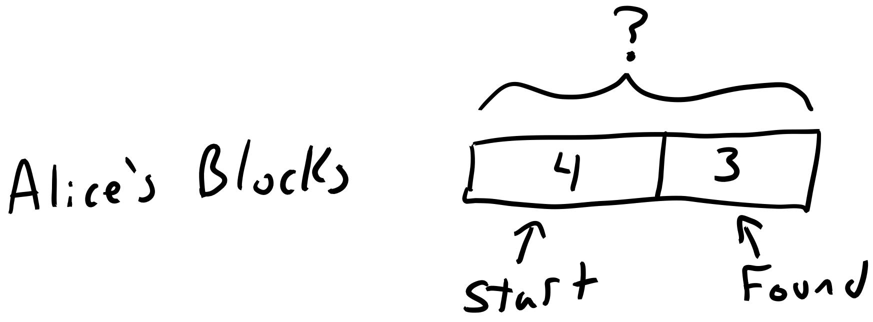
Alice has 2 remaining blocks.

Problem: Alice had 4 blocks, but she found 3 more. How many does she have now?

We draw a bar for Alice's blocks and divide it into two parts representing those she started with and those she found. We label the entire bar with a question mark since we do not know the total number of marbles she has.



Since we know two parts and are looking for the combined whole, we add.

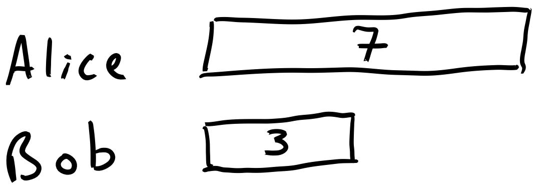


$$? = 4+3=7$$

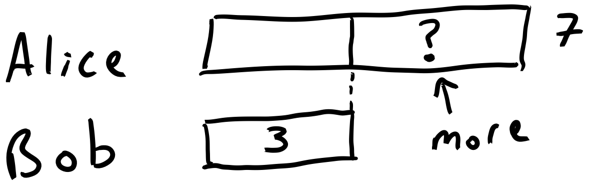
Alice now has 7 blocks.

Problem: Alice has 7 blocks. Bob has 3 blocks. How many more blocks does Alice have than Bob?

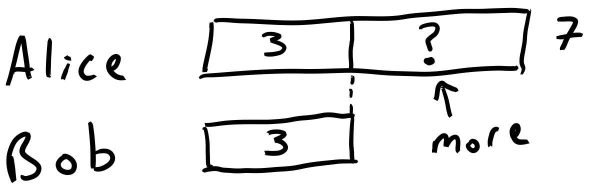
We start by drawing bars for Alice's and Bob's numbers of blocks. We make Alice's bar longer since we know she has more blocks. We initially label the size of the blocks.



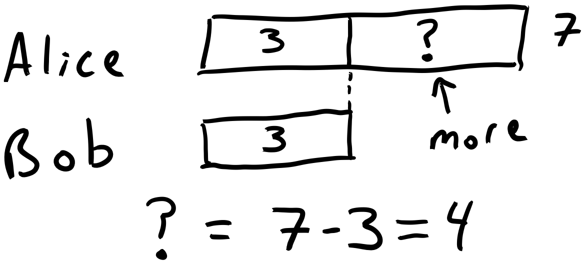
Since this problem asks how many more blocks Alice has, we will end up modifying her bar. Therefore, we move the label 7 outside of the bar. Then, we divide her bar into two parts. One part is the same size as Bob's bar, and we indicate this in the diagram with a dotted line. The other part is the "more" part which we do not know.



We copy the label 3 from Bob to the left part of Alice's bar.



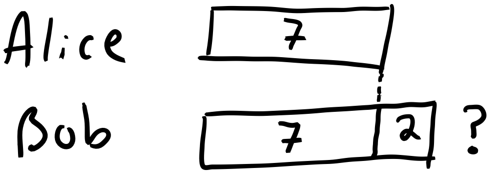
Now, we focus on Alice's bar. It is divided into two parts. One part is of size 3, and the other we do not know. Since we know the size of the whole, we can subtract to find out the "more" part of Alice's bar.



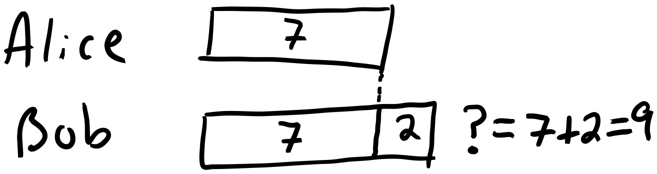
Alice has 4 more blocks than Bob.

Problem: Alice has 7 blocks. Bob has 2 more blocks than Alice. How many blocks does Bob have?

We draw a bar for Alice and one for Bob. Bob's bar we make longer since he has more blocks. We divide Bob's bar into two parts. One part is the same size as Alice's bar. The other contains 2 blocks. We also label Bob's bar with a question mark since we do not yet know how many blocks he has.



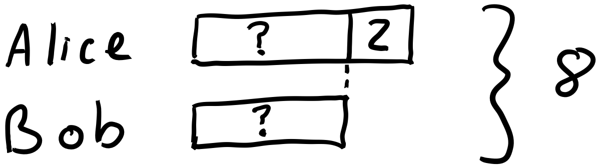
Focusing now on Bob's bar, we see that it has two parts whose sizes we know. To find the number of Bob's blocks, we add.



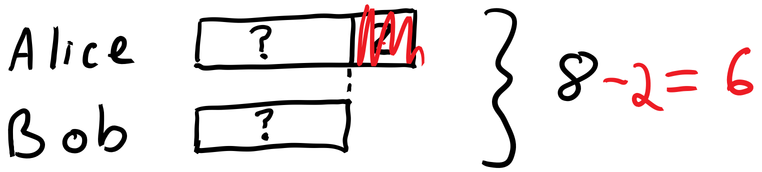
Bob has 9 blocks.

Problem: Alice has 2 more blocks than Bob. Together, they have 8 blocks. How many blocks does each have?

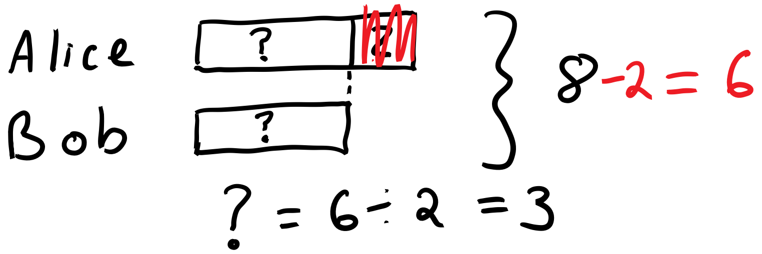
We first draw bars for Alice and Bob, making Alice's 2 blocks longer than Bob's and indicating that the two bars combined include 8 blocks.



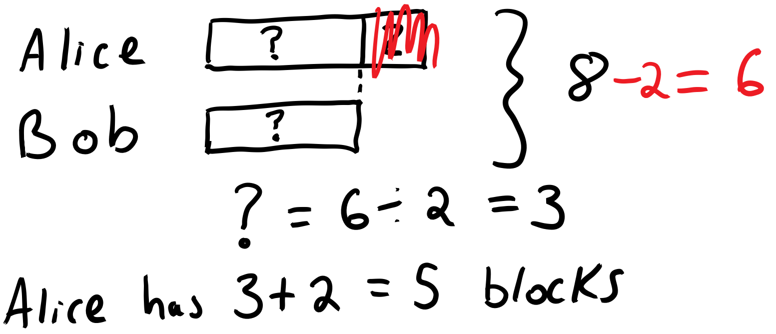
The 2 here is a distraction. We remove the 2 from the diagram and subtract it from 8 to arrive at a new diagram.



We now have two parts labeled with question marks that add up to 6. We can divide to discover the value of the question mark.



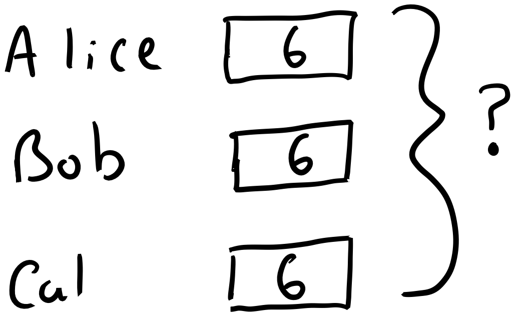
This is the number of Bob's blocks. To find Alice's blocks, we have to return to the original diagram. Alice's bar is now the sum of two parts, one with 2 blocks and one with ?=3 blocks. Adding gives her number of blocks.



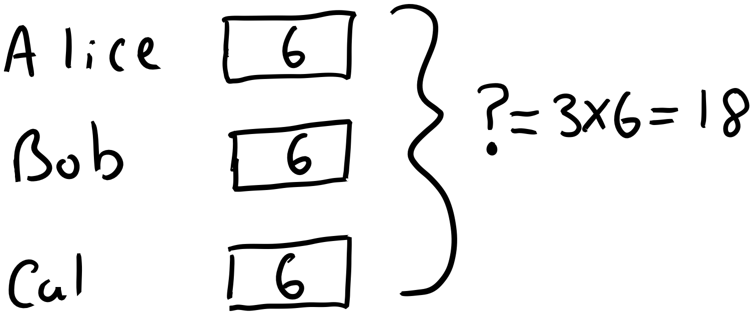
Bob has 3 blocks and Alice has 5 blocks.

Problem: Alice, Bob, and Cal each have 6 blocks. How many blocks do they have together?

We draw a bar for each of Alice, Bob and Cal and label each with a 6. Note that the bars are all the same length. We mark the total of all of the bars with a question mark.



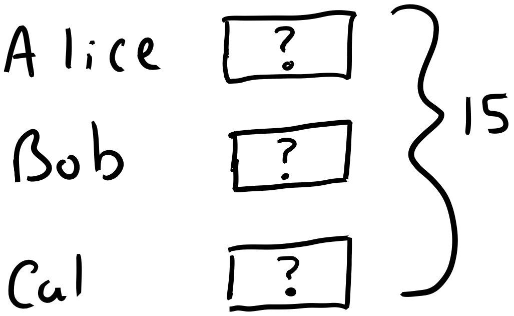
Since we have three groups of size 6, we multiply to find the combined total.



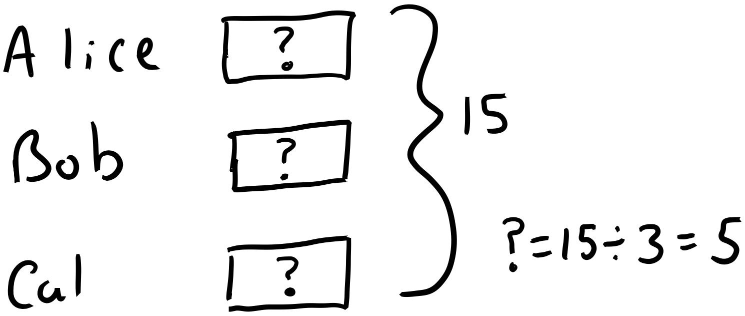
Combined, the children have 18 blocks.

Problem: Alice, Bob, and Cal each have the same number of blocks. They have fifteen blocks together. How many blocks does each have?

We draw bars for each of Alice, Bob, and Calm, making them all the same length and labeling them all with a question mark since we do not know their size. We also indicate that the three groups together add up to 15 blocks.



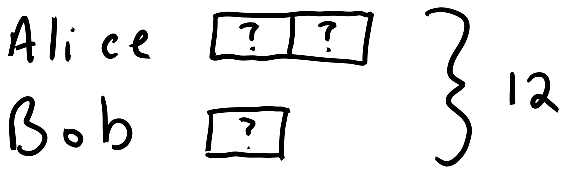
Since we have three equal size groups that add up to 15, we can divide to find the size of each group.



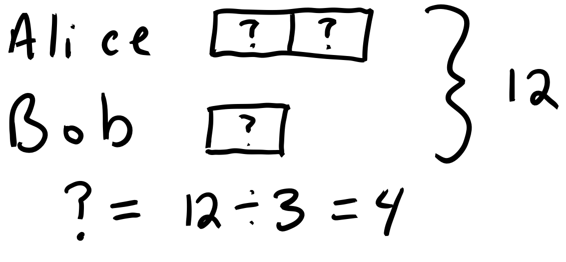
Each child has 5 blocks.

Problem: Alice has twice as many blocks as Bob. Together they have 12 blocks. How many blocks does each person of them have?

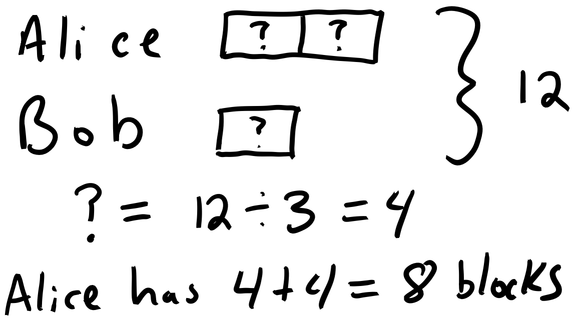
We draw bars for Alice and Bob, making sure that Alice's bar is two copies of Bob's bar. We also indicate that the bars together represent 12 blocks.



Since we have three equal size bars adding up to 12, we can divide to find out how many blocks are represented by each bar.



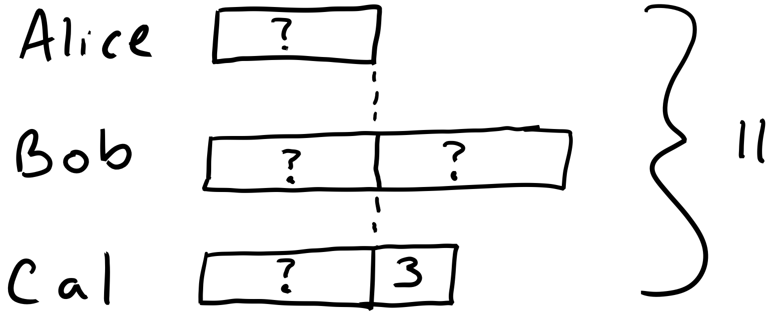
This happens to be the number of blocks that Bob has. For Alice, we note that her bar is composed of two parts which we now know to each be 4 blocks. To find the size of her bar, we add (or multiply by 2).



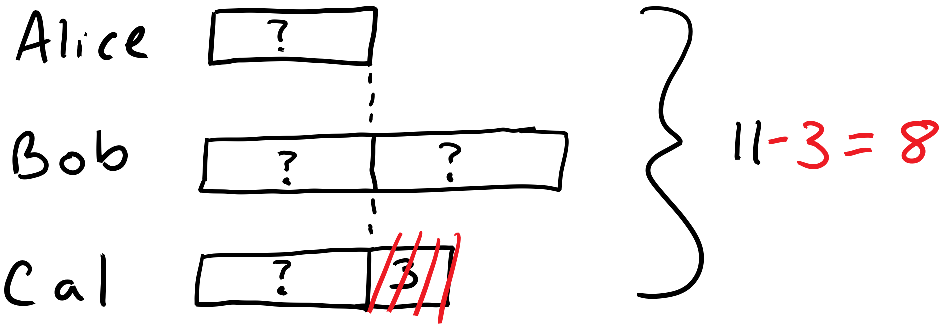
Bob has 4 blocks and Alice has 8 blocks.

Problem: Bob has twice as many blocks as Alice. Cal has 3 more blocks than Alice. Together, they have 11 blocks. How many blocks does each person have?

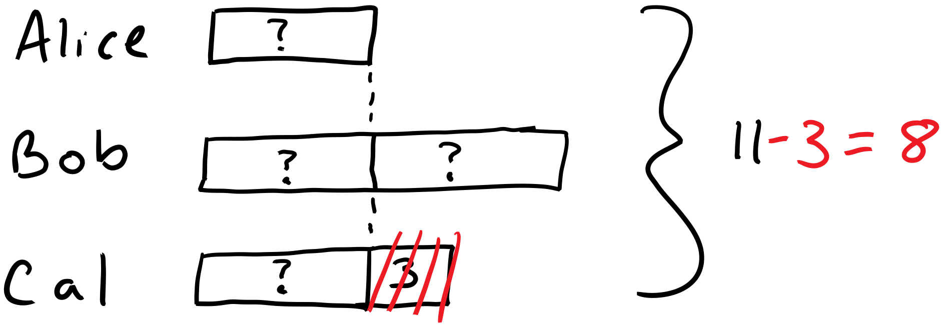
We draw bars for Alice, Bob, and Cal. Bob's bar is two identical copies of Alice's bar. Cal's bar is as long as Alice's bar with a "more" part that is 3 blocks. Combined, the three bars add up to 11.



The 3 is basically a distraction here. We remove it from the diagram and subtract 3 from the total.



What is now left is 4 equal size groups that add up to 8. We can divide to find the size of each group.



Alice: $? = 8 \div 4 = 2$

Bob: $? + ? = 2 + 2 = 4$

Cal: $? + 3 = 2 + 3 = 5$

The question mark we just found happens to be the number of blocks that Alice has. We then add this amount to itself to find out how many blocks Bob has. For Cal, we add 3. Alice has 2 blocks. Bob has 4 blocks, and Cal has 5 blocks.

Properties of Addition and Subtraction

Our arithmetic operations satisfy a number of nice properties that will sometimes allow us to perform computation more easily. Recall that we have defined addition, subtraction, multiplication, and division this way:

Addition: $A + B$ is the number of objects in a group formed by combining a group of A objects with a group of B objects.

Subtraction: $A - B$ is the number of objects left over after B objects are removed from a group of A objects.

Multiplication: $A \times B$ is the number of objects in A groups containing B object each.

Division: $A \div B$ is the number of objects in each group when A objects are placed into B groups which are all the same size.

Properties of Addition

Suppose that we have a box with A marbles in it and another box with B marbles in it. If we pour with box with B marbles into the box with A marbles, adding B to A, then we have $A + B$ marbles in the box. On the other hand, if we pour the box with A marbles into the other box, adding A to B, then we have a box with $B + A$ marbles in it. However, the $A + B$ marbles are the same marbles as the $B + A$ marbles, so it has to be that $A + B = B + A$. This fact is known as the **commutative property of addition**.

Now suppose that we have three boxes with A, B, and C marbles in them. If we combine the first two boxes, we have a group of $A + B$ marbles. If we then combine the third box with this group, we have one group of $(A + B) + C$ marbles. Now suppose that instead of combining the first two boxes we combined the second two boxes to form a group of $B + C$ marbles. Then, when we combine the first box with these marbles, we have a group of $A + (B + C)$ marbles. However, the $(A + B) + C$ marbles and the $A + (B + C)$ marbles are the same marbles, so it has to be that $(A + B) + C = A + (B + C)$. This fact is known as the **associative property of addition**. The associative property of addition implies that it does not matter how we group numbers when we add. Therefore, when we are adding more than two numbers, we usually do not write parentheses. For $(A + B) + C = A + (B + C)$, we will usually write simply $A + B + C$.

Next, suppose that we have a box with 0 marbles and a box with A marbles. If we combine the marbles in the two boxes, all we have are the A marbles from the second box. That is, $0 + A = A$. Similarly, if we start with a box of A marbles and add no marbles to the box, then we have A marbles, so $A + 0 = A$. Because of this, 0 is called the **additive identity**.

Properties of Subtraction

Subtraction is not commutative. Suppose that we have a box with 3 marbles. we could take 2 marbles from the box and be left with $3 - 2 = 1$ marbles. However, if we have a box with 2 marbles, we cannot take 3 marbles out of the box to compute $2 - 3$. At this point, $2 - 3$ is not even defined, so it cannot be that $3 - 2 = 2 - 3$. We will address subtracting a larger number from a smaller number later after we have introduced negative numbers. Even then, this equality cannot be true.

Subtraction is not associative either. To see this, simply consider that

$$(6 - 3) - 2 = 3 - 2 = 1$$

but

$$6 - (3 - 2) = 6 - 1 = 5.$$

Thus subtraction is not associative.

The number 0 is almost but not quite an identity for subtraction. If we have a box of A marbles and take no marbles out of the box, we still have A marbles. This means that $A - 0 = A$. However, as with the commutativity discussion above, it does not make sense to try to compute $0 - A$ at this point. This computation will be important to our introduction of negative numbers later.

Interaction of Addition and Subtraction

Suppose that we have a box containing A marbles and that we take B marbles out of the box. That leaves us with $A - B$ marbles in the box. What happens when we put the B marbles back into the box? If we put the B marbles back into the box, then we are back to having all A marbles in the box. However, when we put the B marbles back into the box, we are combining a group of $A - B$ marbles with a group of B marbles. This means we are calculating $(A - B) + B$. Since we have already seen this leaves us with A marbles, we have that $(A - B) + B = A$. Thus $A - B$ is the number we can add to B to get A. Some books use this as the definition of subtraction. We can express this with an equation by saying that if $A - B = C$ then $A = B + C$. A similar discussion will show that $(A + B) - B = A$. Adding B and subtracting B are **inverses** of each other. If we add B and then subtract B we get back to where we started, and if we subtract B and then add B we get back to where we started.

We have now established these properties of arithmetic.

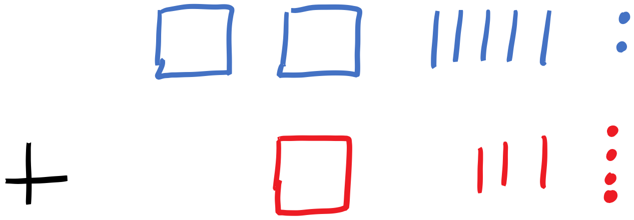
Properties of Addition and Subtraction	
Commutative	$A + B = B + A$
Associative	$(A + B) + C = A + (B + C)$
Identity	$A + 0 = 0 + A = A$ $A - 0 = A$
Inverse	$(A - B) + B = (A + B) - B = A$

Addition Algorithm

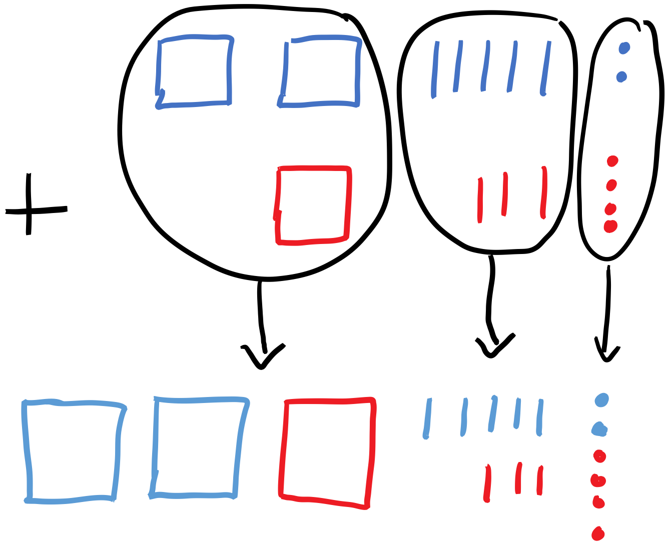
Base ten notation allows us to extend single digit arithmetic facts to numbers with more than one digit. That is, if we know how to add, subtract, multiply, and divide one digit numbers, then we can extend what we know to numbers with more than one digit. In this section, we start with base ten bundling diagrams and explain the standard algorithm for addition. We do so through two examples.

Problem: Add $252 + 134$.

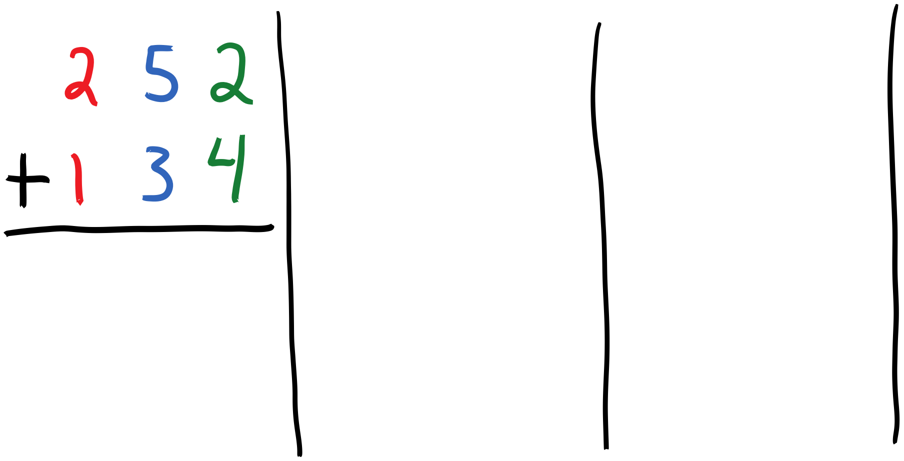
We start by drawing base ten bundling diagrams for 252 and 134.



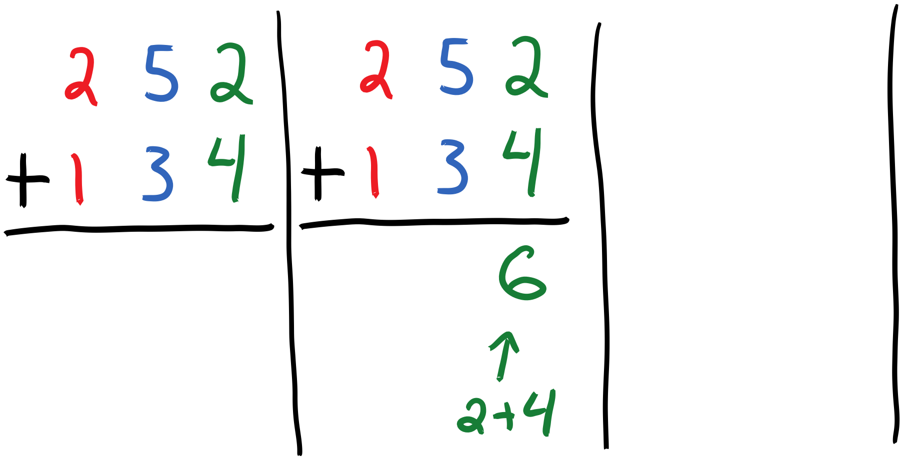
We then combine the two diagrams, grouping all of the ones, tens, and hundreds together.



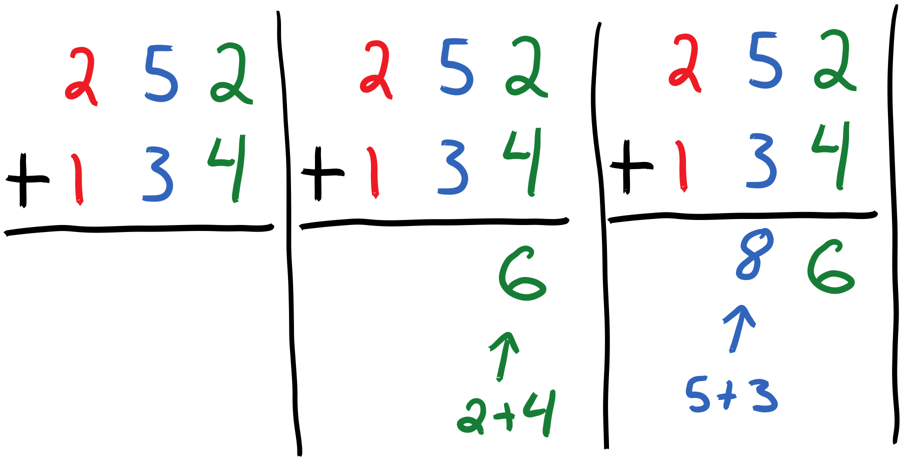
The result is a perfectly acceptable base ten bundling diagram with three hundreds, eight tens, and six ones. It appears that $252 + 134 = 386$. We now demonstrate the same steps with digits rather than bundling diagrams. First, we stack the two numbers on top of each other, being careful to line up the ones, tens, hundreds, and so forth.



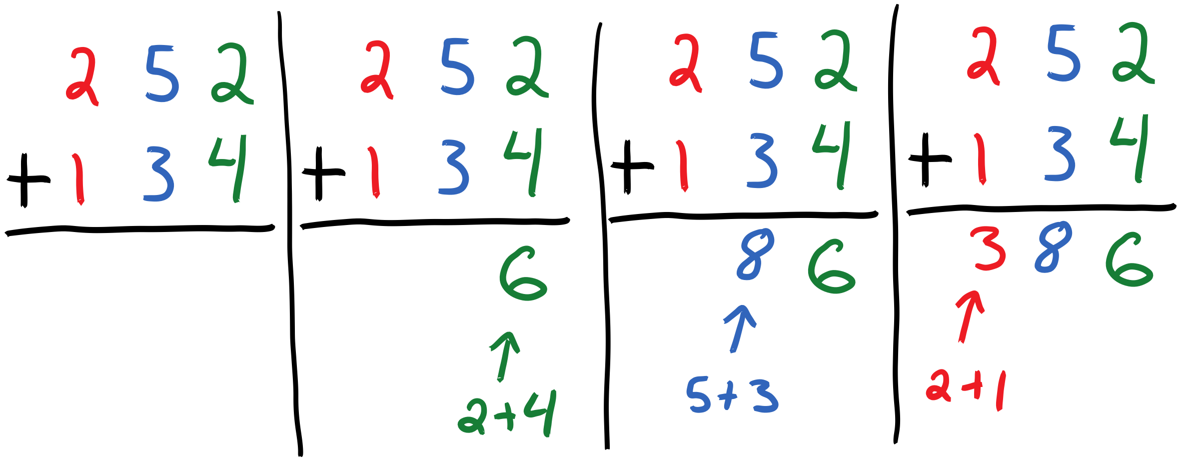
We add the ones digits.



We add the tens digits.



We add the hundreds digits.



As long as each sum of digits is not too big, this process works beautifully and simply. Things are only slightly more complex if any of the sums of digits is too big. We see this in the next example. The computation we just made with bundling diagrams can also be written in expanded notation. First, here is expanded notation with words.

$$\begin{aligned} 252+134 &= 2\ hundreds + 5\ tens + 2 + 1\ hundred + 3\ tens + 4 \\ &= 2\ hundreds + 1\ hundred + 5\ tens + 3\ tens + 2 + 4 \\ &= 3\ hundreds + 8\ tens + 6 \\ &= 386 \end{aligned}$$

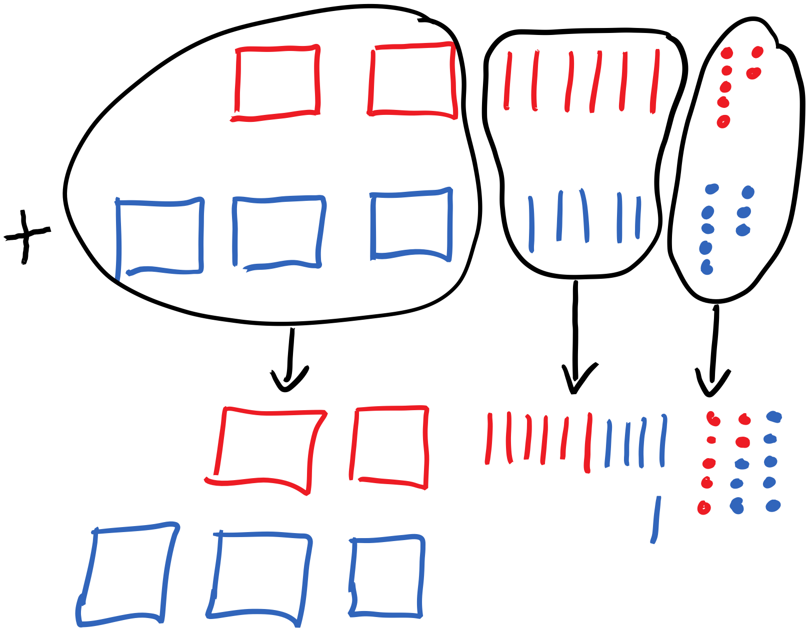
Next, here is the same computation using multiplicative notation.

$$\begin{aligned} 252+134 &= (2\times 100)+(5\times 10)+2+(1\times 100)+(3\times 10)+4 \\ &= (2\times 100)+(1\times 100)+(5\times 10)+(3\times 10)+2+4 \\ &= (3\times 100)+(8\times 10)+6 \\ &= 386 \end{aligned}$$

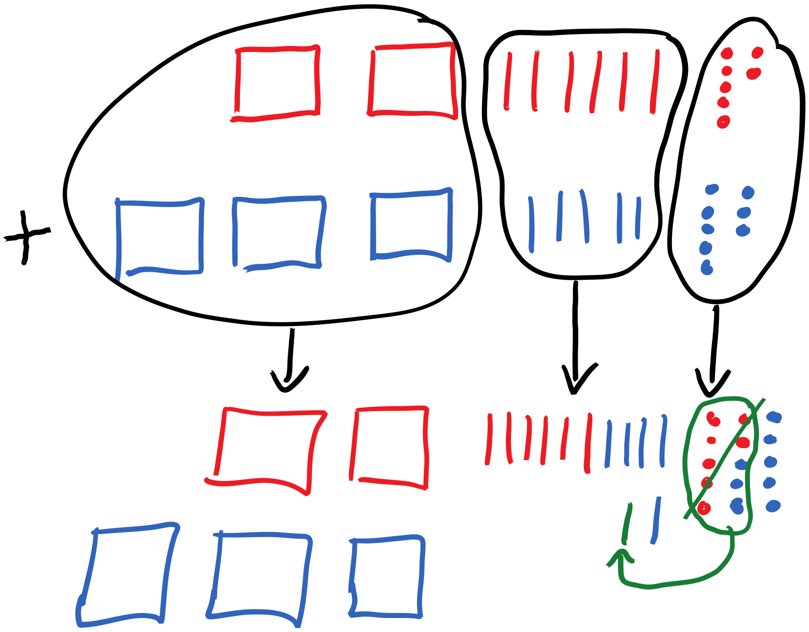
In both of these computations, the first equal sign is just expanding the base ten notation. The second equal sign could be thought of as using the associative and commutative properties of addition (which we will discuss more later). However, it is simpler at this point to note that when combining different groups, it does not matter in what order we combine them. This was noted in the section on tally arithmetic. The third equal sign could be considered an application of the distributive property of multiplication (again, we will discuss this later). However, it is easier at this point to note that 2 hundreds plus 1 hundred is 3 hundreds just like 2 apples plus 1 apple is three apples. The last equal sign is again just base ten notation.

Problem: Add 267 + 358.

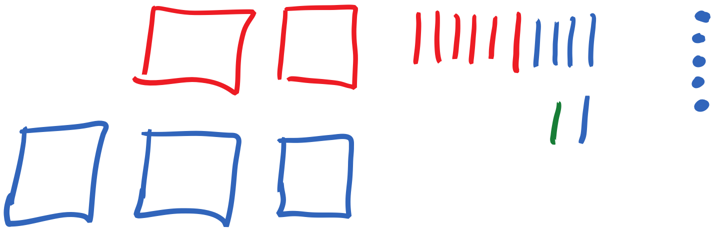
We again begin this problem by drawing bundling diagrams for 2567 and 358 and combining the separate diagrams into one.



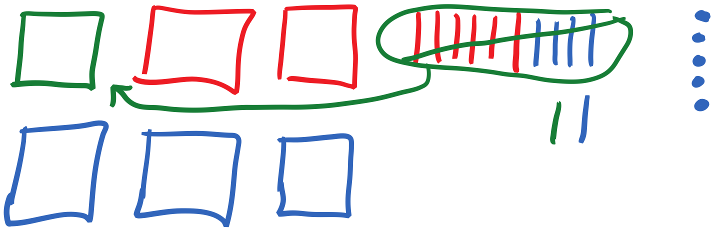
At first glance, it appears as if combining bundles worked as well as last time. However, if we look at the ones, we see that there are 15 single dots. In base ten bundling, there can be no more than nine dots, so this is not legal base ten bundling. To fix this problem, we will group ten of the dots together, remove them and replace them with a line representing a ten.



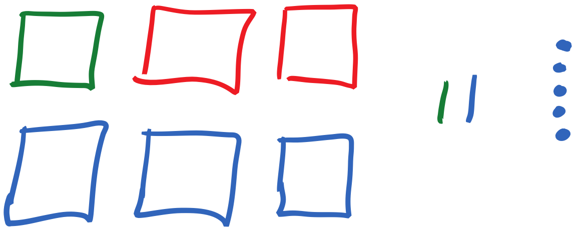
A cleaned up bundling diagram now looks like this.



If we now look at the tens, we see that there are again too many tens for base ten bundling. We group ten of the lines together, remove them, and replace them with a square for a hundred.



After a little cleaning up, our diagram now looks like this.

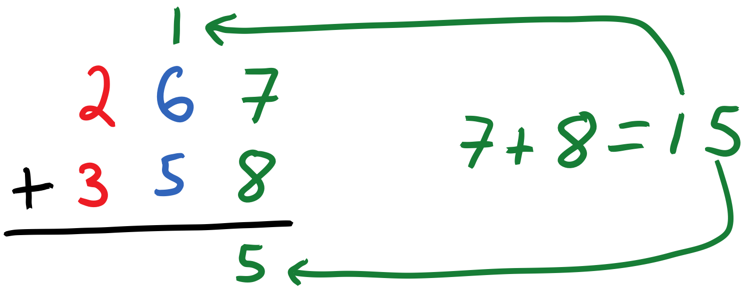


This is now a legal base ten bundling diagram for 625, so it appears that $267 + 358 = 625$. The process of removing ten of the objects from one place value and increasing the number of objects in the next place value is usually called **carrying**. Here is how the process looks when we write out digits rather than using bundles. First, we stack the numbers on top of each other, carefully lining up the ones, tens, and hundreds.

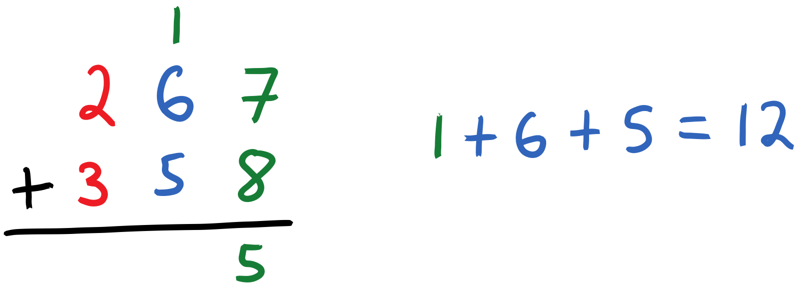
$$\begin{array}{r} 2\;6\;7 \\ +\,3\;5\;8 \\ \hline \end{array}$$

Next, we add the digits in the ones place.

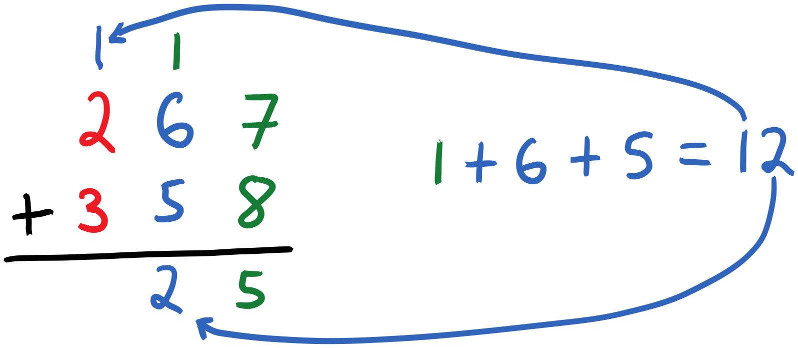
Since we are not allowed in base ten notation to have 15 ones, we take ten of the ones and increase the number of tens in the problem. We do this by writing a 1 over the tens column. The remaining 5 ones go in the ones place of our answer.



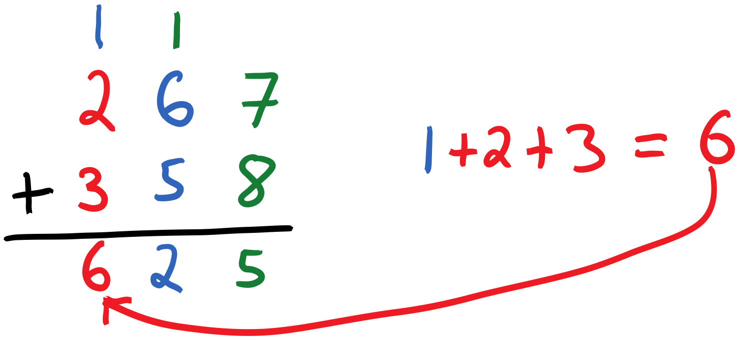
We now add all of the tens, including the extra 1 that we carried over from adding the ones.



Since we cannot have 12 tens in base ten notation, we take ten of them and make a hundred to add to the hundreds place (as a new 1 above the hundreds column). The remaining 2 go into the tens place of our answer.



Finally, we add all of the hundreds (including the carry).

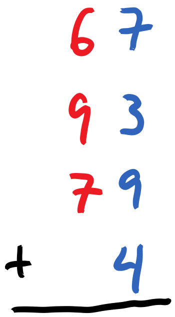


Since 6 is a legal digit to have in base ten notation, we do not have to carry. If we had a sum larger than 9, such as 13, then we would carry 1 over to the thousands place (which would cause us to start writing a thousands place). In expanded form, our computations look like this.

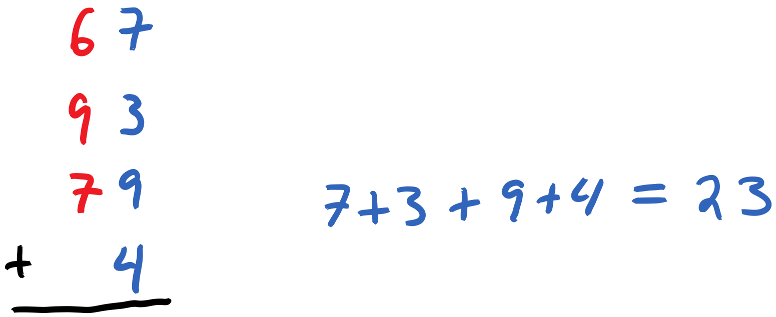
$$\begin{aligned}
267 + 358 &= 2\ \mathit{hundreds} + 6\ \mathit{tens} + 7 + 3\ \mathit{hundreds} + 5\ \mathit{tens} + 8 \\
&= 2\ \mathit{hundreds} + 3\ \mathit{hundreds} + 6\ \mathit{tens} + 5\ \mathit{tens} + 7 + 8 \\
&= 5\ \mathit{hundreds} + 11\ \mathit{tens} + 15 \\
&= 5\ \mathit{hundreds} + 11\ \mathit{tens} + 10 + 5 \\
&= 5\ \mathit{hundreds} + 11\ \mathit{tens} + 1\ \mathit{ten} + 5 \\
&= 5\ \mathit{hundreds} + 12\ \mathit{tens} + 5 \\
&= 5\ \mathit{hundreds} + 10\ \mathit{tens} + 2\ \mathit{tens} + 5 \\
&= 5\ \mathit{hundreds} + 1\ \mathit{hundred} + 2\ \mathit{tens} + 5 \\
&= 6\ \mathit{hundreds} + 2\ \mathit{tens} + 5 \\
&= 625
\end{aligned}$$

Problem: Add $67 + 93 + 79 + 4$

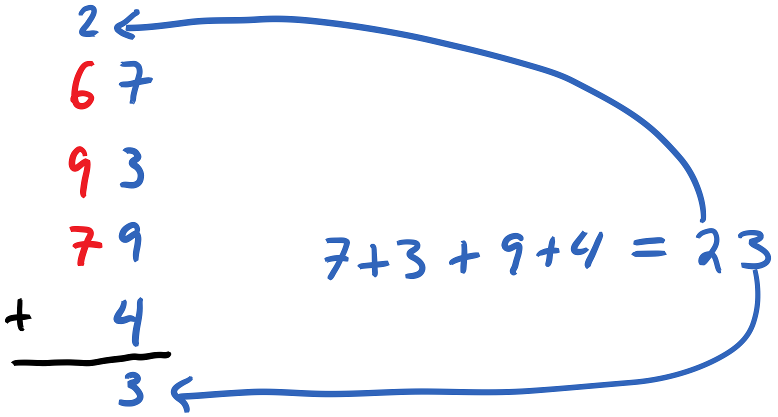
For this problem, we will go straight to the stack-add-and-carry algorithm rather than bundling. Three things of interest will happen in this problem. First, we are adding more than 2 numbers. Second, we will see a carry amount larger than 1. Third, we will have a final carry that causes us to have a hundreds place even though the original numbers did not have a (visible) hundreds place. First, we stack the numbers, being careful to line up tens and hundreds.



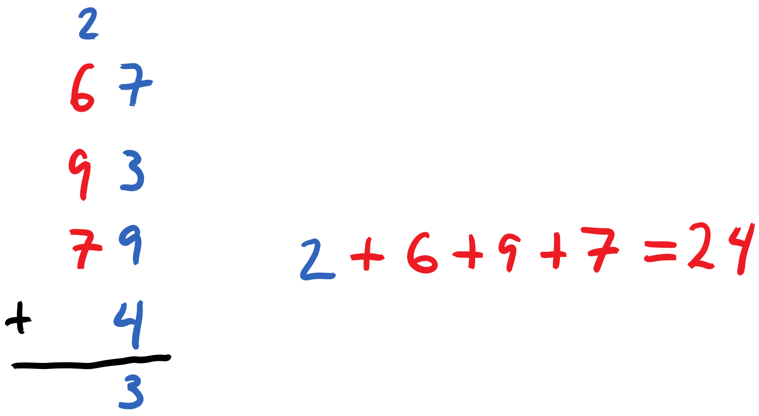
We add all of the ones digits. The ones digits here add to 23. That is twenty, or two tens, and three.



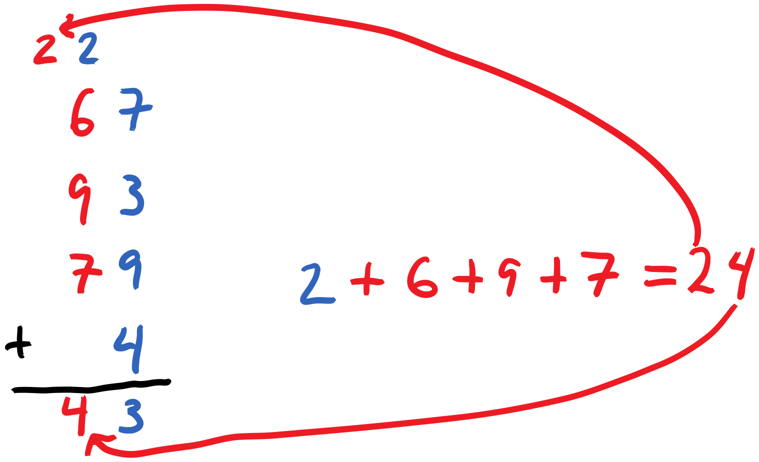
The two tens we carry, placing them in the tens column as a 2 at the top of the column. The 3 becomes the digit in the ones place of our answer.



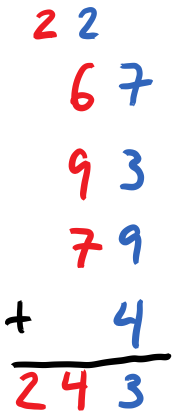
Now we add all of the digits in the tens column, including the carried 2. The tens digits add to 24.



The 2 represents twenty tens or 2 hundreds. We carry these by placing a 2 at the top of the (new) hundreds column. The 4 goes in the tens place of the final answer.



Now we add the hundreds digits. This amounts to only the 2 we carried over from the tens digits.



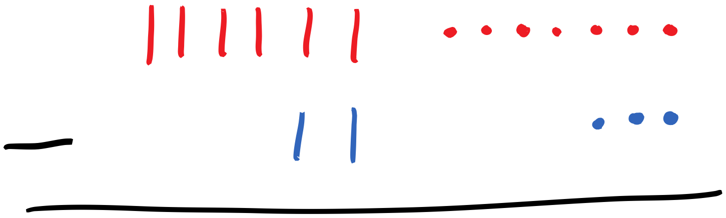
It appears that $67 + 93 + 79 + 4 = 243$.

Subtraction Algorithm

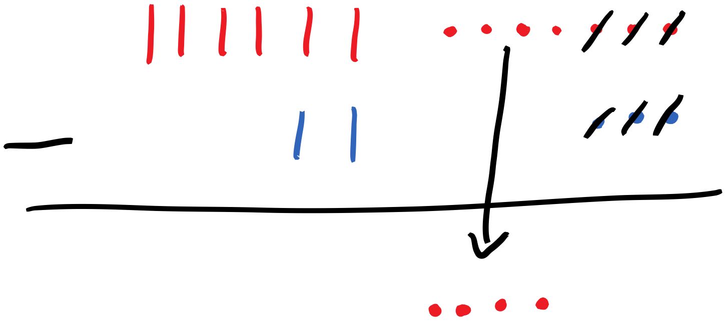
In this section, we start with base ten bundling diagrams and demonstrate the standard algorithm for subtraction.

Problem: Subtract 67 − 23

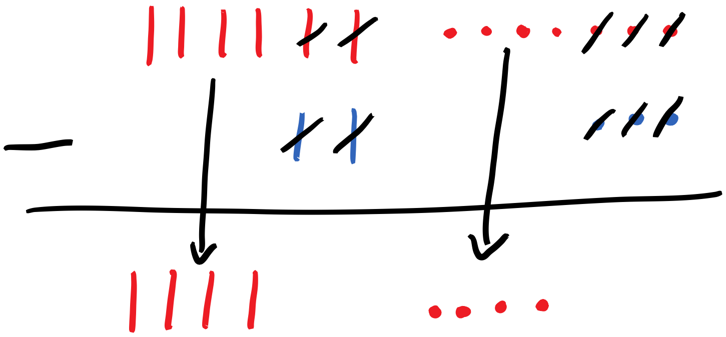
We first address this problem with bundling diagrams. We first draw bundling diagrams for 67 and 23.



Next, for each dot representing a one in 23, we remove a one from 67. This means we cross that dot out along with a dot in 67. This leaves 4 ones left, which we copy into our final answer.

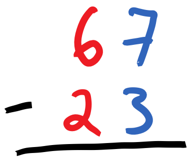


Next, we repeat the process with tens. For each line representing a ten in 23, we remove a ten from 67. That leaves 4 tens from the original 6 in 67. We copy those 4 tens to our final answer.

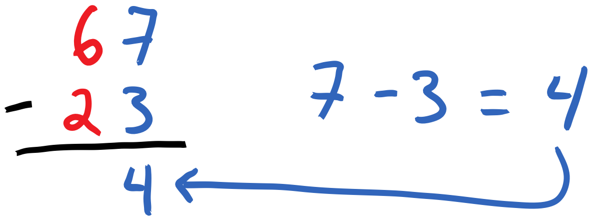


If our numbers had included a hundreds or thousands place, we would continue the process. We have now removed 23 from the original 67, leaving us with a base ten bundling diagram for 44. It appears that $67 - 23 = 44$.

Now we demonstrate the same arithmetic using base ten notation rather than bundling. First, we stack our numbers, carefully lining up the ones and tens.



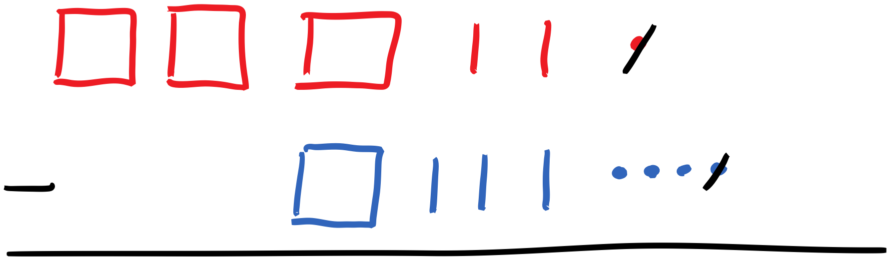
Now we subtract the ones. When the 3 ones from 23 are removed from the 7 ones of 67, we are left with 4 ones.



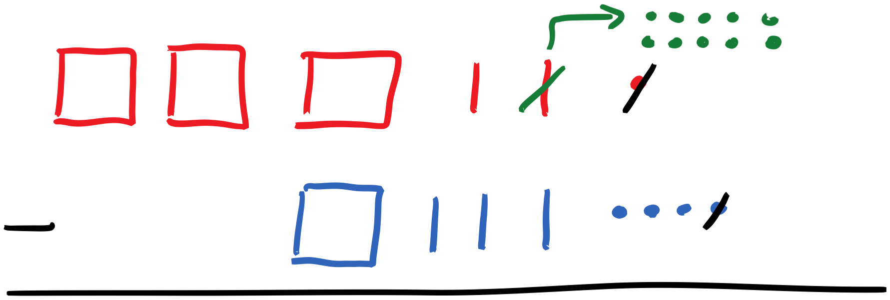
Next, we subtract the tens. When the two tens from 23 are removed from the 6 tens of 67, we are left with 4 tens.

Problem: Subtract $321 - 134$.

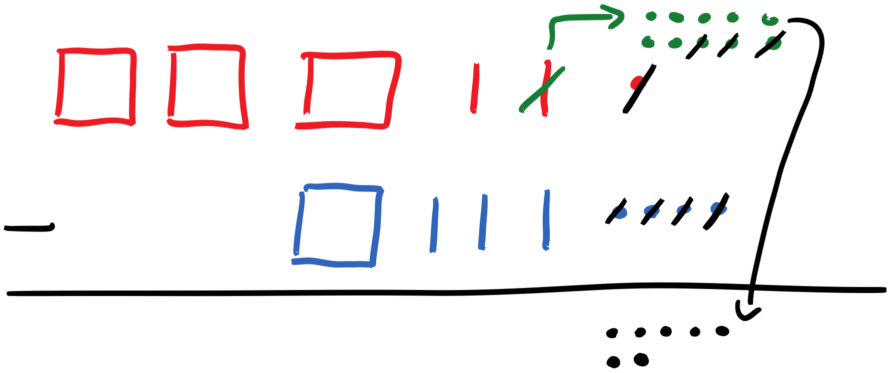
Again, we begin with bundling. We start by drawing bundling diagrams for 321 and 134. Then we start crossing out ones from 134 and corresponding ones from 321.



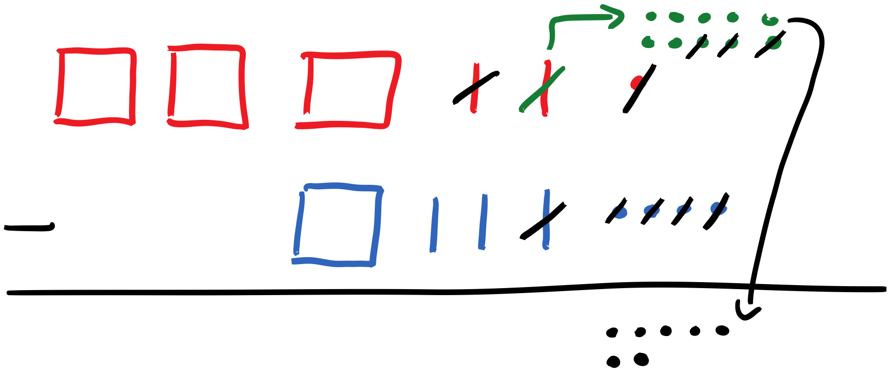
We immediately notice that we have more blue dots than red dots, so we cannot cross out a red dot for each blue dot. Recall that each line segment representing a ten is representing ten ones, or ten dots. Therefore, we can cross out one of the red line segments representing a ten and replace it with ten dots. This process is usually called **borrowing** (although stealing is probably a more appropriate term).



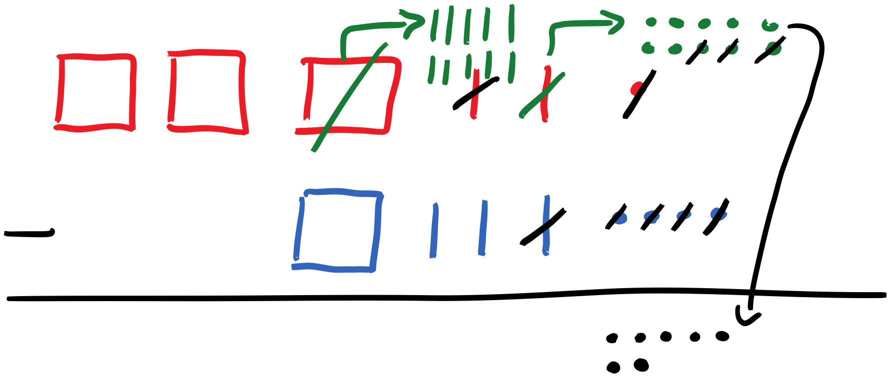
Now that we have enough dots, we can continue crossing out ones until all of the blue dots are used up. This leaves us with 7 dots in the top row, which we copy to our final answer.



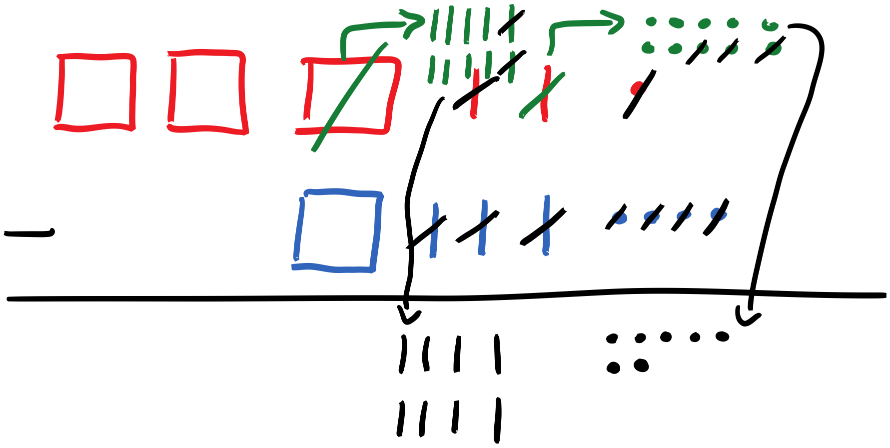
Now we start crossing out tens.



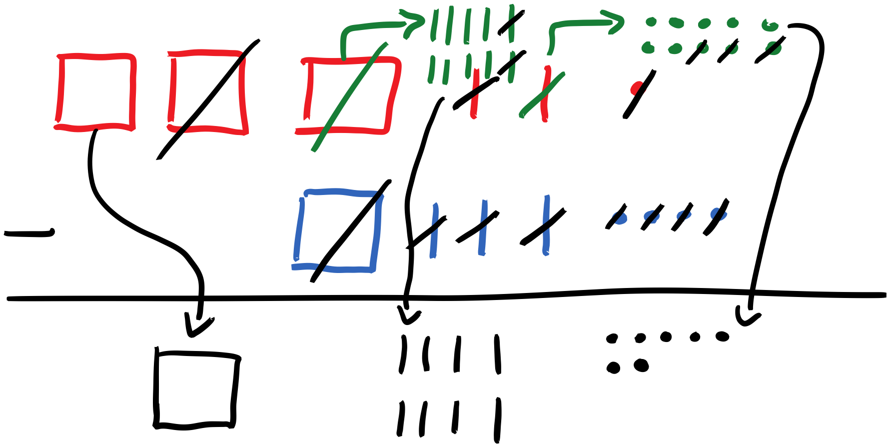
We immediately notice again that we have more blue tens than red tens. To remedy this situation, we again borrow. We cross off one of the red hundreds and replace it with ten tens.



Now we can continue crossing out tens. When all of the blue tens have been crossed off, we are left with 8 tens in the top row, which we copy to our final answer.



Now we cross out the remaining blue hundred along with a red hundred. That leaves one hundred remaining, so we copy it to our final answer.



We have now calculated $321 - 134 = 187$ using bundling. Now we demonstrate the same arithmetic using base ten notation. First, we stack our numbers, being careful to line up ones, tens, and hundreds.

$$\begin{array}{rccc} & 3 & 2 & 1 \\ - & 1 & 3 & 4 \\ \hline \end{array}$$

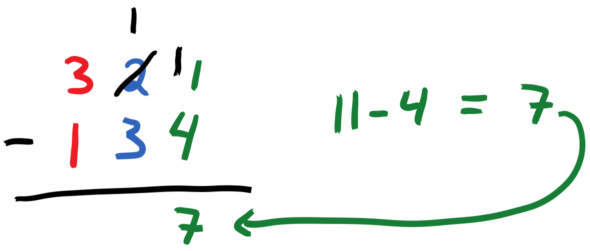
We try to subtract the ones, but we see that the number of ones in the second line is too large to subtract from the number of ones in the first line.

$$\begin{array}{rccc} & 3 & 2 & 1 \\ - & 1 & 3 & 4 \\ \hline \end{array} \qquad 1 - 4 = ?$$

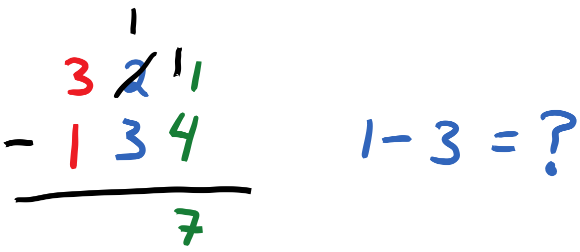
To borrow, we cross out the 2 tens on the top line and replace them with 1 ten. Then we add the ten that we borrowed (stole) to the ones place so that we have 11 ones to subtract from.

$$\begin{array}{rccc} & 3 & \overset{1}{\cancel{2}} & {}^{1}1 \\ - & 1 & 3 & 4 \\ \hline \end{array}$$

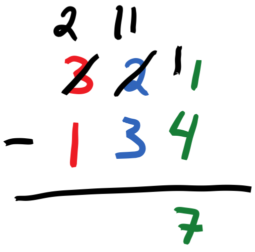
Now we can subtract ones to get $11 - 4 = 7$, which we copy into the ones place of the answer.



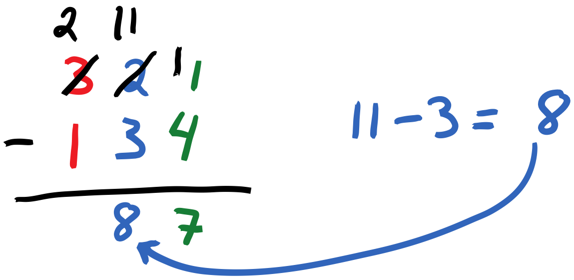
When we try to subtract tens, we again see that we have too many tens on bottom.



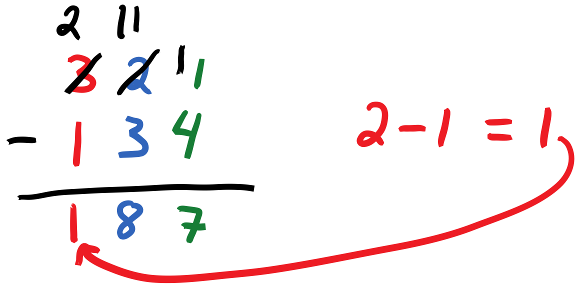
We borrow from the hundreds place by first crossing out the 3 hundreds on top and replacing them with 2 hundreds. We than turn the borrowed hundred into ten tens and add these to the 1 ten to get 11 tens on top.



Now we can subtract to see that we have 8 tens to place in our answer.

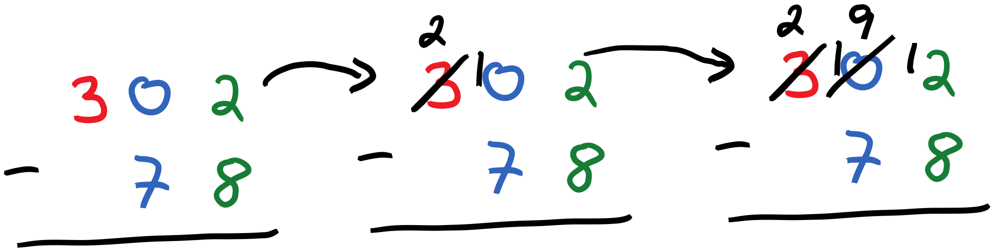


Finally, we subtract the 2 hundreds on the first row minus the 1 hundred on the second row.

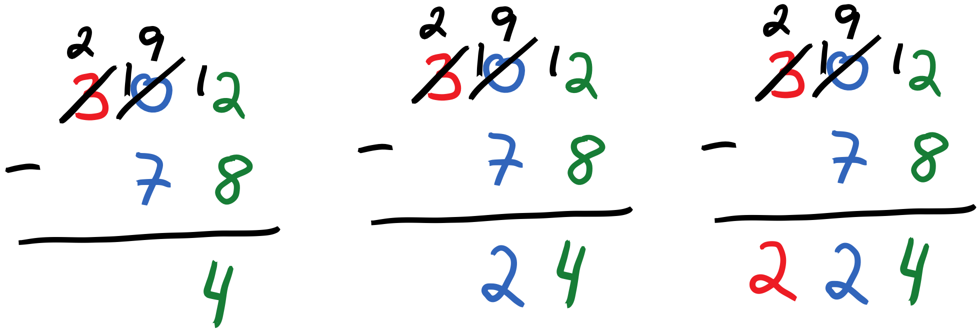


Problem: Subtract $302 - 78$.

For this problem, we go straight to base ten notation and skip the bundling. We immediately notice that we need more ones on the top line in order to subtract 8 ones, so we need to borrow. However, there are no tens to borrow from. The solution here is to first borrow from the 3 hundreds to get 10 tens. Then we can borrow one of these tens to get 10 additional ones.



Now that we have enough ones on top, we can begin subtracting with the ones. We have $12 - 8 = 4$ ones in our answer. Then we move to the tens. We have $9 - 7 = 2$ tens. Finally, we arrive at the hundreds. There are 2 hundreds on top and no hundreds on bottom to remove. Therefore, there are 2 hundreds in our difference.



We now have that $302 - 78 = 224$.

Properties of Multiplication and Division

Our arithmetic operations satisfy a number of nice properties that will sometimes allow us to perform computation more easily. Recall that we have defined addition, subtraction, multiplication, and division this way:

Addition: $A + B$ is the number of objects in a group formed by combining a group of A objects with a group of B objects.

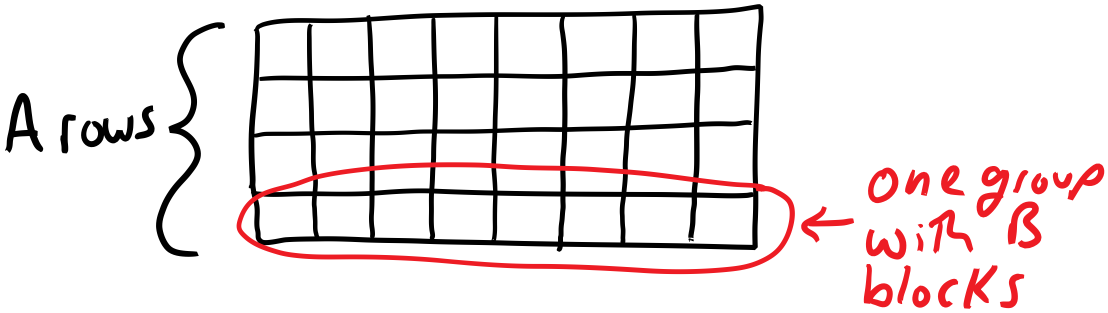
Subtraction: $A - B$ is the number of objects left over after B objects are removed from a group of A objects.

Multiplication: $A \times B$ is the number of objects in A groups containing B object each.

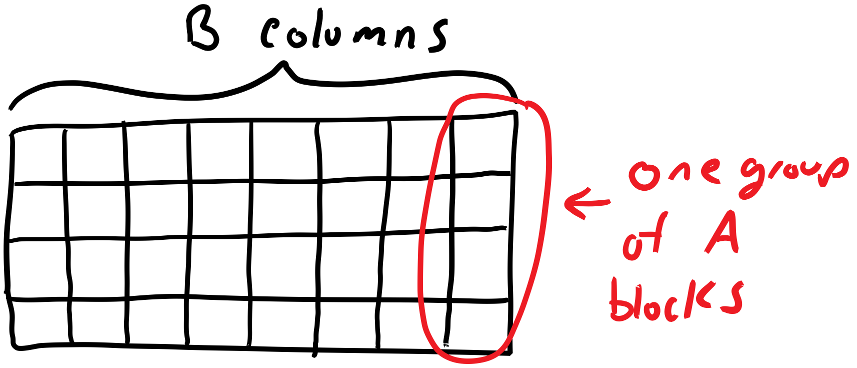
Division: $A \div B$ is the number of objects in each group when A objects are placed into B groups which are all the same size.

Properties of Multiplication

Suppose that we have an array of blocks made of A rows, each containing B blocks. We are going to count the number of blocks in this array in two ways. First, we will focus on rows. Then we will focus on columns.

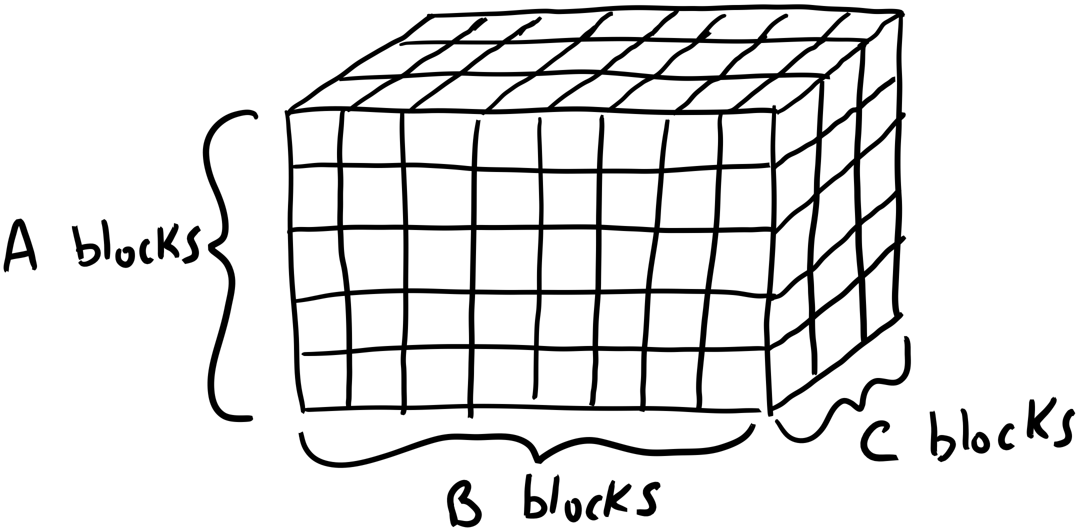


If we view each row as a group, we have A groups of B blocks, or $A \times B$ blocks. On the other hand, we can also focus on the columns of the array. We have B columns of A blocks.

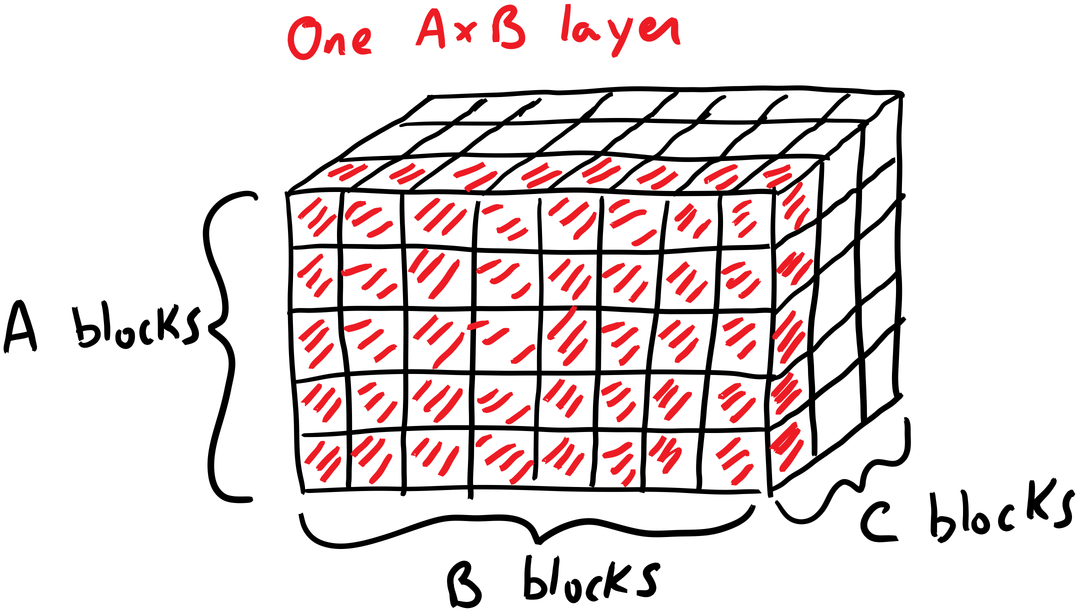


From this viewpoint, we have B groups A blocks, so we have $B \times A$ blocks. However, we are still looking at the same array, so it has to be that $A \times B = B \times A$. This is the **commutative property of multiplication**.

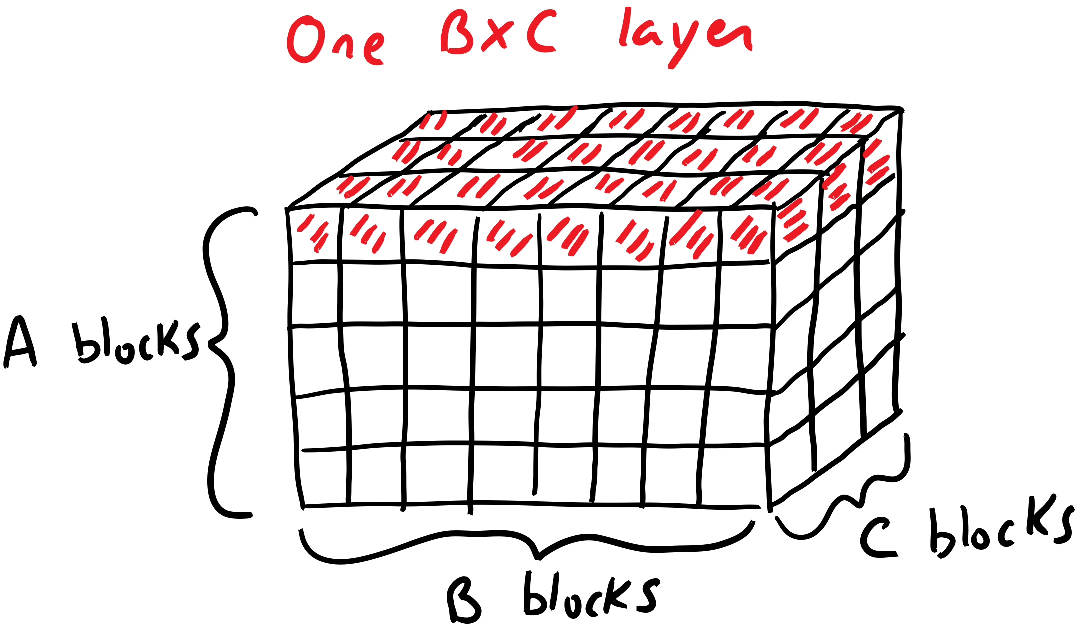
Suppose now that we have a rectangular box of blocks that is A blocks tall, B blocks wide, and C blocks deep.



We could slice this box into layers vertically to form C layers, each with $A \times B$ blocks.



With this view, we have C groups of $A \times B$ or $C \times (A \times B)$ blocks. Since multiplication is commutative, this is the same as $(A \times B) \times C$ blocks in the box. We can also slice this box horizontally to form A layers, each with $B \times C$ blocks.



With this view, we have A groups of $B \times C$ or $A \times (B \times C)$ blocks. We have counted all of our blocks twice. First, we found that the number of blocks was $(A \times B) \times C$. Then we found that the number of blocks was $A \times (B \times C)$. Since the blocks did not change, it has to be that $(A \times B) \times C = A \times (B \times C)$. This is the **associative property of multiplication**.

If we have one group of A marbles, then all we have is A marbles. This implies that $1 \times A = A$. On the other hand, if we have A groups each with one marble, then we still only have A marbles. That is, $A \times 1 = A$. For this reason, 1 is called the **multiplicative identity**.

Suppose we have A boxes and that each box has 0 marbles in it. How many marbles do we have? We have a total of 0 marbles. This implies that $A \times 0 = 0$. On the other hand, suppose that we have 0 boxes of marbles. No matter how many marbles are in the non-existent boxes, we have no marbles, so $0 \times A = 0$ also.

Properties of Division

Division is not commutative. For example, $8 \div 2 = 4$; however, $2 \div 8$ does not even make sense at this point in time because we cannot distribute 2 objects equally among 8 groups. Therefore, it cannot be that $8 \div 2 = 2 \div 8$. We will address this issue a bit more with fractions later; however, even then equality will not hold.

Division is not associative. To see this, simply note that

$$(12 \div 6) \div 2 = 2 \div 2 = 1$$

but

$$12 \div (6 \div 2) = 12 \div 3 = 4.$$

The number 1 is almost but not quite an identity for division. If we evenly distribute A marbles into one box, then that box must contain all A marbles, so $A \div 1 = A$. However, if A is greater than 1, then we cannot distribute 1 marble among A boxes. Thus $1 \div A$ is not even defined. This computation will be important later when we address fractions.

Suppose that we have a nonzero number of marbles A and that we have A boxes. There is only one way to divide the marbles evenly among all of the boxes. That is to put one marble in each box. Therefore, $A \div A = 1$.

Suppose we have 0 marbles and we want to distribute them among A boxes. (Here, A is greater than 0.) Since we have no marbles, we cannot put any in each box. That is $0 \div A = 0$ if A is not 0. On the other hand, suppose that we want to divide A marbles among 0 boxes. We cannot put any marbles into any box because there are no boxes to put marbles in. For this reason, we say that $A \div 0$ is not defined for any A.

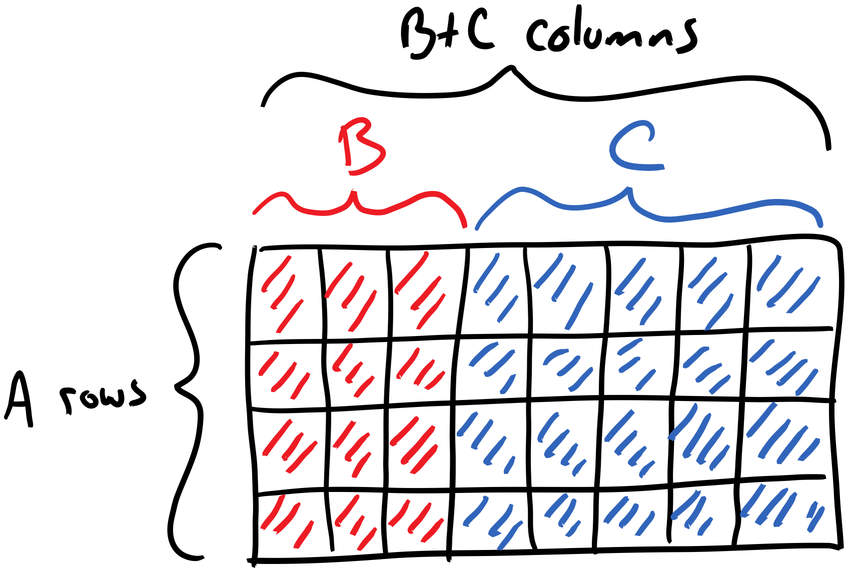
Interaction of Multiplication and Division

Suppose that we have A marbles that we divide among B boxes so that every box contains $A \div B$ marbles. Then we have B boxes, each with $A \div B$ marbles, or B groups of $A \div B$ marbles. The number of marbles here is $B \times (A \div B)$. However, we know that we have A marbles, so $B \times (A \div B) = A$. This means that $A \div B$ is the number we multiply times B to get A. Some books use this as the definition of division. We can express this with equations by saying that if $A \div B = C$ then $A = B \times C$.

Now suppose that we have A boxes of B marbles. We have a total of $A \times B$ marbles. As we saw before, this is the same as $B \times A$ marbles – which represents B boxes of A marbles. If we distribute these B boxes of A marbles into B boxes – calculating $(B \times A) \div B$ – then each box should contain A marbles. Thus, $(A \times B) \div B = (B \times A) \div B = A$. Thus, we see that both $(A \div B) \times B = A$ and $(A \times B) \div B = A$. The operations of dividing by B and multiplying by B are inverses of each other.

Interaction of Multiplication and Addition

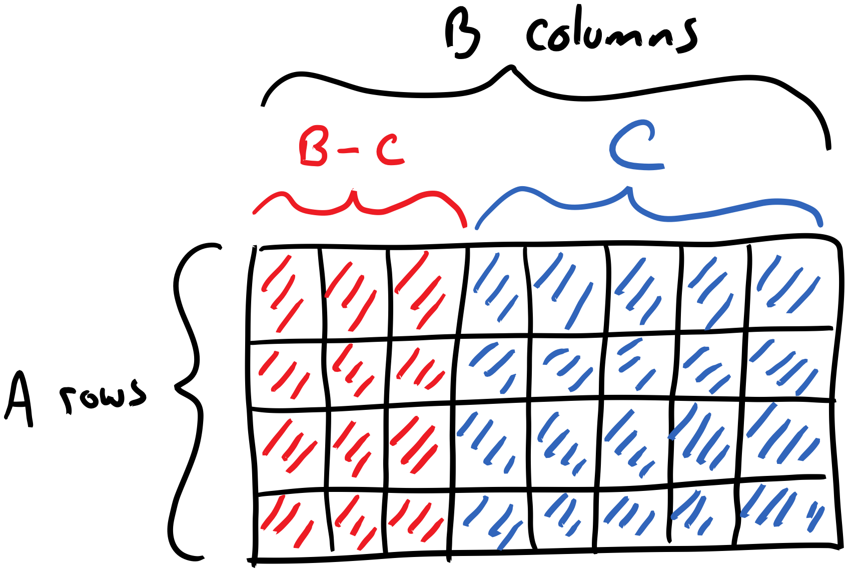
Consider this array of blocks. The array has A rows and $B + C$ columns.



Since the array contains A rows and since each row has $B + C$ blocks, there are $A \times (B + C)$ blocks here. There are A rows of B red blocks, so there are $A \times B$ red blocks. Similarly, there are $A \times C$ blue blocks. If we add the red blocks to the blue blocks, we see that there are $(A \times B) + (A \times C)$ blocks total. Thus it seems that $A \times (B + C) = (A \times B) + (A \times C)$. This is the **distributive property of multiplication over addition**.

Interaction of Multiplication and Subtraction

Consider this array of blocks.



We are going to count the number of red blocks in two different ways to see how multiplication interacts with subtraction. First, notice that since we have B total columns and C blue columns, then the number of red columns is, indeed, $B - C$. Since there are A rows of $B - C$ red blocks, we have $A \times (B - C)$ red blocks. On the other hand, we can count the red blocks by counting the total number of blocks – which is $A \times B$ – and subtracting off the number of blue blocks – which is $A \times C$. Thus, there are $(A \times B) - (A \times C)$ red blocks. It would seem that $A \times (B - C) = (A \times B) - (A \times C)$. This is the **distributive property of multiplication over subtraction**.

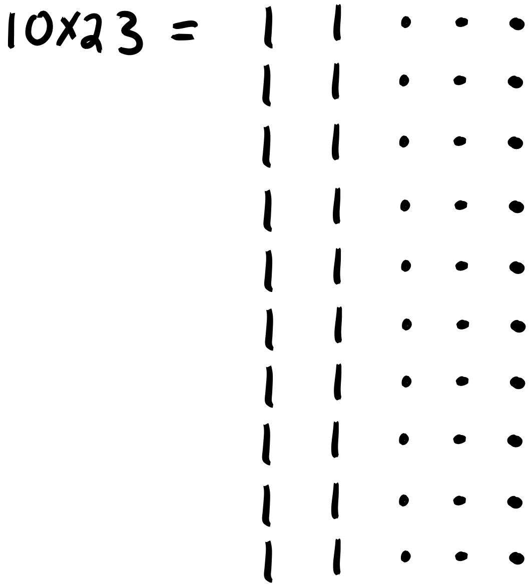
We have now established these properties of arithmetic.

Properties of Addition and Subtraction	
Commutative	$A + B = B + A$
Associative	$(A + B) + C = A + (B + C)$
Identity	$A + 0 = 0 + A = A$ $A - 0 = A$
Inverse	$(A - B) + B = (A + B) - B = A$
Properties of Multiplication and Division	
Commutative	$A \times B = B \times A$
Associative	$(A \times B) \times C = A \times (B \times C)$
Identity	$A \times 1 = 1 \times A = A$ $A \div 1 = A$ $A \div A = 1$
Inverse	$(A \div B) \times B = (A \times B) \div B = A$
Distributive Properties	
$A \times (B + C) = (A \times B) + (A \times C)$	$A \times (B - C) = (A \times B) - (A \times C)$

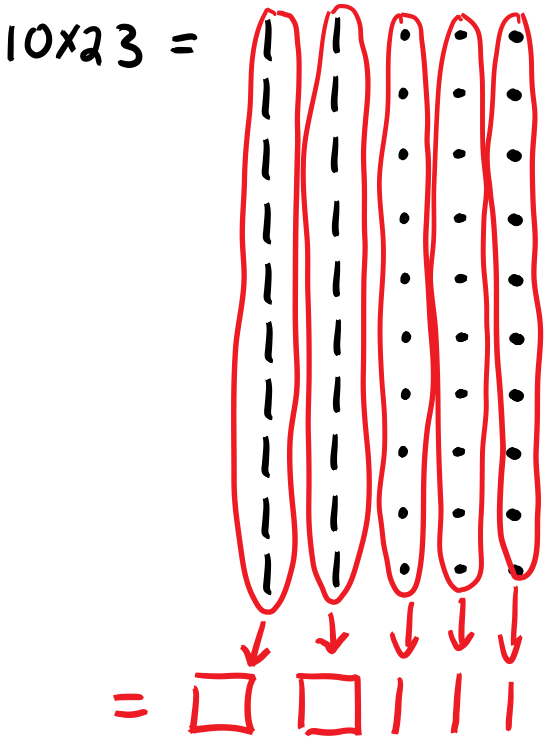
Multiplication Algorithms

In this section we demonstrate why the standard multiplication algorithm works. We first need to know how to multiply by ten. Consider the product 10×23. This is the number of objects in 10 groups of 23 objects. First we draw a bundling diagram for 23.

Next, we draw 10 groups of 23.



Now, each of the original dots representing a one in the original diagram for 23 has become 10 dots. These 10 dots (or ten ones) can be combined into a single line segment representing a ten. We do this for all 3 of the original dots. Also, each line segment representing a ten in the original diagram for 23 has become 10 line segments. These 10 tens can be grouped together to make a square representing 1 hundred. We do this for both original tens.



Our diagram for 10×23 has now become a diagram for 230. Each one gets duplicated 10 times to become a ten, so the ones digit of the original number becomes the tens digit of the product. Similarly, each ten gets duplicated 10 times to become a hundred, so the original tens digit becomes the hundreds digit of the product. This process continues with numbers that have more digits. Moreover, since every symbol in the diagram is duplicated 10 times, there are no ones left by themselves. Therefore, the ones digit of the product is 0. What we are seeing is that to multiply a number by 10, one simply adds a 0 to the right end of the number. Since $100 = 10 \times 10$, multiplication by 100 amounts to adding a 0 to the right end of a number twice, or adding two 0s. Similarly, multiplying by 1000 amounts to adding three 0s, and so forth.

Before we can continue with a general multiplication algorithm, we have to address how to multiply numbers which are a single non-zero digit followed by zeros. Consider 200×3000. We will use the facts that $200 = 2 \times 100$ and $3000 = 3 \times 1000$ along with the associative and commutative properties of multiplication to multiply 200×3000.

$$\begin{aligned} 200 \times 3000 &= 2 \times 100 \times 3 \times 1000 \\ &= 2 \times 3 \times 100 \times 1000 \\ &= 6 \times 100 \times 1000 \\ &= 600 \times 1000 \\ &= 600000 \end{aligned}$$

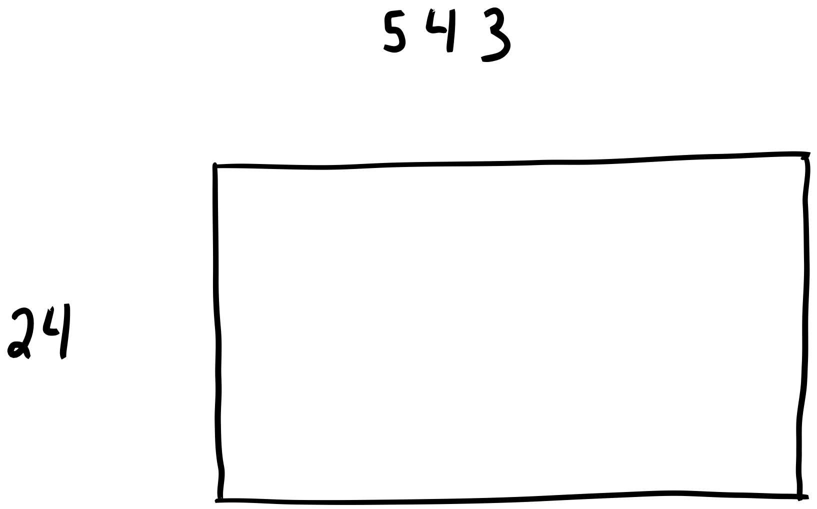
Notice that the result of the multiplication is the product of the first two digits followed by the total number of zeros in both numbers.

Partial Products Multiplication

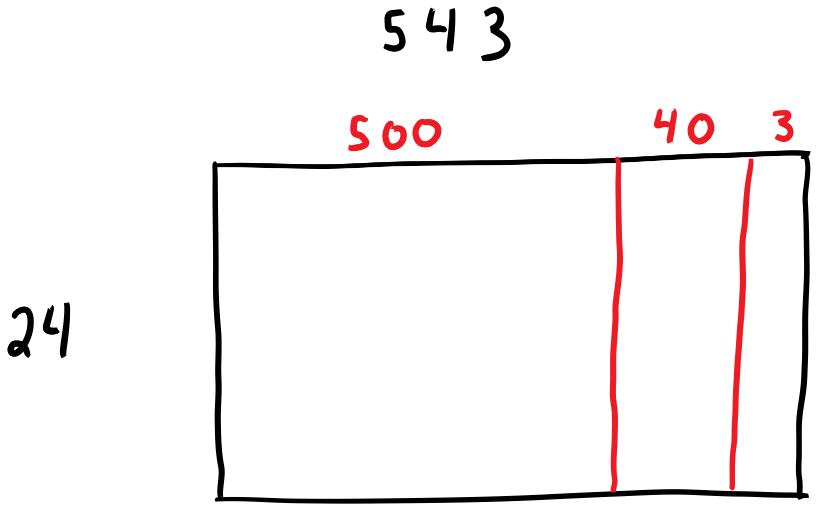
The standard algorithm for multiplying multidigit numbers is really just short-hand for what we call partial product multiplication. We demonstrate partial products with this problem.

Problem: Multiply 24×543.

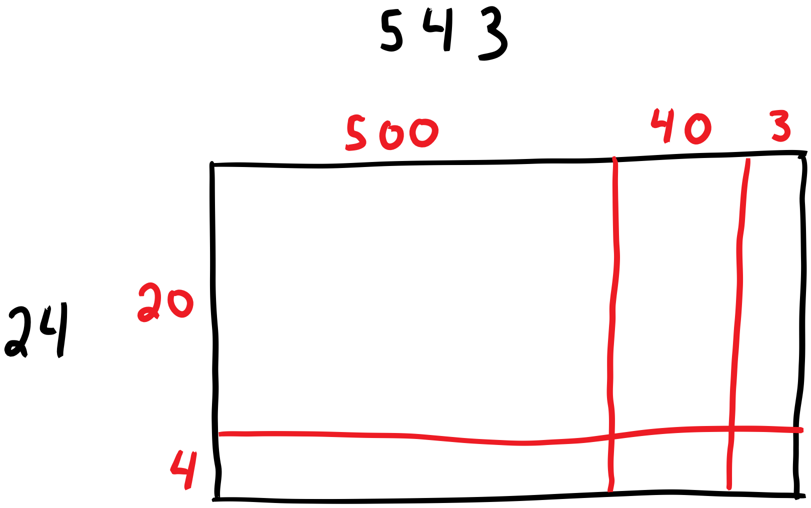
We begin with a pictorial approach that demonstrates the ingredients of the partial products approach. To calculate 24×543 we will count the number of blocks in an array with 24 rows and 543 columns. That would be a lot of blocks to draw (just how many we will not know until after we multiply), so we simply draw labeled rectangles rather than real arrays. Note that our rectangles will in no way be drawn to scale. First we draw an array with 24 rows and 543 columns. (Imagine the blocks in the array.)



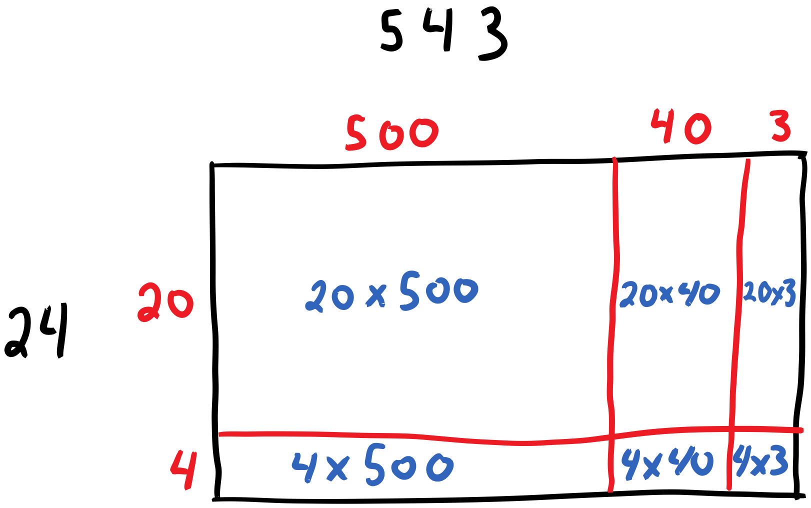
Next, noting that $543 = 500 + 40 + 3$, we divide our columns into groups of 500, 40, and 3.



Similarly, we divide our rows into groups of 20 and 4 since $24 = 20 + 4$.



Our array is now composed of six smaller arrays. If we want to know the size of the large array, we simply need to count the number of blocks in each smaller array and add the results. Luckily, multiplying to count the blocks in each smaller array is easy because the numbers of rows and columns in each smaller array are in a simple form – single digits followed by zeros.



We now can find 24 × 543

$$\begin{aligned} 24 \times 543 &= 20 \times 500 \\ &= (20 \times 500) + (20 \times 40) + (20 \times 3) + (4 \times 500) + (4 \times 40) + (4 \times 3) \\ &= 10000 + 800 + 60 + 2000 + 160 + 12 \\ &= 13032 \end{aligned}$$

The addition in the last step here may be easier if we stack the numbers being added. That is exactly what partial product does. First, we stack the numbers being multiplied. Usually, we put the number with the fewest digits on bottom. Then, we multiply the value represented by each bottom digit times the value represented by each top digit, stacking the results. After all of the multiplying is done, we add the results. Here is the process for this problem.

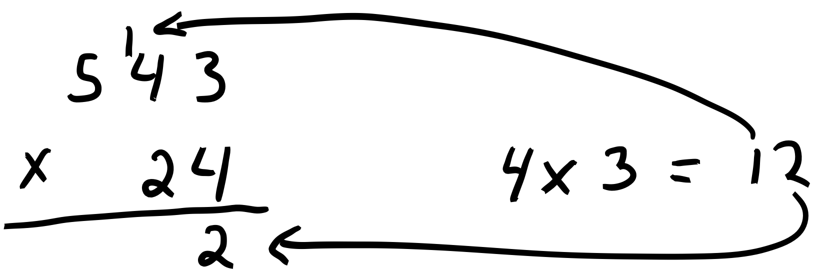
$$\begin{array}{r l}
543 & \\
\times\quad 24 & \\
\hline
{}^{1\;1}\quad 12 & \leftarrow 4\times 3\\
160 & \leftarrow 4\times 40\\
2000 & \leftarrow 4\times 500\\
60 & \leftarrow 20\times 3\\
800 & \leftarrow 20\times 40\\
+\ 10000 & \quad 20\times 500\\
\hline
13032 &
\end{array}$$

The Standard Multiplication Algorithm

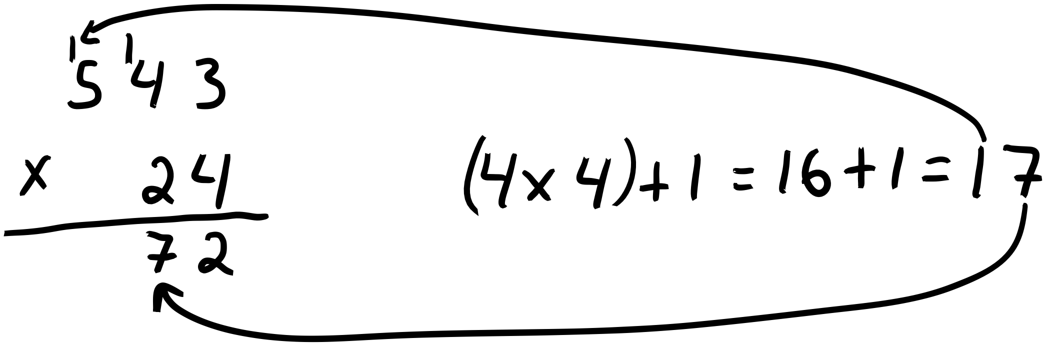
The standard multiplication algorithm follows the steps of the partial products method, but adds the products as multiplication progresses. We first demonstrate the standard algorithm on the same product as above and then show how the two procedures are related. First, we stack the numbers to multiply. Again, we usually place the shorter number on bottom.

$$\begin{array}{r}
543\\
\times\quad 24\\
\hline
\end{array}$$

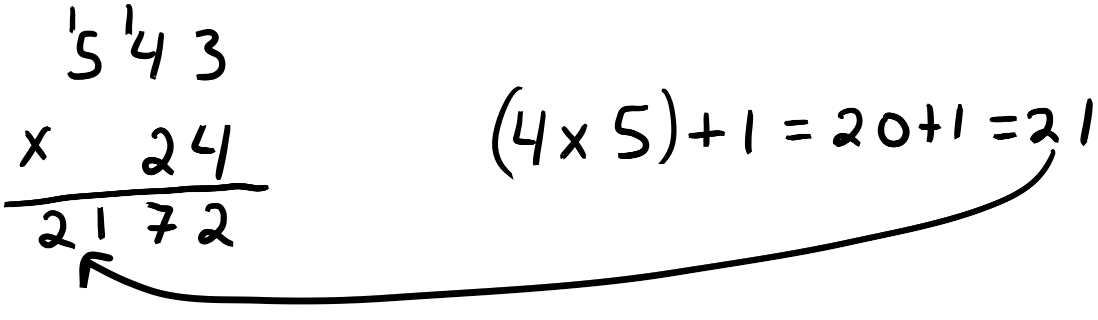
Now we are going to take the 4 from the bottom (the ones digit) and multiply it times each digit from the top number, starting on the right and working toward the left. First, $4 \times 3 = 12$. The 1 here is a ten, so we carry the 1 to the tens column and write the 2 in the ones place under the bar.



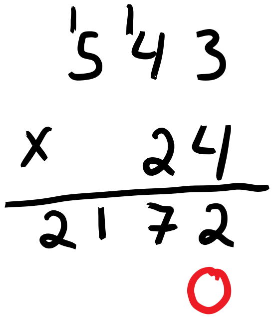
Now we multiply the 4 from the bottom number times the 4 in the top row. After we multiply, we have to be sure to add the 1 that we carried. Now, that 1 is actually a 10 and that 4 is actually a 40. To keep track that we are multiplying tens, we will place (most of) our result in the tens place beneath the bar. Since $(4 \times 4) + 1 = 17$, we carry the 1 to the hundreds column, and we write the 7 in the tens place beneath the bar.



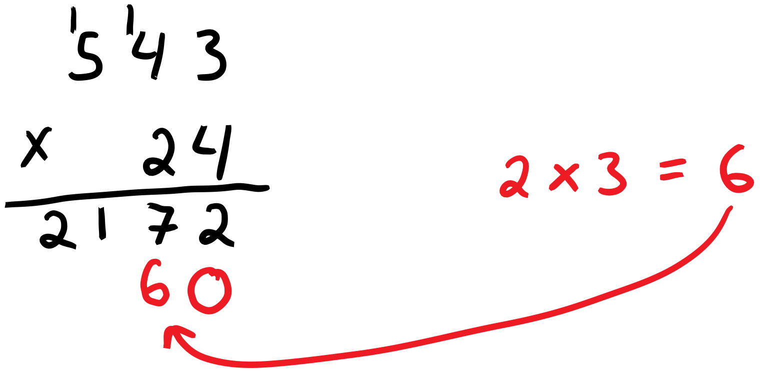
Next, we multiply the 4 times the 5 from the first number, and we add the 1 that was carried. The result is 21. If we had more digits to multiply, the 2 would be carried to the thousands column. Since there are no more digits to multiply by 4, we simply place the 1 in the hundreds place and the two in the thousands place beneath the bar.



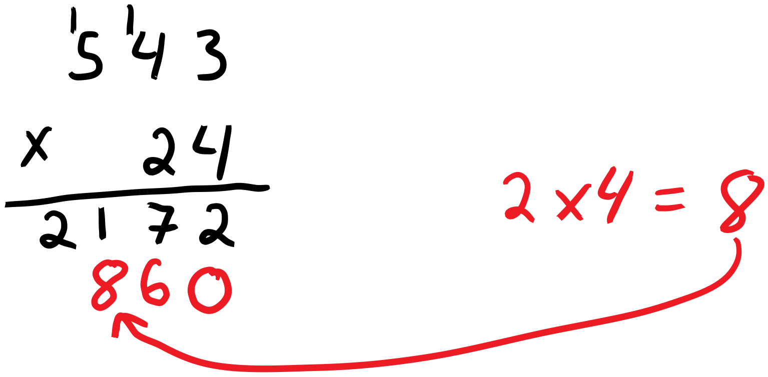
What we have now calculated is 4×543. Now we need to turn our attention to 20×543. To calculate this product, we will multiply by 2×10. We handle the 10 first. Since multiplying by 10 is the same as adding a 0 to the right of a number, we simply write a 0 in the right-most place where we will put our product.



We first multiply $2 \times 3 = 6$. Really, that 2 represents 20, so the product is 60. Notice that since we have already written down the 0, the 6 automatically ends up in the correct place. If our product had been 10 or larger, we would carry to the hundreds column.



Now we multiply $2 \times 4 = 8$. Really, we are multiplying 20 and 40, so this product is 800. Note how the 8 automatically ends up in the correct place because of the 0 that we started with.



Finally, we multiply $2 \times 5 = 10$. Again, the place value takes care of itself. If we had more digits to multiply by 2, we would carry the 1, but we do not have to worry about that in this example.

$$\begin{array}{r} {}^{1}5\,{}^{1}4\,3 \\ \times \quad 24 \\ \hline 2\,1\,7\,2 \\ 1\,0\,8\,6\,0 \end{array} \qquad 2 \times 5 = 10$$

Now that we have multiplied by 4 and by 20, we simply add to find our total product.

$$\begin{array}{r} {}^{1}5\,{}^{1}4\,3 \\ \times \quad 24 \\ \hline {}^{1}2\,{}^{1}1\,7\,2 \\ +\,1\,0\,8\,6\,0 \\ \hline 1\,3\,0\,3\,2 \end{array}$$

Here is a picture to show how the partial products and standard algorithms compare for calculating this product. Note that the three black partial products are simply added together to correspond to 2172 in the standard algorithm. Also, the three red partial products are added to correspond to 10860 in the standard algorithm.

$$\begin{array}{rl}
543 & \\
\times\quad 24 & \\ \hline
{}^{1}\ {}^{1}\quad 12 & \leftarrow 4\times 3 \\
160 & \leftarrow 4\times 40 \\
2000 & \leftarrow 4\times 500 \\
60 & \leftarrow 20\times 3 \\
800 & \leftarrow 20\times 40 \\
+\ 10000 & \quad 20\times 500 \\ \hline
13032 &
\end{array}
\qquad
\begin{array}{r}
{}^{1}5\,{}^{1}4\,3 \\
\times\quad 24 \\ \hline
{}^{1}2\,{}^{1}172 \\
+\ 10860 \\ \hline
13032
\end{array}$$

Problem: Multiply 567 × 345

We will do this problem with both partial products and the standard algorithm.

$$\begin{array}{rl}
567 & \\
\times 345 & \\ \hline
{}^{1}\ 35 & \leftarrow 5\times 7 \\
{}^{1}\ 300 & \leftarrow 5\times 60 \\
2500 & \leftarrow 5\times 500 \\
280 & \leftarrow 40\times 7 \\
2400 & \leftarrow 40\times 60 \\
{}^{1}20000 & \leftarrow 40\times 500 \\
2100 & \quad 300\times 7 \\
18000 & \quad 300\times 60 \\
+\ 150000 & \quad 300\times 500 \\ \hline
195615 &
\end{array}
\qquad
\begin{array}{r}
\overset{2}{\overset{2}{\overset{3}{5}}}\,\overset{2}{\overset{2}{\overset{3}{6}}}\,7 \\
\times 345 \\ \hline
{}^{1}2\,{}^{1}835 \\
22680 \\
+\ 170100 \\ \hline
195615
\end{array}$$

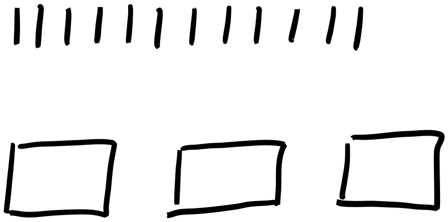
Note with the standard algorithm that most of the products involved carrying. We color coded the carries here to keep from mixing them up. That is a common place to make clerical errors. Also notice in the blue product we are multiplying by 300, so that product ends with two zeros. For the red product, we are multiplying by 20, so that product ends in only one zero. Also notice that while each step on the partial products method might be simpler, the standard algorithm is much shorter for this product.

Division Algorithms

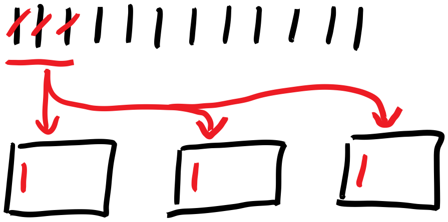
In this section, we demonstrate the standard division algorithm.

Problem: Use tallies to divide 12 ÷ 3.

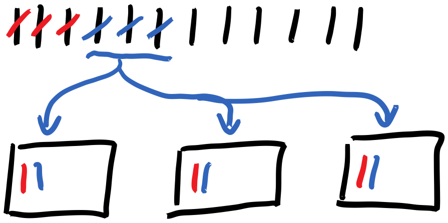
We draw 12 tallies and 3 boxes in which to place the tallies. The objective is to divide the tallies among the boxes so that every box has the same number of tallies.



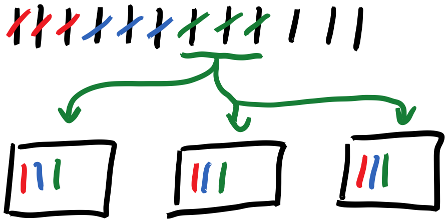
Now we cross out three tallies (because we have 3 boxes) and place one in each box.



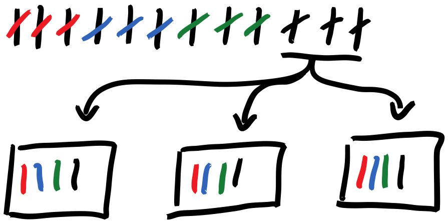
We cross out another three tallies and place them in the boxes.



And we repeat.



And we do it one more time.



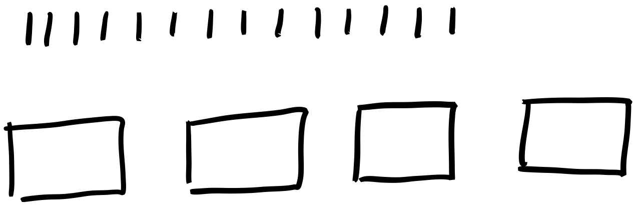
At this point, we have placed all of the tallies into boxes, and we have the same number in each box. We see that $12 \div 3 = 4$.

Division with Remainder

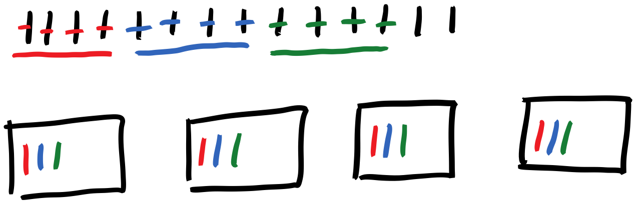
Sometimes (usually, perhaps) when we try to divide, we cannot distribute the tallies evenly among the boxes. When this happens, we have tallies left over that we call the **remainder**.

Problem: Divide $15 \div 4$.

We start by drawing 15 tallies and 4 boxes into which to place them.



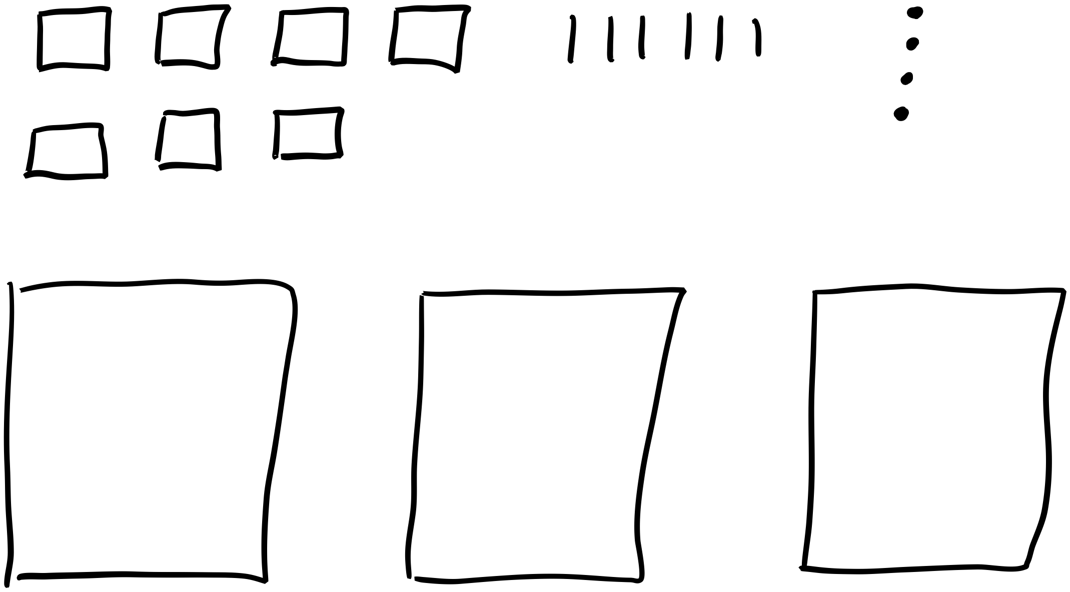
When we try to place the tallies in the boxes, things go well until we have 2 tallies left over but 4 boxes.



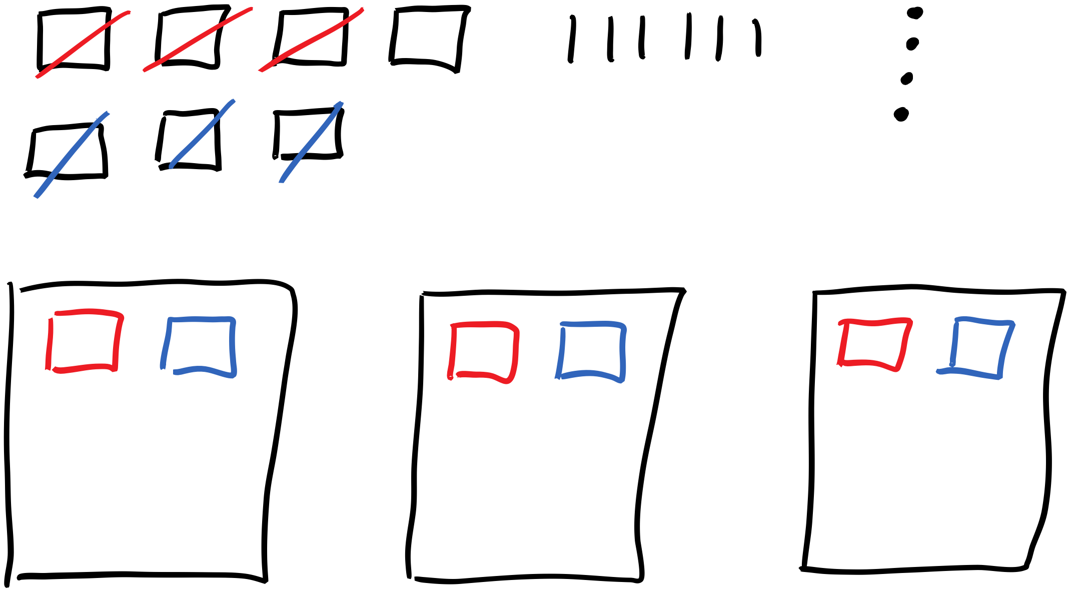
Since we no longer have enough tallies to place a tally in each box, we must stop. We say that $14 \div 4$ is 3 with a remainder of 2. Our notation for this is $14 \div 4 = 3R2$.

Problem: Divide $764 \div 3$.

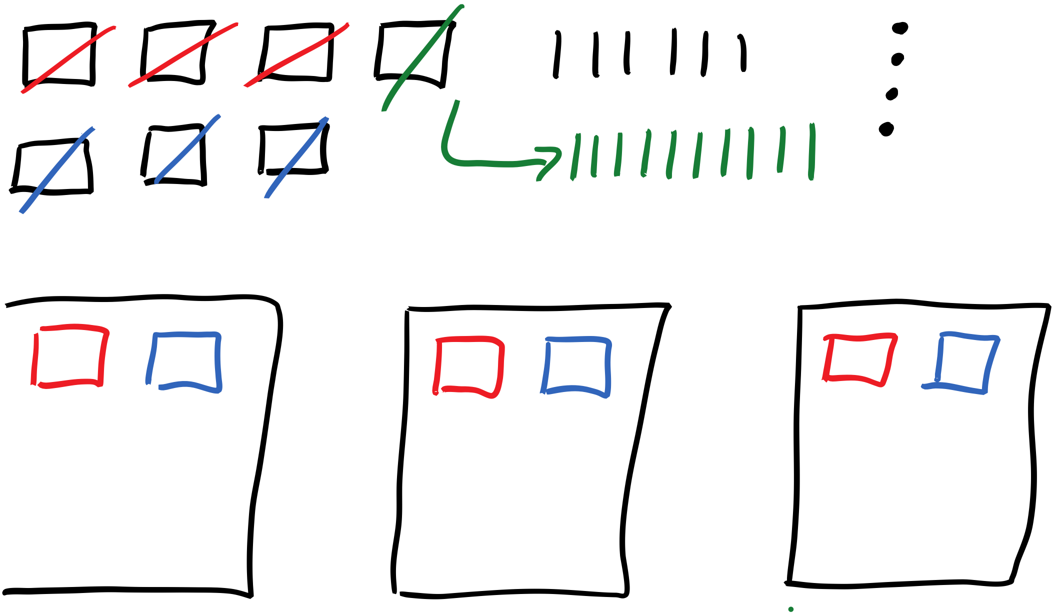
It would be unreasonable to draw 764 tallies to perform this division, so we resort to base ten bundles. We draw a bundling diagram for 764, and we draw 3 boxes to divide the bundles among.



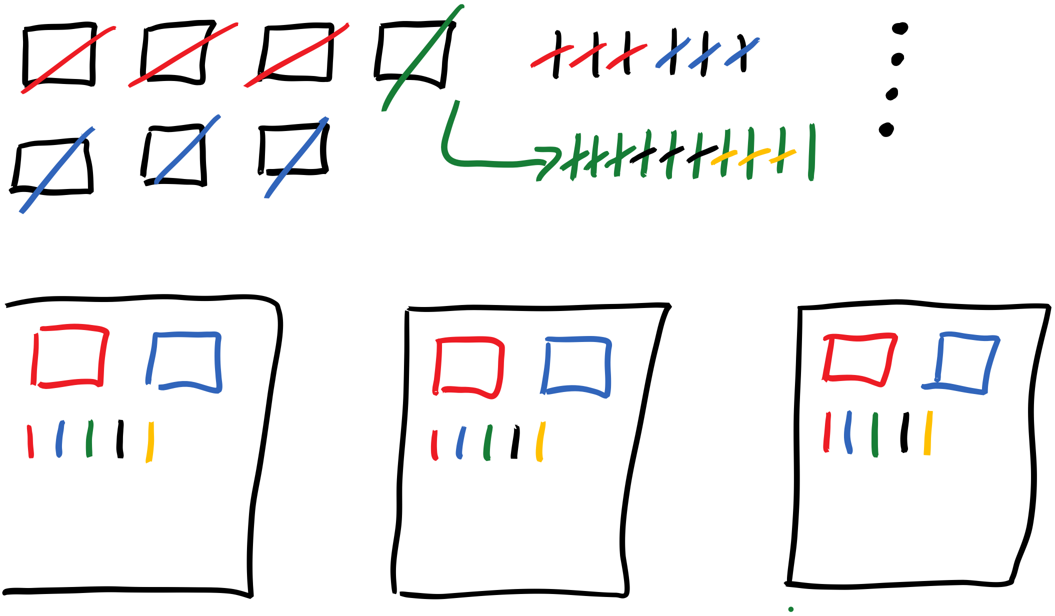
We begin by distributing each of the squares representing hundreds among the three boxes. This is the equivalent of distributing 100 tallies at a time in each box.



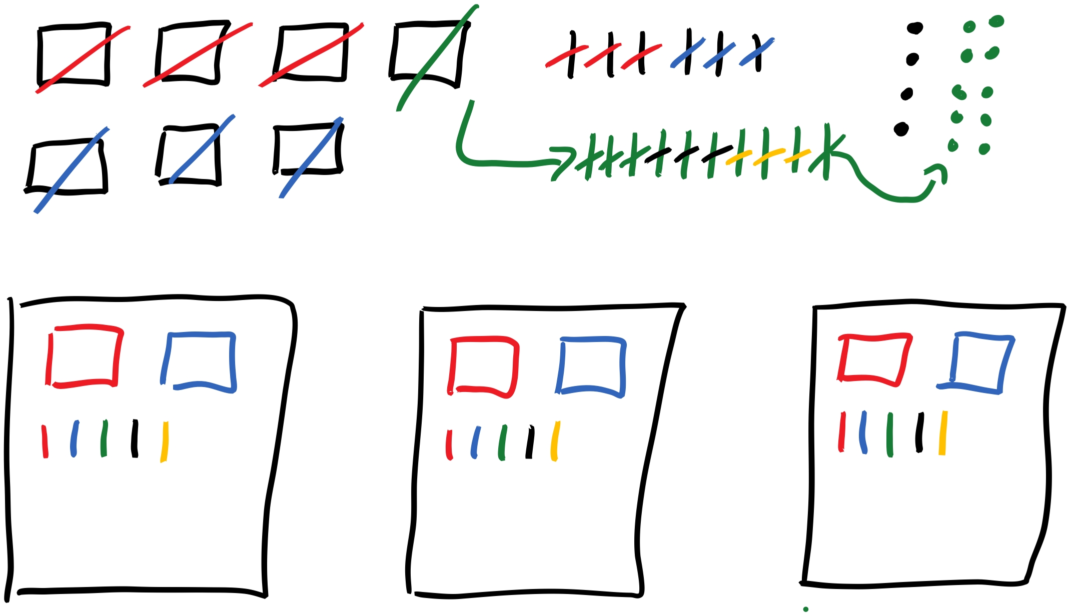
We successfully distribute 6 of the hundreds, but we are left with one square. We break that square up into 10 line segments, representing 10 tens.



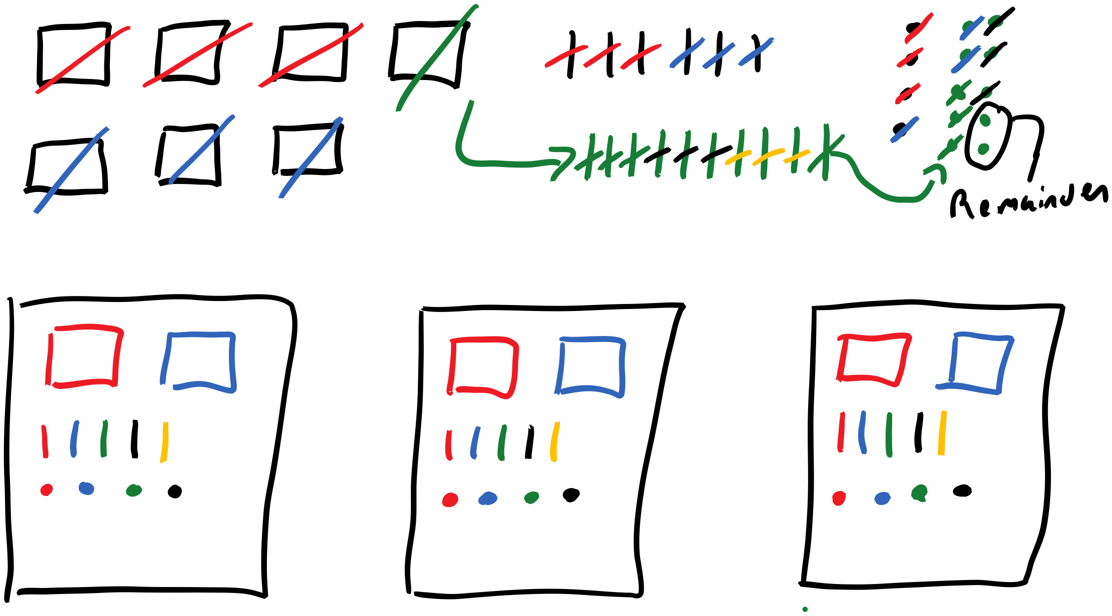
Now we distribute the line segments representing tens among the boxes. This is the equivalent of placing ten tallies at a time in each box.



We are successful in placing all but one of the tens. That last ten we break up into 10 ones.



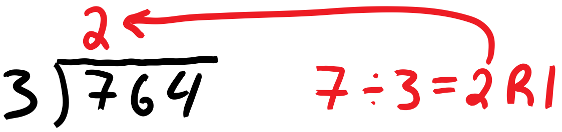
Now we will distribute the ones among the boxes.



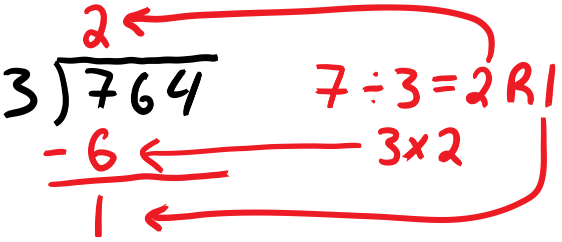
After placing as many ones as possible in the boxes, we are left with 2 ones. That is our remainder. In each box, we have a base ten bundling diagram for 254. Therefore, $764 \div 3 = 254R2$.

Notice the process we just followed which seemed to happen naturally because of the base ten bundling. We first divided the hundreds digit by 3 and converted the remainder to tens. This contributed hundreds to the final quotient. We then divided the total number of tens (the digit plus the remainder from the hundreds) by 3 and converted the remainder to ones. This contributed tens to the final quotient. Finally, we divided the ones by 3. We have special notation for division which we use to expedite this process. We start by writing the dividend (the number being divided) under a symbol we call a **table** and writing the divisor (the number of containers) outside the table.

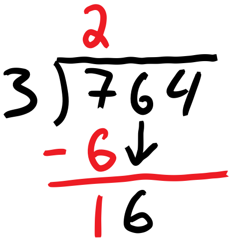
Next, just as we did with the base ten bundles, we concern ourselves only with the hundreds place. We know that $7 \div 3 = 2R1$, so our quotient will have a 200 in it. We will write our quotient above the table. We place a 2 (which will eventually become 200) above the hundreds place.



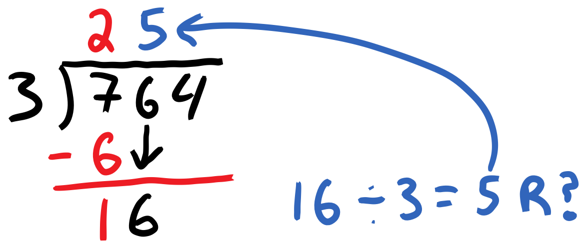
It happens to be here that our numbers are small enough that we already know the remainder when 7 is divided by 3. This remainder will usually be something we need to compute. To compute the remainder at this point, we multiply 3 by 2 and subtract the result from 7. Comparing this to the base ten bundling, we placed 2 hundreds in each of 3 boxes, so the computation 3×2 is telling us how many hundreds we have used so far.



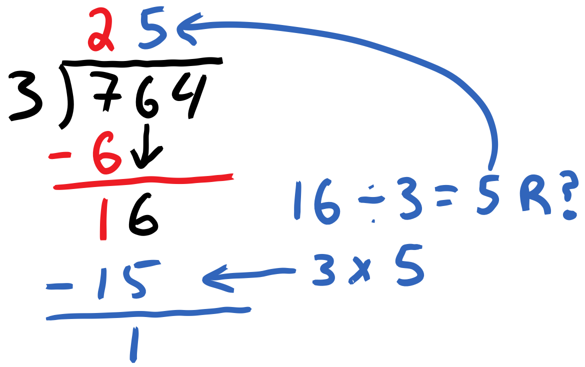
In the base ten bundling approach to this problem, we converted the remaining hundred to tens and added that to the 6 tens we already have. In this algorithm, that is easily accomplished by copying the 6 next to the 1 remainder.



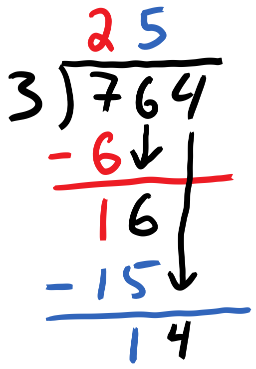
Now we divide 16 (the number of tens) by 3. Since the numbers are small, we can probably tell that $16 \div 3$ is at least 5. Since we are going to subtract in a moment to find a remainder, we do not bother finding the remainder yet. We place 5 in the tens place (remember, we are actually dividing 160 by 3).



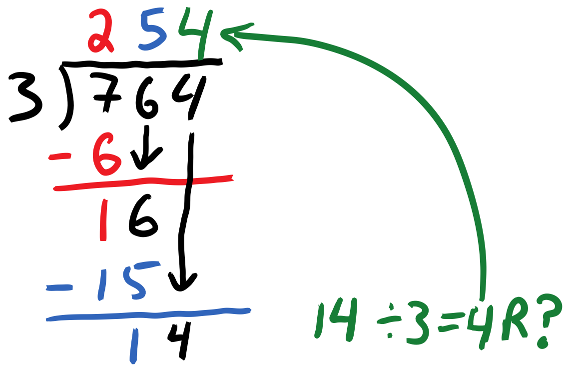
We multiply 3 by five and subtract from 16. Here, again, the numbers are small, so there are no surprises. If 5 had been too large, we would have ended up with a product larger than 16. If this had happened, we would try a number smaller than 5. If 5 had been too small, the difference here (the remainder) would have been larger than 3. Then we would have tried a number larger than 5.



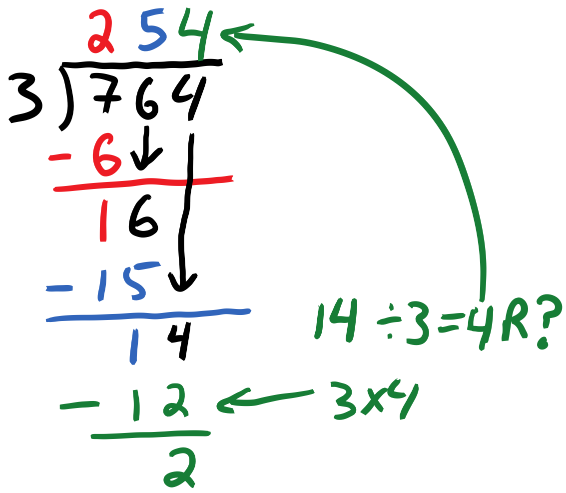
In terms of our bundling diagrams above, we have now distributed as many hundreds and tens as possible. We have one ten remaining. We add that ten to the 4 ones to continue dividing. Adding ten to the 4 ones is easy, we simply copy the 4 next to the 1.



Now we try to divide 14 by 3. The quotient is surely 4, but we hold off on the remainder because we will subtract in a moment.



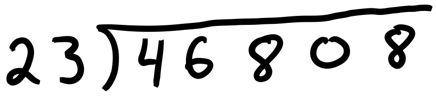
We multiply 3 times 4 and subtract that from 14 to get a remainder of 2.



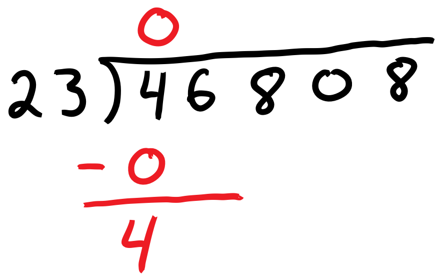
Since in this last step we were finally dividing our ones by 3, we are finished. We see that $764 \div 3 = 254R2$.

Problem: Divide $46808 \div 23$.

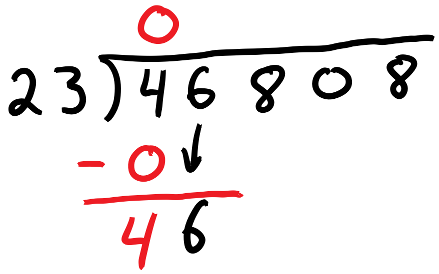
Our algorithms for addition and multiplication reduce all computation to adding and multiplying one digit numbers. Our algorithm for subtraction is almost as good. It reduces all computation to subtracting a one digit number from a number that is at most 18. Our division algorithm is not quite as good. Computation does not reduce to one digit number division, but our base ten notation does let us focus on a few digits at once. We begin this problem by setting up a table for the division.



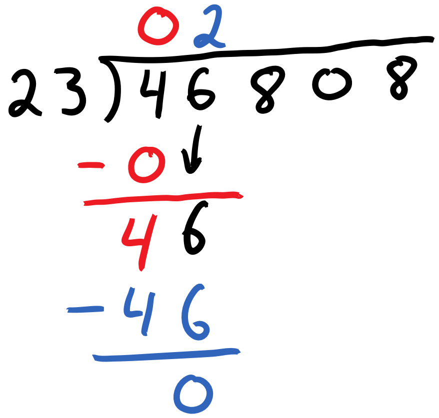
We try to divide our first digit 4 (ten thousands) by 23; however, 4 is too small to divide by 23. We could say that $4 \div 23 = 0R4$, and we will do that here. Most people would skip this multiplication by 0. We will come back to this after this time through the example. For now, we put 0 above the ten thousands place, multiply 23 by 0, and subtract.



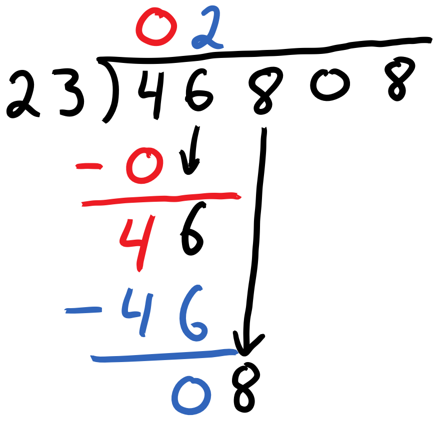
We need to add those 4 ten thousands to the thousands place to continue. This is easy. Just copy the 6 (thousands) next to the 4.



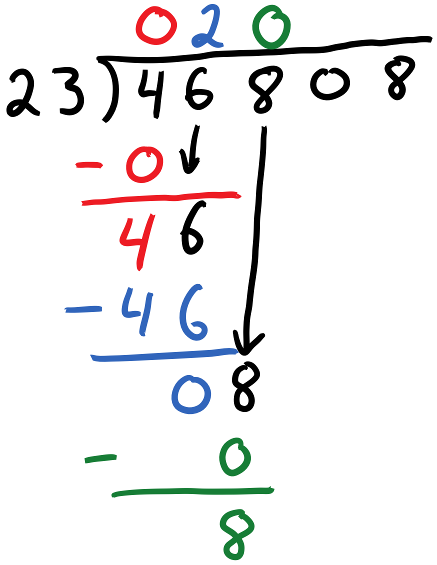
We now divide 46 by 23. We guess that the quotient should be 2, so we place a 2 in the thousands place (remember that we are considering 46 thousands), multiply 23 by 2 and subtract.



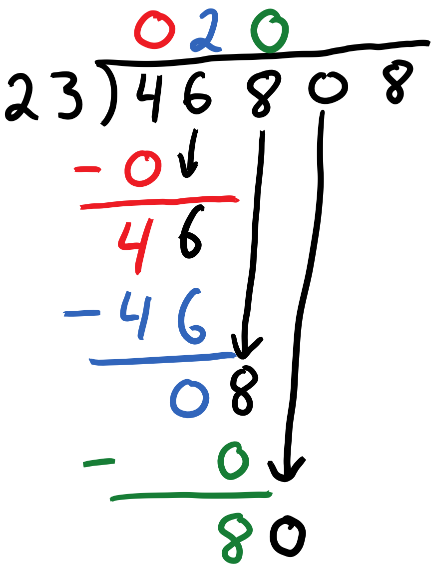
We copy the 8 (hundreds) next to the current remainder in order to continue.



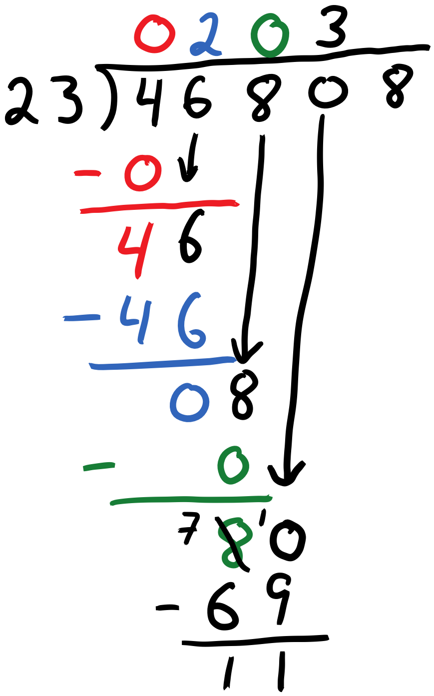
The number 8 is not divisible by 23, so we place a 0 in the thousands place, multiply 23 by 0, and subtract. This is like our very first step of the problem. We can (and will) make this step shorter later.



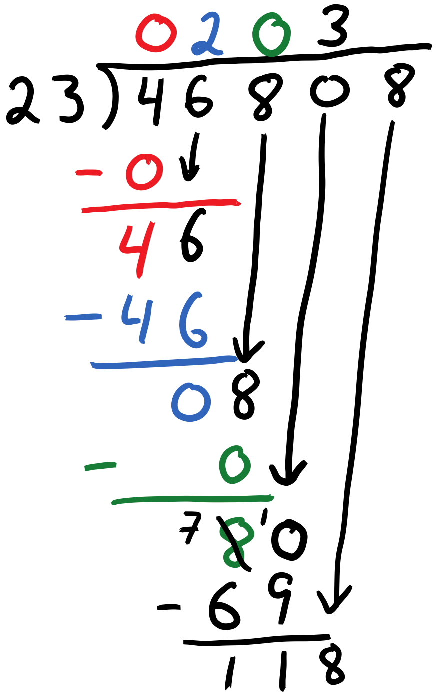
Now, we copy the 0 (tens) next to the current remainder of 8 (hundreds).



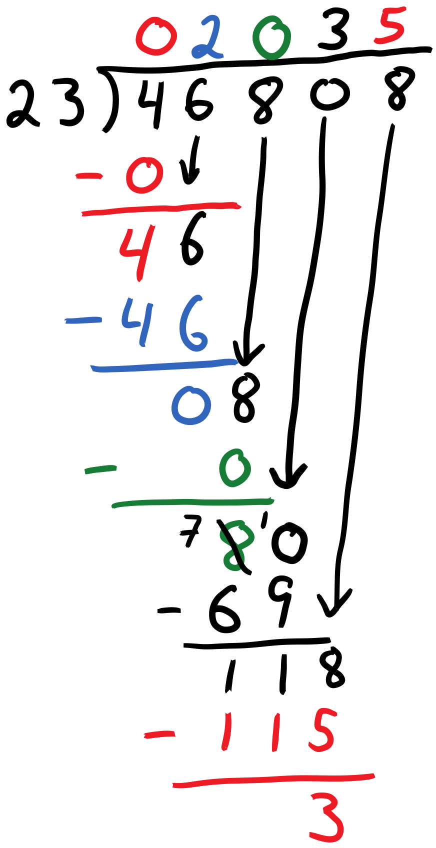
We need to divide 80 by 23. To see what our quotient should be, we might just look at the first digits. It happens to be that $8 \div 2 = 4$, so a good first guess would be 4. However, $23 \times 4 = 92$, which is larger than 80. Therefore, we try something smaller than 4, like 3. We place 3 in the tens place, multiply 23 by 3, and subtract.



At this point our remainder is 11 tens, we add this to the 8 ones by copying the 8 next to the 11.

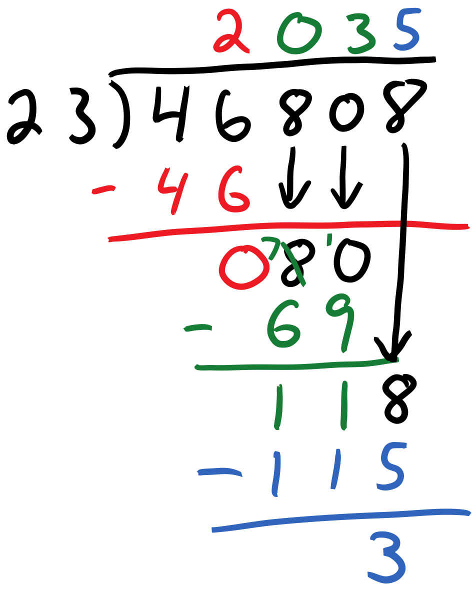


Now we need to divide 118 by 23. Since $11 \div 2 = 5R1$, a good guess to try is 5. We place 5 in the ones place, multiply 23 by 5, and subtract.



We now have a final answer $46808 \div 23 = 2035R3$.

Our work on this problem might be compressed to the arithmetic below. At the very beginning of the problem, we note that 4 is not divisible by 23, so we do not try to divide 4 by 23. Instead, we go straight to dividing 46 by 23. After we have copied the 8 down, we note that $08 = 8$ is not divisible by 23, so we place a 0 above the 8 as before. However, instead of going through multiplying by 0 and subtracting, we simply copy the next digit (0) down next to the 8. The rest of the arithmetic is identical.



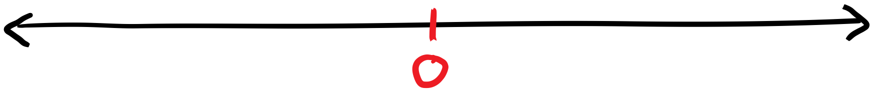
Long Division

The division algorithm we have been working through is often called **long division**. I prefer simply calling it **division**. It is the only real division algorithm students will be taught, so it is no longer than another algorithm. Also, attaching the adjective long to the algorithm unnecessarily predisposes students to dread it.

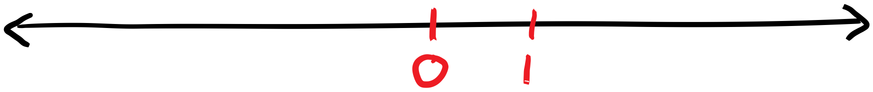
Number Lines and Negative Numbers

We have used tallies, base ten bundling, and bar models to visualize numbers. Another way to visualize numbers is through a number line. To build a number, start by drawing a line.

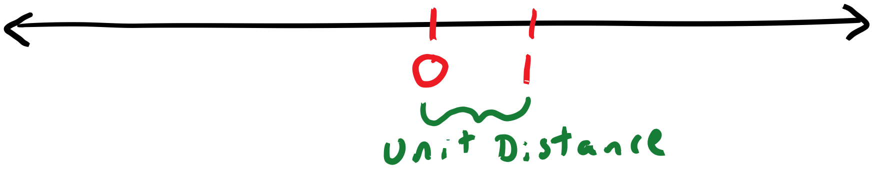
This line extends forever in both directions. Select any point on the line and call it 0.



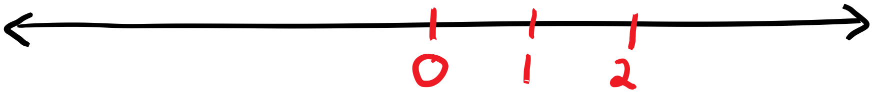
Select any point on the line to the right of 0 and call it 1.



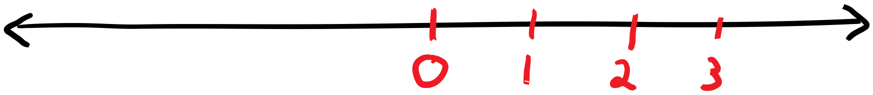
The distance from 0 to 1 we will call a **unit distance** or **unit step**.



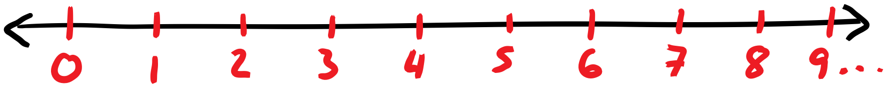
If we start at 0 and move two unit steps to the right, the point we arrive at we call 2.



The number 3 is associated with the point three unit steps to the right of 0.

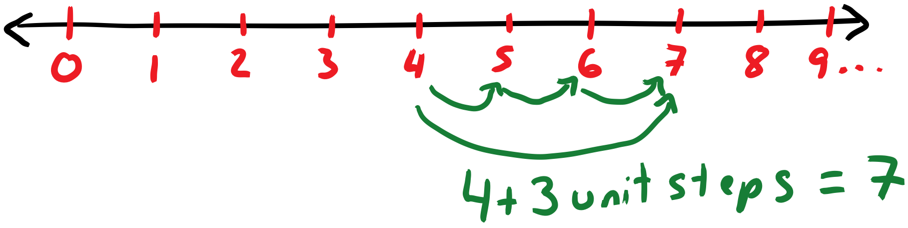


This process continues indefinitely through all of the whole numbers.

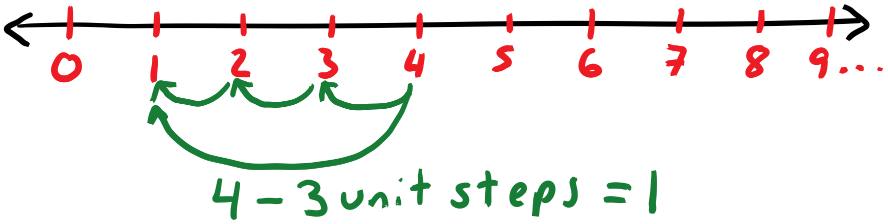


Operations on the Number Line

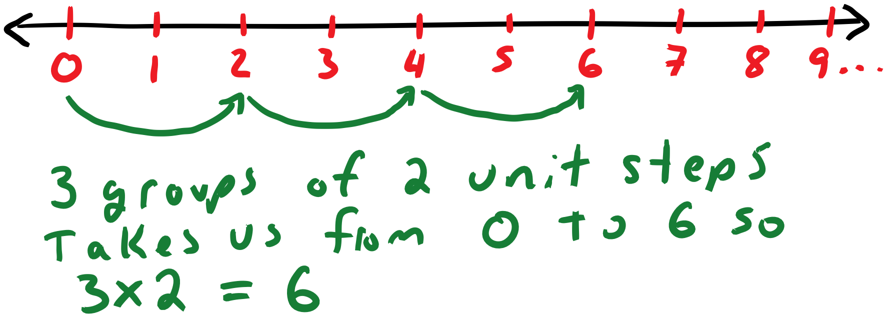
We can visualize all of our arithmetic operations on the number line. To add two numbers A and B, start at A on the number line and move B unit steps to the right. The point we arrive at is $A + B$. For example, if we start at 4 and move 3 unit steps right, then we land at 7, $4 + 3 = 7$.



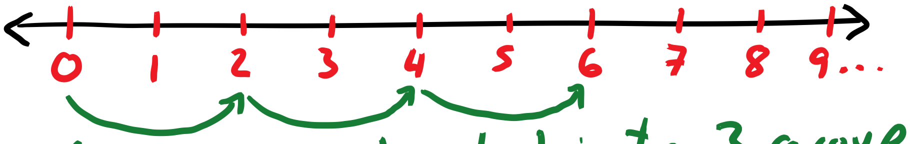
If a number A on the number line is greater than B, then we can subtract B from A by starting at the point A on the number line and moving B unit steps to the left. The point we arrive at is $A - B$. For example, if we start at 4 and take 3 unit steps left, we land at 1, $4 - 3 = 1$.



To multiply two numbers A and B on the number line, start at 0 and take Bunit steps to the right A times. For example, if we start at 0 and take 2 steps right 3 times, we land at 6, $3 \times 2 = 6$.



To divide A by B on the number line, we ask the question, if we take B steps from 0 to the right and land at A, then how long was each step? Alternatively, we could ask, if we start at 0 and take steps of length B until we arrive at A, then how many steps should we take?

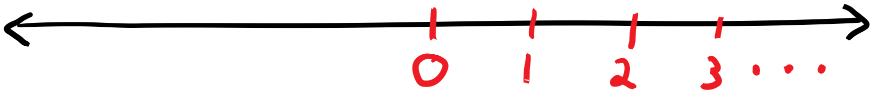


6 can be divided into 3 groups of 2 unit steps so

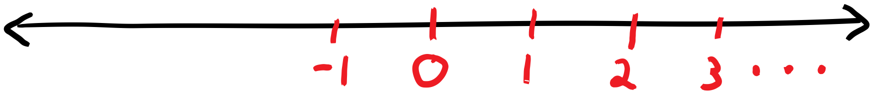
$6 \div 3 = 2$ or $6 \div 2 = 3$

Negative Numbers

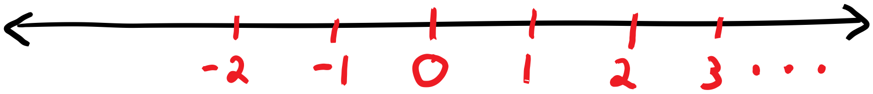
Our construction of a number line started with a line that extends forever in both directions, but we only selected points to represent numbers on the right half of the line.



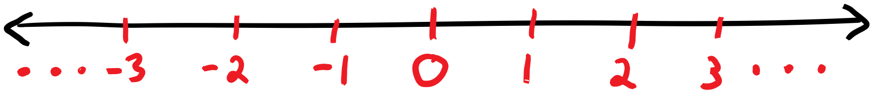
If we repeat the same process moving to the left, we get the negative numbers. If we start at 0 and take one unit step left, then we arrive at a point we call **negative** 1. We denote this as -1. The symbol in front of the 1 that looks unfortunately like a subtraction symbol is a **negative sign**.



If we start at 0 and take two unit steps left, we arrive at negative 2, -2.



We can continue this process through -3 and -4 and so forth.



These new number to the left of 0 are called **negative numbers**. The numbers to the right of 0 are called **positive numbers**. The number 0 is not negative or positive. We are associating negative numbers with numbers to the left of 0 on the number line. However, we could also associate negative numbers with quantities like altitude. A mountain which is 1000 feet above sea level has an altitude of 1000 feet. A valley under the ocean which is 1000 feet below sea level has an altitude of -1000 feet. We can also associate negative numbers with debt. Someone who is in debt 1000 dollars might be said to have -1000 dollars.

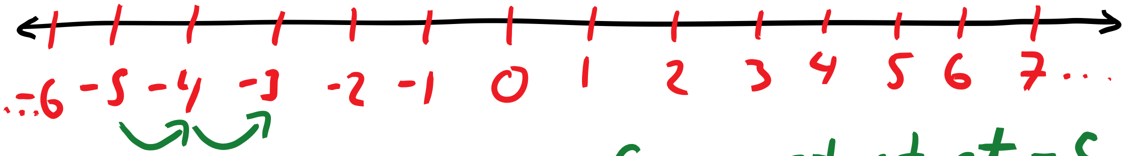
Arithmetic with Negative Numbers

We now have three number systems. The **counting numbers** are the numbers 1, 2, 3… The **whole numbers** are the numbers 0, 1, 2, 3… These are the counting numbers along with 0. Note that we still have no reason to call these whole numbers since we have not seen any partial numbers. When we add the negative counting numbers into the whole numbers, we call the resulting set of numbers the **integers**. The integers are the numbers …$-3, -2, -1, 0, 1, 2, 3, 4$ … All of the arithmetic we have done so far has been with the whole numbers. Now that we have negative numbers too, we have to say what it means to add, subtract, multiply, and divide negative numbers. We will largely motivate how we handle extending our operations to negative numbers by referring to direction on the number line. We will associate positive with right and negative with left. What is really underlying everything we do here however is that the arithmetic operations satisfy nice properties (commutativity, associativity, distributivity) when applied to whole numbers. We want the extension of these operations to the integers to satisfy the same nice properties.

To extend our operations to negative numbers, we will need to be able to refer to the size of a number – that is, how large the number is aside from its being positive or negative. We use **absolute values** to accomplish this. The absolute value of a number A is the positive number which is the same number of steps from 0 as A. Thus, $|-2| = 2$ because 2 is positive and both 2 and -2 are 2 steps from 0. On the other hand, $|2|$ is also 2.

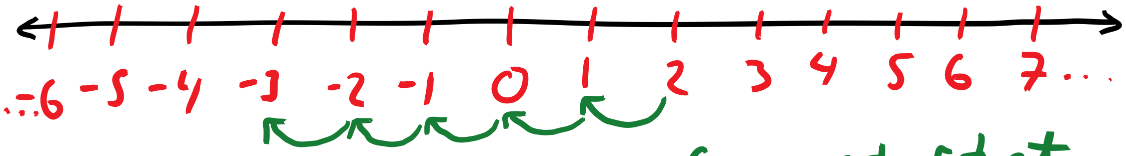
Addition

If B is positive, then we said above that to calculate $A + B$ we should start at A and move B steps to the right. This interpretation of how to add when B is positive even works when A is negative. This is demonstrated in the next figure. If we start at -5 and move 2 units right, we land at -3, so $-5 + 2 = -3$.



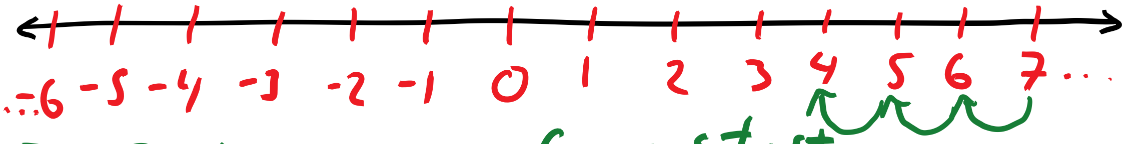
$-5+2 = -3$ because if we start at -5 and take 2 steps right we land at -3

The natural extension of this to the case when B is negative is to move left instead of right. If B is negative, then to $A + B$ is the number we arrive at by starting at A and moving $|B|$ steps to the left. Notice that this is the same as subtracting $|B|$. If we start at 2 and move 5 steps left, we land at -3, so $2 + -5 = -3$. Some would write this as $2 + (-5) = -3$. The parentheses around the -5 are optional but can improve readability. Notice that moving 5 steps left is the same as subtracting 5, so we have that $2 - 5 = 2 + -5 = -3$. Notice that $2 - 5$ is the same size as $5 - 2$ but is negative.



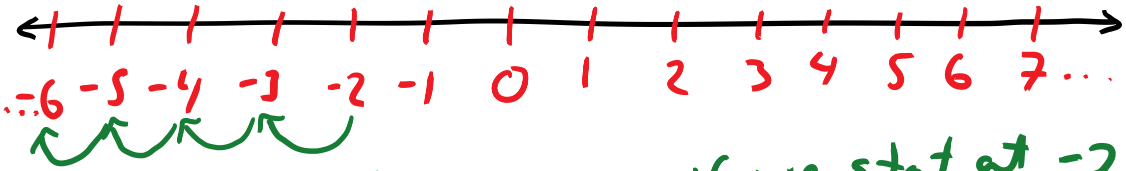
$2 + {}^{-}5 = -3$ because if we start at 2 and take $5 = |-5|$ steps left we land at -3.

Similarly, if we start at 7 and take 3 steps left, then we land at 4, so $7 + -3 = 4$. Note that moving left 3 steps is the same as subtracting 3, so we have $7 + -3 = 7 - 3$.



$7 + {}^{-}3 = 4$ because if we start at 7 and take $3 = |-3|$ steps left we land at 4. Note $7 + {}^{-}3 = 4 = 7 - 3$

If we start at -2 and move 4 steps left, then we land at 6. Thus, $-2 + -6 = -4$. As above, moving 4 steps left is the same as subtracting 4, so $-2 - 4 = -2 + -4 = -6$. Notice that $-2 + -6$ is the same size as $2 + 6$ but negative.



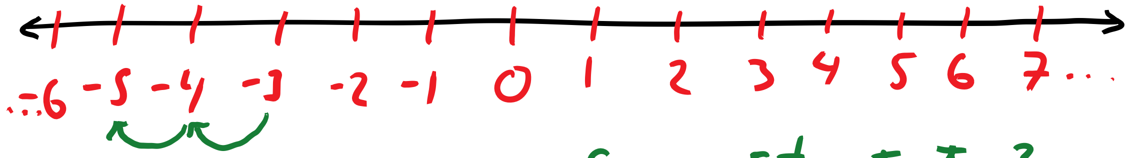
$-2 + {}^{-}4 = -6$ because if we stat at -2 and take 4 steps left we land at -6

Addition

To add $A + B$, locate A on the number line and move $|B|$ steps in the direction of B – left if B is negative and right if B is positive.

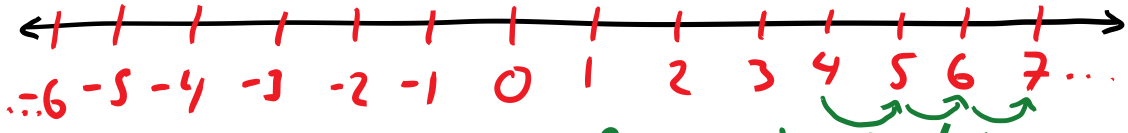
Subtraction

We said above that if B is positive then we can calculate $A - B$ by finding A on the number line and moving B steps left. This works even if A is negative. If we start at -3 and move two steps left, we land at -5, so $-3 - 2 = -5$. Notice that this is the same size as $3 + 2$ but negative.



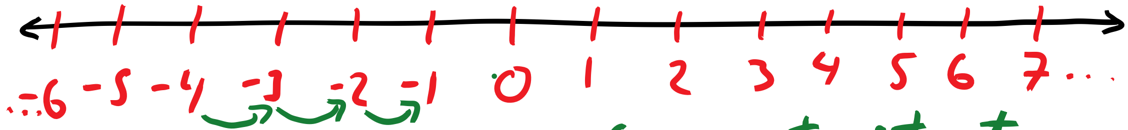
$-3-2 = -5$ because if we start at -3
and move 2 steps left we land at -5

To subtract a negative number, we change direction. Instead of moving left to subtract, we move right. If B is negative, then $A - B$ is the number we end at if we start at A and move $|B|$ steps right. Notice that this is the same as adding $|B|$. If we start at 4 and move 3 steps right, then we land at 7; therefore, $4 - -3 = 7$. Note that this is the same as $4 + 3$.



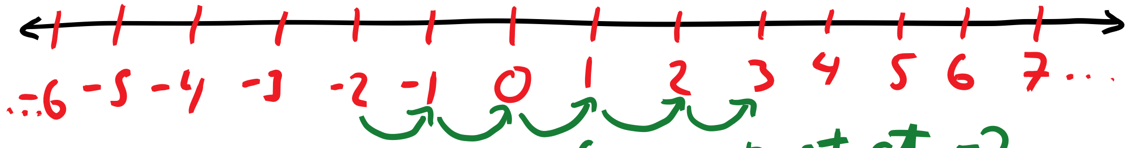
$4 - {}^{-}3 = 7$ because if we start at 4
and move 3 steps right we end at 7
Note $4 - {}^{-}3 = 7 = 4+3$

If we start at -4 and move 3 steps right, then we land at -1, so $-4 - -3 = -1$. Notice that this is the same size as $4 - 3$ but negative.



$-4 - {}^{-}3 = -1$ because if we start at
-4 and take 3 steps right we end at -1

If we start at -2 and move 5 steps right, we land at 3, so $-2 - -5 = 3$. Notice that this is the same as $5 - 2 = 5 + -2 = -2 + 5$.



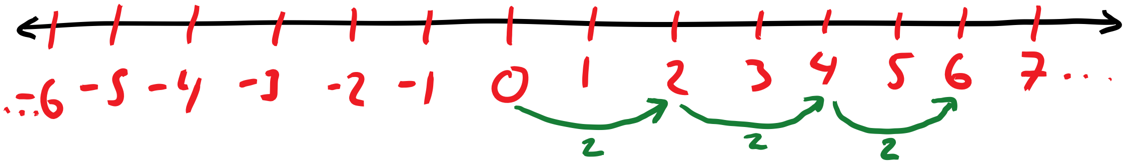
$-2 - {}^{-}5 = 3$ because if we start at -2
and take 5 steps right we end at 3

Subtraction

To calculate $A - B$, locate A on the number line and move $|B|$ steps in the direction opposite of B – right if B is negative and left if B is positive.

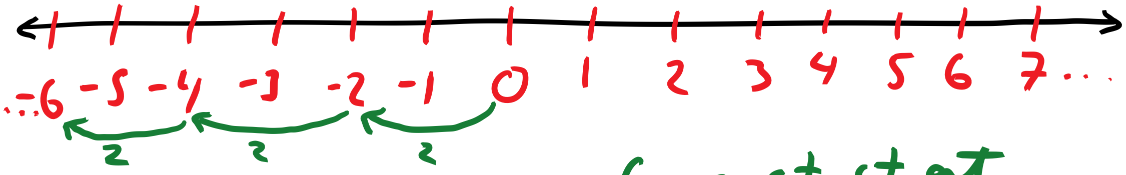
Multiplication

Earlier, we said that to calculate $A \times B$ when A and B are both positive we start at 0 and take a total of A sets of B steps to the right. To adapt this to negative numbers, we add direction. If B is negative, then we take steps to the left. If A is negative, then we take steps in the direction opposite of B. To calculate 3×2, we start at 0 and take 2 steps right a total of 3 times, ending at 6.



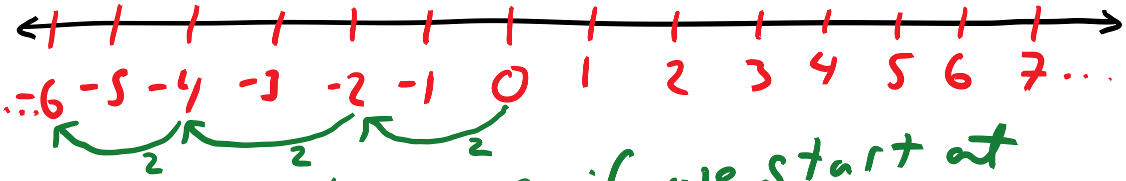
3x2=6 because if we start at 0 and move 2 steps right 3 times, we end at 6

To calculate 3×-2, we will move to the left rather than the right because the -2 is negative. We start at 0 and take 2 steps left a total of 3 times. Notice that we end up at $-6 = -(3 \times 2)$.



3x -2 = -6 because if we start at 0 and move 2 steps left 3 times we end at -6. Note (3x-2) = -(3x2).

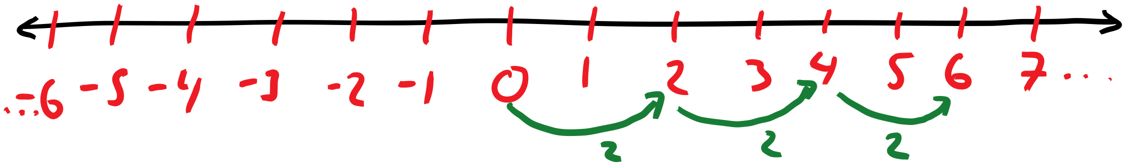
Next, we calculate -3×2. Since the -3 is negative, we move in the direction opposite 2. We move left 2 steps a total of 3 times. When we do so, we end up at -6. Notice that $-3 \times 2 = 3 \times -2 = -(3 \times 2)$. In all three cases, the one negative sign changes direction once from right to left.



$-3\times2=-6$ because if we start at 0 and Take 2 steps left 3 times then we end at -6.

Note That $-3\times2=3\times-2=-(3\times2)$.

Next we calculate -3×-2. Since the -3 is negative, our steps will be in the opposite direction of -2, that is, our steps will be to the right. We start at 0 and take 2 steps to the right 3 times. We end at 6. Notice that $-3\times-2=3\times2$. The two negative signs cause us to change direction twice.



$-3\times-2=6$ because if we start at 0 and Take 2 steps right (opposite -2) 3 times we land at 6.

Note That $-3\times-2=3\times2$

Multiplication

To calculate $A\times B$, start at 0 and take $|B|$ steps a total of $|A|$ times. If A is positive, the steps should be in the direction of B. If A is negative, the steps should be in the direction opposite B.

Notice how the signs work out when we multiply.

- If both of A and B are positive, then we are taking steps to the right (the direction of B), so $A\times B$ is positive.
- If A is positive and B is negative, then we are taking steps to the left (the direction of B), so $A\times B$ is negative.
- If A is negative and B is positive, then we are taking steps to the left (the direction opposite B), so $A\times B$ is negative.
- If both of A and B are negative, then we are taking steps to the right (the direction opposite B), so $A\times B$ is positive.

Division

To calculate $A \div B$ on the number line when A and B are both positive, we asked the question, if we start at 0 and take B steps to end up at A, then how long is each step? To accommodate negative numbers, we simply ask how long was each step, and in what direction? If A and B are both positive, then our steps are to the right, and $A \div B$ will be positive. If A is negative, but B is positive, then we must take steps to the left to get to A. In this case, $A \div B$ is negative. Our interpretation is slightly more complex if B is negative. According to our interpretation, this means we are taking a negative number of steps. We will take this to mean that we are taking steps in the opposite direction. If A is positive and B is negative, then $A \div B$ will have to be negative so that when we take steps in the opposite direction we are moving toward the positive A. If both A and B are negative, then $A \div B$ will have to be negative so that when we take steps in the opposite direction we are moving toward the negative A.

Another way to view the signs when dividing positive and negative numbers is to recall that $A \div B$ should be the number we multiply by B to get A. It follows from this that if A and B are either both positive or both negative, then $A \div B$ will have to be positive. Otherwise, $A \div B$ will be negative.

Negation

Every positive number A is paired with a negative number $-A$. We call A and $-A$ **opposites** or **additive inverses**. If we add a number and its opposite then we get 0. In fact, some books define negative numbers without a number line so that $-A$ is a number that we add to A to get 0. Suppose that A is a positive number. Then $-1 \times A$ is the number that is one set of A unit steps from 0 in the direction opposite from A. This is exactly $-A$, so A and $-1 \times A$ are opposites when A is positive. Similarly, if A is negative, then $-1 \times A$ will also be the opposite of A. For this reason, we usually abbreviate our notation and use $-A$ to mean $-1 \times A$. From what we have just said about opposites we know that $A + -A = -A + A = 0$. From our discussions above about addition and subtraction, it follows that $A + -B = A - B$ and $A - -B = A + B$. From the distributive property of multiplication, we know that $-A - B = -(A + B)$.

We have now extended addition, subtraction, multiplication, and division to negative numbers by referring to the number line. We will take it for granted that these extensions satisfy the properties of arithmetic that we outlined earlier. We summarize here these operations along with some new properties.

Addition

To add $A + B$, locate A on the number line and move $|B|$ steps in the direction of B – left if B is negative and right if B is positive.

- If B is negative, then $A + B = A - |B|$.
- If A is negative, then $A + B = B - |A|$.

Subtraction

To calculate $A - B$, locate A on the number line and move $|B|$ steps in the direction opposite of B – right if B is negative and left if B is positive.

- If B is negative, then $A - B = A + B$.

Multiplication

To calculate $A \times B$, start at 0 and take $|B|$ steps a total of $|A|$ times. If A is positive, the steps should be in the direction of B. If A is negative, the steps should be in the direction opposite B. $A \times B$ will be the same size as $|A| \times |B|$.

- If both of A and B are positive, then we are taking steps to the right (the direction of B), so $A \times B$ is positive.
- If A is positive and B is negative, then we are taking steps to the left (the direction of B), so $A \times B$ is negative.
- If A is negative and B is positive, then we are taking steps to the left (the direction opposite B), so $A \times B$ is negative.
- If both of A and B are negative, then we are taking steps to the right (the direction opposite B), so $A \times B$ is positive.

Division

To calculate $A \div B$, we asked the question, if we start at 0 and take B steps to end up at A, then how long is each step? And in what direction? $A \div B$ will be the same size as $|A| \div |B|$. The sign of $A \div B$ follows the same rules as for division.

Negation

The opposite or negation of A is $-A = -1 \times A$.

- $A + -A = -A + A = 0$.
- $A + -B = A - B$.
- $A - -B = A + B$.
- $-A - B = -(A + B)$.

Helpful Hints for Adding Positive and Negative Numbers

If A and B are both negative, then $A + B$ can be found by adding their absolute values and negating it. For example $-5 + -7 = -(5 + 7) = -12$.

If one of A and B is positive, and the other is negative, then to calculate $A + B$ subtract the smaller absolute value from the larger and give the answer the sign of the number with the larger absolute value. For example, if we want to find $-12 + 7$. We first calculate $12 - 7 = 5$. Then, since -12 has a larger absolute value than 7, we know that the answer should be negative. Therefore, $-12 + 7 = -5$.

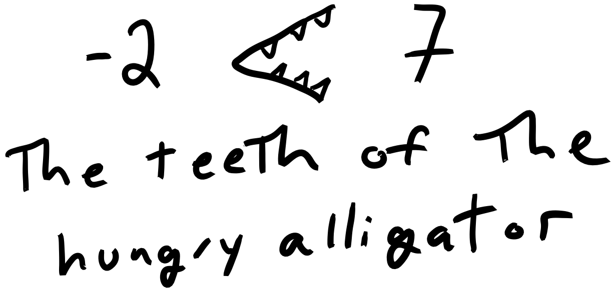
If B is negative, then calculate $A - B$ as $A + |B|$. For example, $-5 - -7 = -5 + 7$. Here, we first calculate $7 - 5 = 2$. Then, since 7 has the larger absolute value, the answer should be positive, so $-5 - -7 = -5 + 7 = 2$.

If A is negative and B is positive, then calculate $A - B$ as $-(|A| + |B|)$. For example, $-5 - 7 = -(5 + 7) = -12$.

Comparing Numbers

The whole numbers $0, 1, 2, 3, 4$ … come with a memorized order. This order can be extended naturally to the integers …-3, -2, -1, 0, 1, 2, 3, 4… by using the number line. When a number A is to the right of a number B on the number line, we say that A is **less than** B. In symbols we write this as $A < B$. We can also say that B is **greater than** A and write $B > A$. For example, $2 < 5$ because 5 is farther right on the number line than 2. We could also write this as $5 > 2$. As another example, $-5 < 2$ because -5 is farther to the left on the number line than 2. This is because -5 is to the left of 0, while 2 is to the right of 0. Finally, $-5 < -2$ because -5 is farther left than -2 on the number line. This is because -5 is 5 steps to the left of 0 while -2 is only 2 steps to the left of 0.

Some people have trouble remembering which way the symbols $<$ and $>$ should point. A common way to remember is the *hungry alligator* analogy. Imagine that the less than or greater than symbol is the mouth of a hungry alligator. Since the alligator is hungry (and greedy) its mouth is always opened toward the larger number (which it wants to eat first). Alternatively, the larger/taller side of the symbol is closest to the larger number. The smaller/shorter side of the symbol is closer to the smaller number.



We will use the symbol ≤ to mean **less than or equal to**. For example, 1 is less than 2, so 1 is less than or equal to 2. In this case, we can write $1 \leq 2$. Also, 1 is equal to 1, so 1 is less than or equal to 1. Therefore, we can write $1 \leq 1$. On the other hand, we cannot write $1 < 1$ because 1 is not less than 1.

We can summarize how to compare two different integers A and B this way:

- If A and B are both greater than or equal to 0 then use the memorized order of the whole numbers to compare the numbers (or the base ten technique used below).
- If A is 0 and B is positive, then $A < B$.
- If A is negative and B is positive, then $A < B$.
- If A is positive and B is 0, then $A > B$.
- If A is negative and B is 0, then $A < B$.
- If A is positive and B is negative, then $A > B$.
- If A is 0 and B is negative, then $A > B$.
- If A and B are both negative an $|A| < |B|$, then $A > B$.
- If A and B are both negative an $|A| > |B|$, then $A < B$.

Comparing Numbers in Base Ten

Base ten notation provides an easy way to compare numbers having only memorized the order on the digits $0 < 1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9$. We demonstrate why the method works with an example.

Problem: Suppose that $5AB$ and $6XY$ are positive three digit numbers. We do not know what the digits A, B, X, Y are. Explain why $5AB < 6XY$.

The largest $5AB$ can be is 599. The smallest $6XY$ can be is 600. Since $599 < 600$ we know that $5AB < 6XY$. This idea is the basis for how we compare base ten numbers. To compare two numbers in base ten, find the left-most place where the numbers are different. The number with the larger digit in that place is the larger number.

Problem: Which is larger, 236547 or 234179?

The left-most place where these numbers are different is the thousands place. The number 236547 has a 6 in this place. The number 234179 has a 4 in this place. Since $6 > 4$, it follows that $236547 > 234179$.

Problem: Which is smaller, 123456 or 9876?

Since these two numbers are different lengths, it may be good to pad 9876 with some 0s on the left to make them the same length. This way, we are comparing 123456 and 009876. The left-most digit where these numbers differ is the hundred thousands digit. The number 123456 has a 1 in this place. The number 009876 has a 0 in this place. Since $0 < 1$, it follows that $009876 < 123456$.

Problem: Which is smaller, -123456 or -9876?

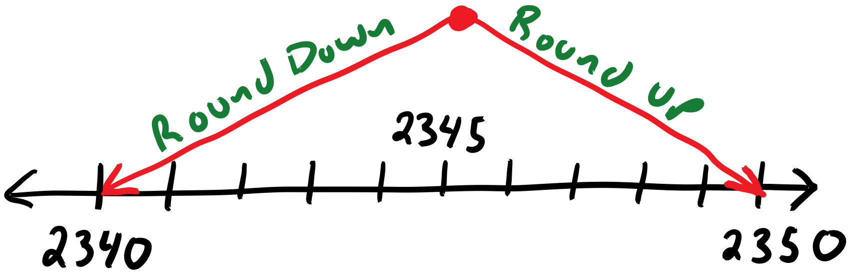
We just saw that $9876 < 123456$, so $-123456 < -9876$.

Rounding Numbers

Sometimes we may need to do arithmetic with a number we do not know exactly. For example, if we are doing computations with the populations of cities, we may not know the populations exactly, and those populations may be changing daily. As another example, we may be doing computations with distances between galaxies. These, again, would be numbers that we do not know exactly and which are changing constantly. Also, numbers such as distances between galaxies may be so enormous that computations with exact values may be beyond tedious. When we are doing computations with numbers we do not know exactly or computations which are exceptionally tedious, we might approximate answers. The basis for approximation is rounding. To **round** a number to the nearest ten, or hundred, or thousand, or so forth is to find the multiple of 10, or 100, or 1000, or so forth that is closest to the number. We illustrate the process with examples. Our method is the **5-up** method which is almost universal.

Problem: Round 2347 to the nearest ten.

We know that $2340 < 2347 < 2350$. The middle number between 2340 and 2350 is 2345. Any number larger than 2345 is closer to 2350 than 2340. Any number below 2345 is closer to 2340. Since 2347 is larger than 2345, 2347 is closer to 2350. Therefore, we **round up** to 2350. In this discussion, there is the question of what to do with 2345 since it is right in the middle between 2340 and 2350. We (rather arbitrarily) decide that 2345 must round up to 2350. This is how we round to the nearest ten: Look at the ones digit. If the ones digit is 5 or larger, round up. If the ones digit is less than 5 round down. Some people like to draw a number line like the one below to round. We have a mountain between 2340 and 2350 with a peak at 2345. Rounding is simply letting numbers slide down the slopes of the mountain. Every number to the left of the peak rounds (slides) down to 2340. Every number on the peak or to the right rounds (slides) up to 2350.



Problem: Round 874239 to the nearest hundred.

First, note that $874200 < 874239 < 874300$. The middle number between 874200 and 874300 is 874250. Since $874239 < 874250$, we **round down** to 874200. Again, we can do this looking only at the tens place (the digit immediately right of the hundreds place). The tens place is 3, which is less than 5. To round to the hundreds place, we round down.

Problem: Round 87,512,343 to the nearest million.

To round to the nearest million, we look at the digit immediately to the right, the hundred thousands place. Since the hundred thousands digit is 5, we round up to 88,000,000.

Rounding with Negative Numbers

There is some disagreement when it comes to rounding negative numbers. We illustrate the confusion with rounding to the nearest 10. There is no debate that -10, -11, -12, -13, and -14 should all round to -10, which is technically rounding up. There also is no debate that -20, -19, -18, -17, and -16 should round to -20, which is technically rounding down. However, what about -15? Some books round this to -10, sticking with the ideas that 5s go up and that up means right on the number line. Other books ignore the negative sign, round 15 to 20, and then put the negative sign back so that they round -15 to -20. These books are really rounding 5s away from 0.

Approximate Arithmetic

One of the uses of rounding is to approximate computations. In approximating computations, rounding numbers before computing decreases the number of non-zero digits and, hence, makes computation easier – but less accurate.

Problem: There are 87 people at a party. Each person's dinner costs \$18. Use rounding to approximate the total cost of the dinners.

There are about 90 people at the party (rounding to the nearest ten). Each person's dinner cost about \$20. Therefore, the total cost is about $90 \times \$20 = \$1800$. This is a very rough approximation. The actual cost is $87 \times \$18 = \$1566$, which rounds to $\$1600$. For our approximation, we chose to round both the number of people and the cost to the tens place. We might also have rounded just the cost to get the approximation $87 \times \$20 = \$1740$, which is better.

Problem: Smallville has a population of 1972. Mediumville has a population of 27321. Approximate the combined population of the two towns by rounding first.

We first need to decide how to round. Rounding to the nearest ten or hundred will not save us much arithmetic. We could round both populations to the nearest thousand. That gives an approximation of $2000 + 27000 = 29000$. The actual value is $1972 + 27321 = 29293$.

Order of Operations

We now have the arithmetic operations addition, subtraction, multiplication, division, and negation. If we use parentheses to explicitly detail in what order a computation should be done, our work will be tedious and will appear complex. For example, consider

$$\big((-8) \times 4\big) + (2 \times 3) + \left(4 \times \left(2^{(7+2)}\right)\right) + \left(3 \times \big((1+2) - 3\big)\right)$$

There are a lot of parentheses here. In this section, we declare an accepted order of operations which will allow us to dispense with many of these parentheses. We note that the order of operations is somewhat arbitrary, but it is universally accepted.

Exponents

Before we dive into our order of operations, we add one more operation to our list – exponents. If N is a counting number (positive integer) and A is an integer, then A^N is the product of N copies of A. In this notation, N is called an **exponent** and A is called the **base** of the exponent. The entire expression A^N is called an **exponential**. For example, $5^3 = 5 \times 5 \times 5 = 125$. In this computation, 5 is the base and 3 is the exponent. Similarly,

$$2^{10} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 1024.$$

Note that if $N = 1$ there is just 1 A in the product A^N (so it really is not a product). Therefore, $A^1 = A$. Exponentiation satisfies some nice properties that we need to be aware of. First, suppose we multiply $A^N \times A^M$. The number A^N is a product of N copies of A, and A^M is a product of M copies of A. Therefore, $A^N \times A^M$ is a product of $N + M$ copies of A. Therefore, $A^N \times A^M = A^{(N+M)}$. For example,

$$2^3 \times 2^4 = (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2) = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7.$$

Thus, when we multiply exponentials with like bases, we add exponents. For example,

$$3^{12} \times 3^{13} = 3^{(12+13)} = 3^{25}.$$

For the next property, recall that we have an inverse property of multiplication and division that says that $(A \times B) \div B = B$. That is, if we multiply by B and then divide by B, we end up where we started. Suppose now that N and M are counting numbers and that $N > M$. Then $N = M + (N - M)$. We want to consider $A^N \div A^M$.

$$\begin{aligned} A^N \div A^M &= A^{(M+(N-M))} \div A^M \\ &= \left(A^M \times A^{(N-M)}\right) \div A^M \\ &= \left(A^{(N-M)} \times A^M\right) \div A^M \\ &= A^{(N-M)} \end{aligned}$$

Thus, when we divide exponentials with like bases, we subtract exponents. For example

$$3^{13} \div 3^{12} = 3^{(13-12)} = 3^1 = 3.$$

Or definition of exponents so far requires the exponent to be a positive integer. We now consider what happens if the exponent is 0. If we want our subtraction/division property to hold, then the following equalities must be true.

$$A^0 = A^1 \div A^1 = A \div A = 1.$$

Thus, it makes sense that $A^0 = 1$. At least this makes sense if we can divide by A. Since we cannot divide by 0, we avoid defining 0^0. There are some isolated situations in which we would accept 0^0 to be 1. However, there are problems with this in general. These problems do not arise until calculus and a topic known as limits, so we can avoid them. Just be aware that 0^0 is not defined. We can now handle exponents that are whole numbers – positive integers. For negative integer exponents, we will have to wait until after our discussion of fractions. This table summarizes what we know about exponents.

$A^N = A \times A \times A \times \cdots \times A$	A can be any integer. N must be positive.
$A^1 = A$	A can be any integer.
$A^0 = 1$	A can be any integer other than 0.
$A^N \times A^M = A^{(N+M)}$	Be sure to avoid 0^0.
$A^N \div A^M = A^{(N-M)}$	$N > M$ and avoid 0^0

Order of Operations

We now introduce an agreed upon order of operations that allows us to dispense with many parenthesis in order to make arithmetic seem simple. We perform arithmetic in this order:

1. Parentheses: Start inside the inner-most parentheses and perform arithmetic following the order specified below.
2. Exponents: Calculate all exponentials whose base and exponents are not in parentheses (which should be all exponentials if we did parentheses first).
3. Multiplication and Division: Perform all multiplications and divisions working from left to right.
4. Addition and Subtraction: Perform all additions and subtractions working from left to right.

Some people remember the order Parentheses, Exponents, Multiplication, Division, Addition, Subtraction through the acronym PEMDAS. There are a number of phrases that can be memorized to help remember this order of letters. One of the most common is:

Please explain, my dear Aunt Sally.

Slightly less kind to Aunt Sally is:

Please excuse my dear Aunt Sally.

We can leave Aunt Sally out of it and use:

People eat more donuts after tea.

You may have noticed that there is no negation in the order of operations. Recall that $-A = -1 \times A$, so negation takes the same priority as multiplication.

Problem: Calculate $2 \times \big(3 + 4 \times (7 - 2^2)\big)$.

We start inside the inner-most parentheses. Inside these parentheses is an exponential, so we start there. In the next series of equal signs, we perform one operation per equal sign, and we color in red before the equal sign the piece of arithmetic we perform.

$$\begin{aligned} 2 \times \big(3 + 4 \times (7 - 2^2)\big) &= 2 \times \big(3 + 4 \times (7 - 4)\big) \\ &= 2 \times (3 + 4 \times 3) \\ &= 2 \times (3 + 12) \\ &= 2 \times 15 \\ &= 30 \end{aligned}$$

Problem: Calculate $5 + 3 - 3$.

This question has only addition and subtraction, so we work from left to right.

$$5 + 3 - 3 = 8 - 3 = 5$$

Notice that it would be very easy to mistakenly calculate $5 + (3 - 3)$ instead.

Problem: Calculate $6 \times 2 \div 3 \times 2$.

Since this only involves multiplication and division, we work from left to right.

$$\color{red}{6 \times 2} \color{black}{\div 3 \times 2 =} \color{red}{12 \div 3} \color{black}{\times 2 =} \color{red}{4 \times 2} \color{black}{= 8}$$

It is extremely common for students to mistake this problem for $(6 \times 2) \div (3 \times 2)$. This is particularly common among people who use a slash / for division. Later, when we are working with fractions, we might encounter a fraction with 6×2 on top and 3×2 on bottom. If the fraction is written with a horizontal line, then no parentheses are necessary. If the fraction is written with a slash, parentheses are necessary.

Problem: Compare -2^4 and $(-2)^4$.

The expression -2^4 is the same as $-(2^4) = -(2 \times 2 \times 2 \times 2)$. It requires that we calculate $2^4 = 16$ and then negate it to get -16. On the other hand, $(-2)^4$ is $-2 \times -2 \times -2 \times -2$. Since a negative times a negative is positive, the result here is going to be positive, $(-2)^4 = 16$.

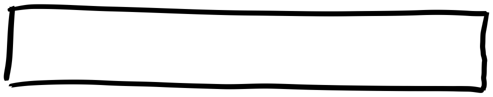
Fractions

In this section we introduce the notion of fraction in two different but equivalent ways. First, we discuss fractions as parts of a whole. This is the most common definition of fraction and will serve as our definition in later sections. Then, we will derive fractions starting with a number line.

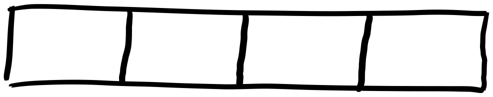
Parts of a Whole

Most often when we encounter a fraction, we are speaking of a fraction of something – a part of a whole. Suppose that B is a counting number (in particular $B \neq 0$) and that A is a nonnegative integer. If an object is divided into B equal size pieces, then each piece is called $\frac{1}{B}$ of the object. Then $\frac{A}{B}$ of the object is the size of A parts where each part is $\frac{1}{B}$ of the object. In this case, the object being divided is our **whole** or **unit**. The symbols $\frac{1}{B}$ and $\frac{A}{B}$ are called **fractions**. The fraction $\frac{1}{B}$ is called a **unit fraction**. In the fraction $\frac{A}{B}$, A is called the **numerator** of the fraction, and B is called the denominator of the **fraction**. (I personally prefer the words **top** and **bottom**.)

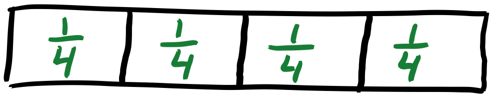
Here is a bar which we will use to represent one unit.



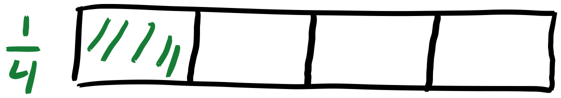
Here is the same bar divided into 4 equal size pieces.



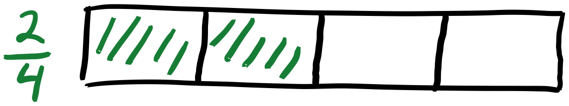
Each of the 4 equal size pieces is $\frac{1}{4}$ of the bar.



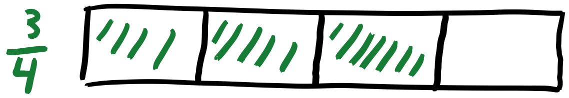
Here, $\frac{1}{4}$ of the bar is shaded.



Here, $\frac{2}{4}$ of the bar is shaded.

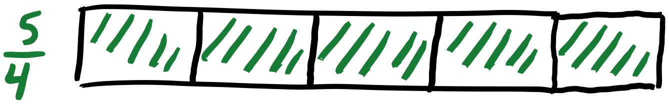


Here, $\frac{3}{4}$ of the bar is shaded.

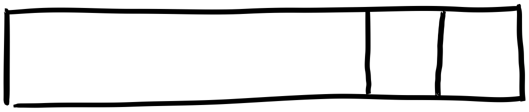


Here, $\frac{4}{4}$ of the bar is shaded. Notice that $\frac{4}{4}$ of the bar is the whole bar. In general, if we divide a whole into B equal size parts, then B of those parts together make the whole, so $\frac{B}{B} = 1$.

Here, we have 5 parts that are each the same size as $\frac{1}{4}$ of the bar. We say that $\frac{5}{4}$ of the bar is shaded.

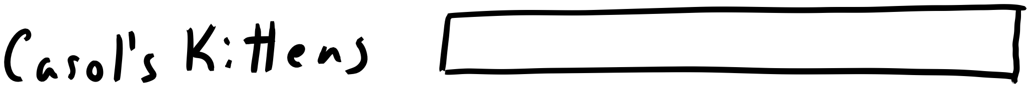


It is important to remember that when working with fractions, the parts should all be the same size. For example, the bar below is divided into 3 parts. However, the parts are not all the same size, so we cannot say that each part is $\frac{1}{3}$ of the bar.

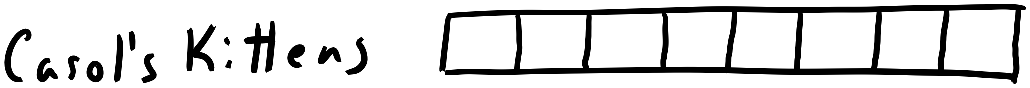


Problem: Carol had 16 kittens. She gave $\frac{3}{8}$ of her kittens to Claire. How many kittens did Carol give to Claire?

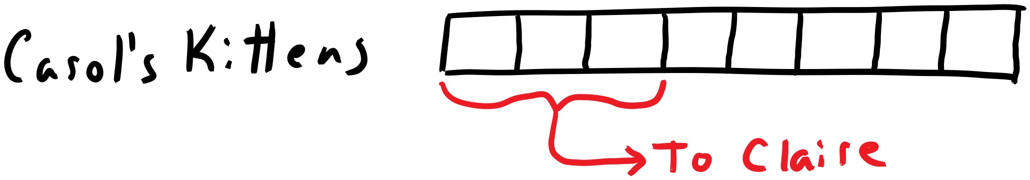
We approach this problem with a bar model. We first draw a bar representing all of Carol's kittens.



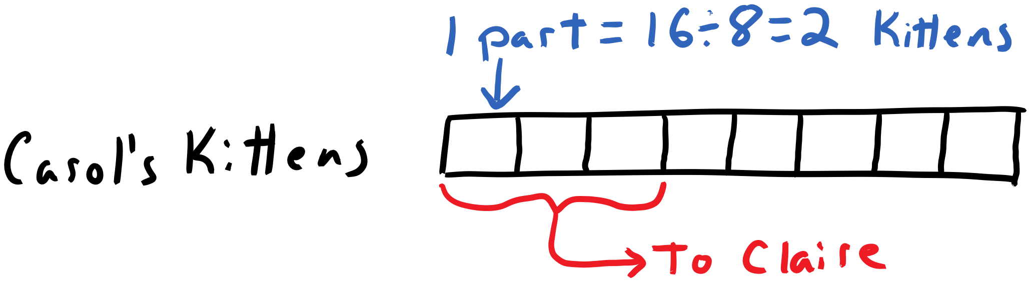
Since the problem refers to $\frac{3}{8}$ of Carol's kittens, we divide the bar into 8 equal size parts.



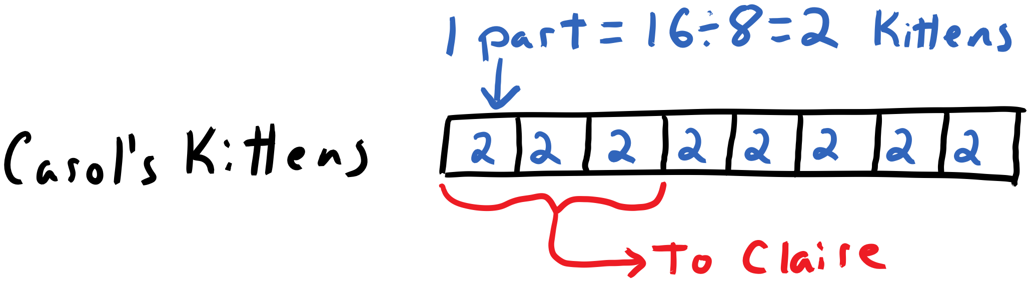
We indicate in our bar model that 3 of these parts are given to Claire.



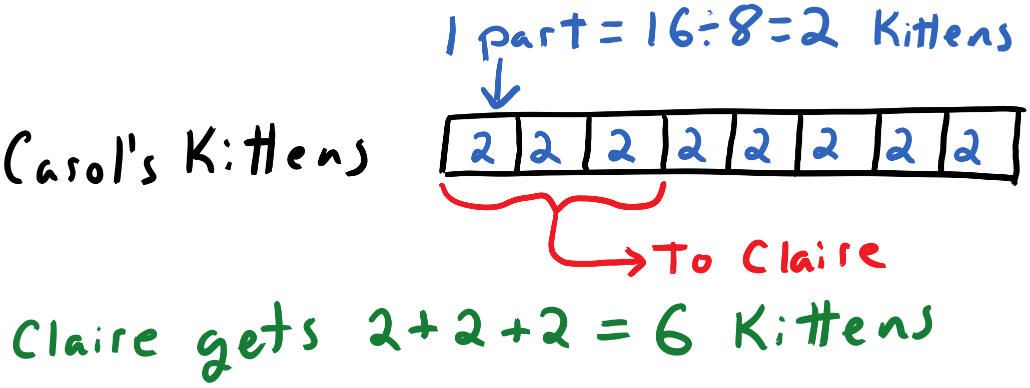
Now, this bar for Carol's kittens is divided into 8 equal size parts that add up to 16 kittens. We can find the size of each part by dividing, $16 \div 8 = 2$.



Each part represents 2 kittens.



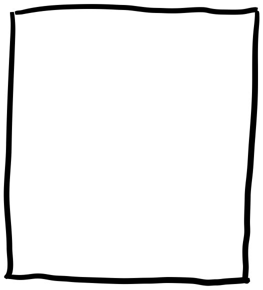
Since Claire received 3 parts, she received $3 \times 2 = 2 + 2 + 2 = 6$ kittens.



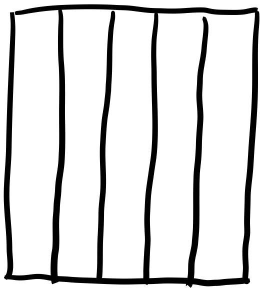
Notice that a result of the work in this problem is that $\frac{3}{8}$ of 16 is 6.

Problem: Carol's cats ate $\frac{2}{5}$ of a bag of cat food. Carol's dog ate $\frac{3}{8}$ of what was left over. What fraction of the bag of cat food did the dog eat?

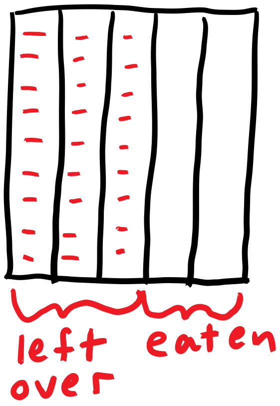
When we work problems in which a quantity is divided up in more than one way, it will often be useful to draw a rectangle or box that is tall enough that we can divide it both with vertical lines and horizontal lines. We do this here. First, we draw a rectangle to represent the entire bag of cat food.



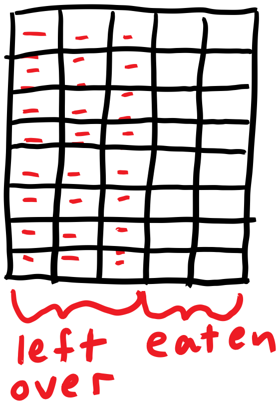
Since the cats ate $\frac{2}{5}$ of the cat food, we divide the box into 5 equal size parts using vertical lines.



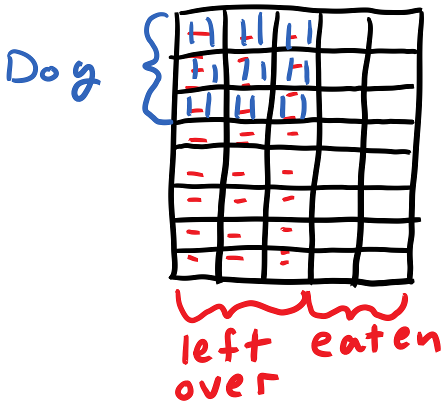
The cat ate 2 of these 5 parts. That leaves 3 parts left over. We indicate this in our model and lightly shade those 3 parts in red.



Now we address the dog. The dog ate $\frac{3}{8}$ of the left over, so we want to divide the 3 red bars into 8 equal size parts and shade 3 of them. We divide this time with horizontal lines. We go ahead and divide the entire rectangle into 8 parts (partly because we can) even though at this point we are really focused on the 3 red bars.



Now that the 3 red bars are divided into 8 parts, we shade 3 of those 8 parts blue. When we do so, we ignore the vertical lines, and we ignore the 2 vertical parts that are not shaded red.



We can now address the fraction of the entire bag of cat food eaten by the dog. The entire bag is now divided into 40 equal size parts. Of these parts, 9 are shaded blue. Therefore, the blue part eaten by the dog is $\frac{9}{40}$ of the bag of cat food. Notice here where the numbers 9 and 40 come from. The 40 comes from 5 rows of 8 parts each. The 9 comes from 3 rows of 3 parts each.

Problem: The rectangle below is $\frac{3}{7}$ of a larger rectangle. How many *s are in the larger rectangle?

* * * * * *
* * * * * *
* * * * * *

First, we will attempt to draw the original rectangle. Since this is $\frac{3}{7}$ of the original rectangle, it should be 3 parts, each of which is $\frac{1}{7}$ of the original. Therefore, we divide it into 3 parts.

* *	* *	* *
* *	* *	* *
* *	* *	* *

Now, each of these smaller rectangles is $\frac{1}{7}$ of a large rectangle, so we draw 7 copies of one of these parts.

* *	* *	* *	* *	* *	* *	* *
* *	* *	* *	* *	* *	* *	* *
* *	* *	* *	* *	* *	* *	* *

We now have 7 parts, each containing 6 *s, so there are a total of $7 \times 6 = 42$ *s.

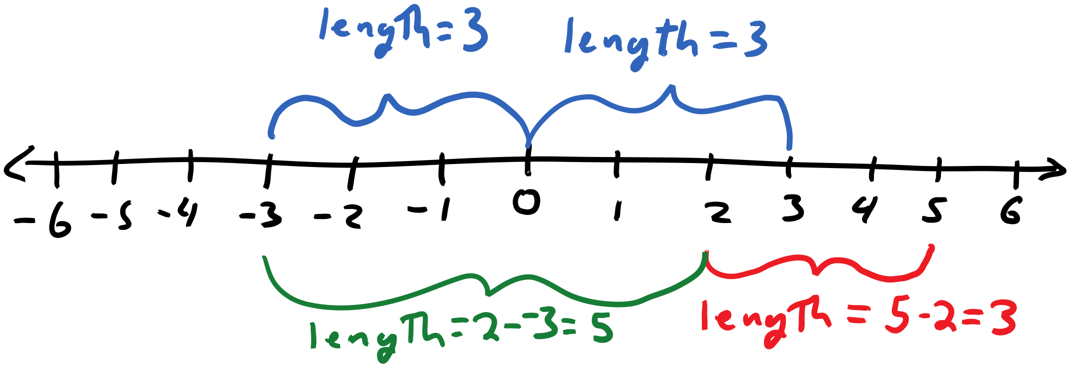
An alternative way to work this problem is to divide the smaller rectangle into 3 parts horizontally.

* * * * * *
* * * * * *
* * * * * *

Then, each part, which is $\frac{1}{7}$ of the larger rectangle is one row. The large rectangle should be 7 parts or 7 rows. Now, 7 rows of 6 *s adds up to $7 \times 6 = 42$ *s.

Lengths of Intervals

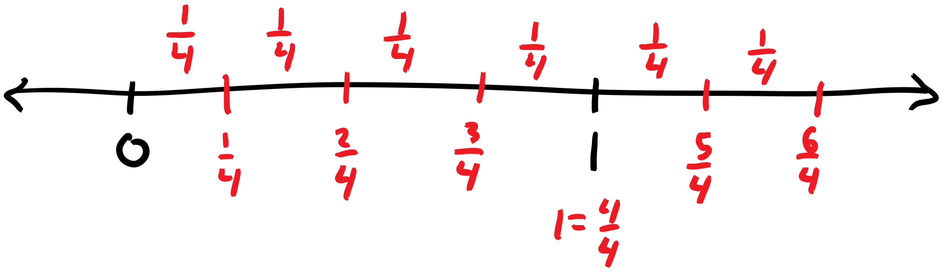
Our definition of fraction above relies on dividing some object into equal sized pieces. We are now going to address fractions in terms of a number line. In this case, the object being divided will always be a part of the number line. First, we need to make a comment about length. For us, numbers are points on the number line. We can use those numbers to measure the length of pieces of the number line. If $A < B$ are numbers on the number line, then the segment of the number line from A up to B is called an **interval**. The length of the interval from A to B is the **difference** $B - A$. Notice that we always subtract the larger number minus the smaller number so that the length is positive. If we do not know which of the two numbers is larger, we can say that the length of the interval between A and B is $|A - B| = |B - A|$. For example, the length of the interval from 2 to 5 is $5 - 2 = 3$. The length of the interval from -3 to 2 is $2 - -3 = 5$. The length of the interval from 0 to 3 is $3 - 0 = 3$. The length of the interval from -3 to 0 is $0 - -3 = 3$.



Note that the length of the interval from 0 to A is $|A|$.

Fractions and Number Lines

For addressing fractions on the number line, we will initially divide the interval from 0 to 1 into equal size parts. We call this interval the **unit interval**. Suppose now that B is a counting number and that A is a nonnegative integer. If we divide the interval from 0 to 1 into B equal length parts, then the length of each part is the fraction $\frac{1}{B}$. The combined length of A intervals each of length $\frac{1}{B}$ is denoted as $\frac{A}{B}$.



Notice that the points (numbers) between 0 and 1 which are a distance of $\frac{1}{B}$ apart are numbered $\frac{1}{B}, \frac{2}{B}, \frac{3}{B}, \dots$

There are benefits to using the number line and fractions of the unit interval. One benefit is that we inherit the natural order of the number line for comparing fractions. Another benefit, is that we automatically get negative fractions. If A and B are positive, then $-\frac{A}{B}$ is the number which is the same distance to the left of 0 as $\frac{A}{B}$ is to the right of 0.

Fraction Notation and Integer Multiplication

Our definition of $\frac{A}{B}$ is that $\frac{A}{B}$ is the size (length) of A parts where each part is $\frac{1}{B}$ of the whole (or unit). When we were simply counting objects in groups rather than measuring size or length, we might have called this value $A \times \frac{1}{B}$ or even $\frac{1}{B} + \frac{1}{B} + \cdots + \frac{1}{B}$ (where there are A copies of $\frac{1}{B}$). Therefore, we can treat the symbol $\frac{A}{B}$ as an abbreviation of these other expressions:

$$\frac{A}{B} = A \times \frac{1}{B} = \frac{1}{B} + \frac{1}{B} + \cdots + \frac{1}{B}.$$

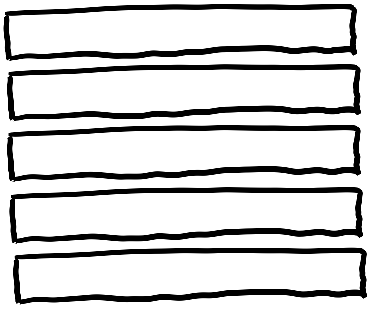
This practice adapts nicely to negative fractions. The fraction $-\frac{A}{B}$ represents a number to the left of 0 the same distance that $\frac{A}{B}$ is to the right. Using the associativity of multiplication, it should be that

$$-\frac{A}{B} = -(A \times B) = -A \times B = \frac{-A}{B}.$$

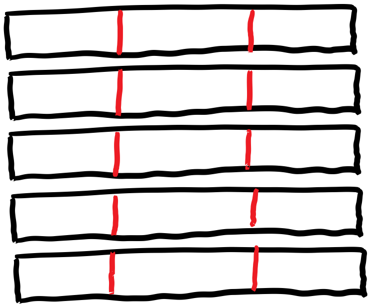
Here, the final fraction could be interpreted as the number which is A copies of $\frac{1}{B}$ to the left of 0. This is consistent with our interpretation of $-\frac{A}{B}$.

Fractions and Division

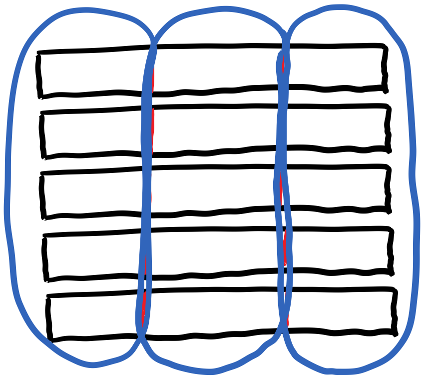
Suppose that we have 5 cakes (which are all the same size) and that we want to give the same amount of cake to each of 3 people. This is a division problem, if we divide 5 objects into three groups, then how many objects are placed in each group? The problem is that 5 is not divisible by 3. We can, however, slice cakes. Here are 5 bars representing the 5 cakes.



To divide the cakes between the 3 people, we first slice each cake into 3 equal size pieces.



This gives us 15 pieces of cake that we need to divide among 3 people. Each person should get $15 \div 3 = 5$ pieces of cake. We distribute the cake this way, so each person gets one piece from each cake.



Consider now how much cake each person gets. Since we have divided 5 things among 3 people, each person should be getting $5 \div 3$ parts of a cake. However, each person gets 5 pieces, and each piece is one of 3 equal size parts of one cake. That is, each person gets 5 pieces, and each piece is $\frac{1}{3}$ of one cake. We call 5 parts where each part is $\frac{1}{3}$ the fraction $\frac{5}{3}$. Thus it seems that $5 \div 3$ (which prior to now had no meaning) should be equal to $\frac{5}{3}$. This is true in general, and we can treat fraction notation $\frac{A}{B}$ simply as another way of writing division $A \div B$. The difference is that now, quotients such as $5 \div 3$ now make sense because we no longer require each group to contain a whole number of objects.

Number Systems

Prior to this section, we had encountered arithmetic in these numbers systems

Counting Numbers	1, 2, 3, 4…
Whole Numbers	0, 1, 2, 3, 4…
Integers	…-4, -3, -2, -1, 0, 1, 2, 3, 4…

We now have fractions. The number system which includes all fractions $\frac{A}{B}$ where A and B are integers and $B \neq 0$ is called the **rational numbers**. A truly formal derivation of the integers and rational numbers from the whole numbers would work something like this. First, addition and multiplication are defined (somehow) on the counting numbers and whole numbers. For each counting number A, let $-A$ be a number so that $-A + A = -A + A = 0$. Such a number is called an additive inverse of A. Then define the integers to include the counting numbers, 0, and the additive inverse of every counting number and extend the operations of addition and multiplication to the integers. Now, for each nonzero integer B define $\frac{1}{B}$ to be a number so that $B \times \frac{1}{B} = \frac{1}{B} \times B = 1$. Such a number is called a multiplicative inverse of B. Extend the definitions of addition and multiplication of integers to include the multiplicative inverse of every nonzero integer. This extension necessarily encounters numbers of the form $A \times \frac{1}{B}$, and we use the notation $\frac{A}{B}$ for these products. The rational numbers are then all numbers of the form $\frac{A}{B}$ where A and B are integers and $B \neq 0$. Note that in this process the only operations that are mentioned are addition and multiplication. There is no such thing as subtraction or division. Subtraction is merely adding an additive inverse. Division is mere multiplying by a multiplicative inverse.

Slashes

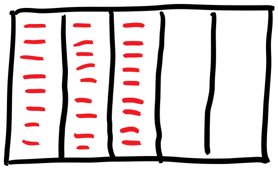
Since fraction notation $\frac{A}{B}$ can be equated to the division $A \div B$, many people use a slash as a compromise notation. The notation A/B means the same thing as $\frac{A}{B} = A \div B$. The benefits of the slash are that A/B is slightly easier to write than $A \div B$ and that A/B is easier to include inline than $\frac{A}{B}$. There are some dangers to using slashes that we should be aware of though. First, it is very easy when you are in a hurry to write $A + B/C + D$ when you mean $\frac{A+B}{C+D}$. However, because of our order of operations, these are very different expressions. If you want to write $\frac{A+B}{C+D}$ with a slash, you need to use parentheses: $(A + B)/(C + D)$. Similarly, while $AB/C = \frac{AB}{C}$, the expressions $\frac{A}{B \times C}$ and $A/B \times C$ are different. To typeset $\frac{A}{B \times C}$ with a slash, you need parentheses: $A/(B + C)$.

Forms of Fractions

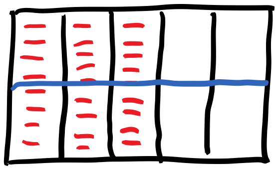
Every fraction can be written in infinitely many different ways. In this sections, we discuss equivalent fractions and converting between improper fractions and mixed numerals.

Equivalent Fractions

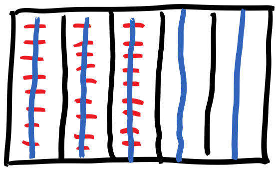
Consider the fraction $\frac{3}{5}$. Here is a rectangle divided into fifths with vertical lines with $\frac{3}{5}$ shaded.



And here is the same rectangle with an addition horizontal divider which divides the entire rectangle (and each of the fifths) into two equal size parts.



In this diagram, the rectangle (the unit or whole) is now divided into 10 equal size parts. Of those parts, 6 are shaded. Therefore, the shading represents the fraction $\frac{6}{10}$. However, the shading did not change between the two diagrams. In the first diagram, the shading represents $\frac{3}{5}$. Thus it has to be that $\frac{3}{5} = \frac{3\times2}{5\times2} = \frac{6}{10}$. When two fractions represent the same value as these do, we say that the fractions are **equal** or **equivalent**. Note here that we could also have divided our original parts vertically into two parts each like so:

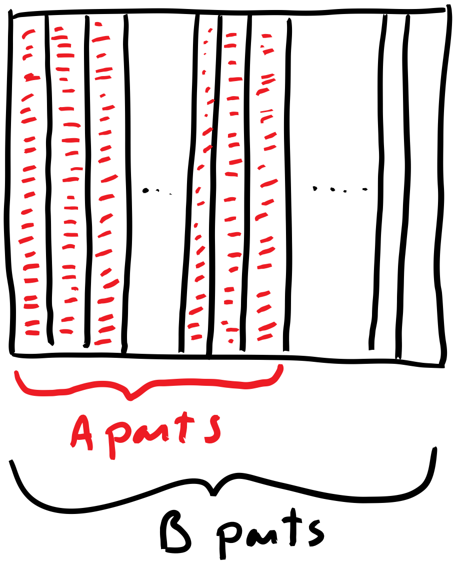


However, when we use horizontal dividers like we did in our first approach, it appears more clear that we are dividing the entire rectangle into equal parts. Also, with the first approach, it is clear that we have an array of parts that consists of 2 rows and 3 columns.

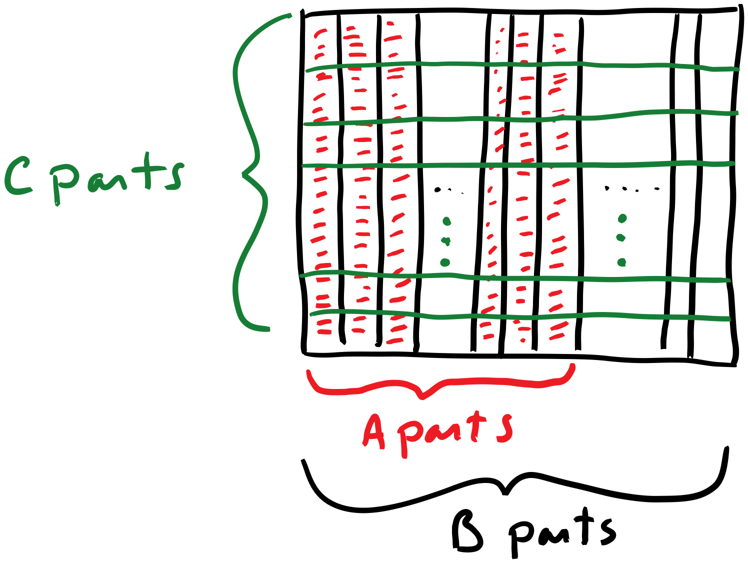
There is nothing special about the fraction $\frac{3}{5}$ here. If we multiply the top and bottom of any fraction by the same number, then we arrive at an equivalent fraction. Suppose that A, B, and C are integers with B and C not zero. Then $\frac{A}{B} = \frac{A\times C}{B\times C}$. In this equality, we can think of multiplying the top and bottom of $\frac{A}{B}$ times C, or we can think of dividing the top and bottom of $\frac{A\times C}{B\times C}$ by C. When we think of dividing, we will often say that we are **cancelling** the C.

$$\begin{tikzcd} \frac{A}{B} \arrow[r, bend left, "\times C"] \arrow[r, bend right, "\times C"'] \arrow[r, phantom, "="] & \frac{A\times C}{B\times C} \end{tikzcd} \qquad \begin{tikzcd} \frac{A\times C}{B\times C} \arrow[r, bend left, "\div C"] \arrow[r, bend right, "\div C"'] \arrow[r, phantom, "="] & \frac{A}{B} \end{tikzcd} \qquad \frac{A\times \cancel{C}}{B\times \cancel{C}} = \frac{A}{B}$$

We can demonstrate why $\frac{A}{B}=\frac{A\times C}{B\times C}$ with diagrams the way we did above. First, we draw a bar or rectangle representing $\frac{A}{B}$ by dividing the bar with vertical lines into B equal size parts and shading A of them.



We then divide the entire bar with horizontal lines into C equal size parts.



At then point, the entire bar has been divided into B columns, each containing C equal size parts. There are $B\times C$ of these parts. The shaded portion of the bar has been divided into A columns, each containing C equal size parts. There are $A \times C$ shaded parts. The shaded region, then, represents the fraction $\frac{A\times C}{B\times C}$. However, this is the same shading that represented $\frac{A}{B}$. Therefore, it has to be that $\frac{A}{B} = \frac{A\times C}{B\times C}$.

Problem: Convert the fraction $\frac{5}{6}$ to an equivalent fraction with a denominator of 18.

Since $18 \div 6 = 3$, we multiply the top and bottom of our fraction by 3.

$$\frac{5}{6} = \frac{5 \times 3}{6 \times 3} = \frac{15}{18}.$$

Problem: Convert the fraction $\frac{20}{36}$ to a fraction with a denominator of 9.

If we divide 36 by 9, we get $36 \div 9 = 4$, so $36 \div 4 = 9$. We simply divide the top and bottom of our fraction by 4.

$$\frac{20}{36} = \frac{20 \div 4}{36 \div 4} = \frac{5}{9}.$$

At this point, there is no number which is a factor of both the top and the bottom of this fraction. Such fractions are called **completely reduced**.

Problem: Convert the fractions $\frac{7}{8}$ and $\frac{7}{12}$ to equivalent fractions with the same denominator.

When two fractions have the same denominator, we call that denominator a **common denominator**. One common denominator for any pair of fractions is the product of the denominators. In this case, we could multiply the top and bottom of $\frac{7}{8}$ by 12 and the top and bottom of $\frac{7}{12}$ by 8. The resulting equivalent fractions have a common denominator.

$$\frac{7}{8} = \frac{7\times 12}{8\times 12} = \frac{84}{96} \text{ and } \frac{7}{12} = \frac{7\times 8}{12\times 8} = \frac{56}{96}.$$

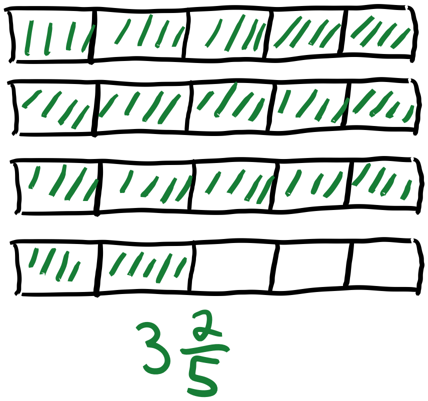
Using the product of the denominators like this as a common denominator is always an option. It may not be the best option, because the product can sometimes get large. Any number which is a multiple of both 8 and 12 will do. For example, $24 = 8 \times 3$ and $24 = 12 \times 2$. Therefore, we can convert our fractions to have a common denominator of 24:

$$\frac{7}{8} = \frac{7\times 3}{8\times 3} = \frac{21}{24} \text{ and } \frac{7}{12} = \frac{7\times 2}{12\times 2} = \frac{14}{24}.$$

The benefit of these fractions is that all of the numbers are smaller. The number 24 is actually the smallest number that can be used as a common denominator between 8 and 12. It is called the **least common denominator**.

Improper Fractions and Mixed Numbers

The notation $3\frac{2}{5}$ is called a mixed number. It is read as “3 and $\frac{2}{5}$,” and it means $3 + \frac{2}{5}$. We can convert mixed numbers to fractions by counting parts. We draw 3 shaded bars to represent the 3, and then we shade $\frac{2}{5}$ of another bar. Since the problem refers to fifths, we divide all of the bars into fifths.



At this point, there are $3 \times 5 + 2 = 17$ parts shaded. Each part is $\frac{1}{5}$ of a whole bar. Therefore, the part shaded represents $\frac{17}{5}$. That is $3\frac{2}{5} = \frac{17}{5}$. Let us examine where this 17 comes from. $17 = 3 \times 5 + 2$. This is the number of wholes (3) times the number of parts in each whole (5) – which is the denominator of the fraction part of the mixed number – plus the numerator of the fraction. In general, we can convert mixed numbers to fractions using the formula

$$A\frac{B}{C} = \frac{A \times C + B}{C}.$$

Problem: Convert $6\frac{7}{9}$ to a fraction.

All we have to do is multiply 6 by 9 and add 7 for our new numerator.

$$6\frac{7}{9} = \frac{6 \times 9 + 7}{9} = \frac{54 + 7}{9} = \frac{61}{9}.$$

When we convert a mixed number to a fraction, we will always get a fraction where the numerator is larger than the denominator. Such fractions are called **improper fractions**. Personally, I think these fractions are poorly named. The name makes one think that there is something wrong with these fractions and that they should be avoided. However, there is nothing wrong with them, and we will see later that we actually prefer improper fractions when doing arithmetic with and comparing fractions.

We need to be able to convert mixed numbers to improper fractions, and we need to be able to convert improper fractions to mixed numbers. To see how to do so, we start with an example. We convert $\frac{14}{3}$ to a mixed number. To do so, we need to know how to answer the question, if 14 objects are placed into groups of 3, then how many groups are there and how many objects are left over? Answering this is exactly dividing. Since $14 \div 3 = 4R2$, we know that $\frac{14}{3}$ is 4 groups of 3 thirds plus 2 thirds left over. Since 3 thirds is one whole, 4 groups of 3 thirds is simply 4. Also, 2 thirds is $\frac{2}{3}$. Thus, $\frac{14}{3} = 4\frac{2}{3}$. All we did here is divide $14 \div 3 = 4R2$. The quotient 4 becomes the whole number part of our mixed number, and the remainder 2 becomes the numerator of the fraction part of our mixed number. In general, if $\frac{A}{B}$ is an improper fraction, and if $A \div B = CRD$ then $\frac{A}{B} = C\frac{D}{B}$.

Problem: Convert $\frac{25}{7}$ to a mixed number.

First, we divide: $25 \div 7 = 3R4$. Then $\frac{25}{7} = 3\frac{4}{7}$.

Negative Mixed Numbers

Some caution should be used with negative mixed numbers. The number $-2\frac{3}{4}$ is the same as $-\left(2\frac{3}{4}\right) = -\left(2 + \frac{3}{4}\right) = -\frac{11}{4}$. It is not the same as $-2 + \frac{3}{4}$. It is definitely not the same as $2\frac{-3}{4}$. In fact this last expression should never be written down. It has no meaning.

Operations with Fractions

We have addressed what fractions are, and we have address how to write fractions in different forms. In this section, we address how to perform arithmetic operations with fractions.

Comparing Fractions

It is easy to see that the fraction $\frac{6}{13}$ is larger than the fraction $\frac{4}{13}$ because 6 of something is more than 4 of something. However, it may be difficult to tell at first glance which is larger between $\frac{6}{13}$ and $\frac{13}{30}$. Both of these numbers are a little less than one half. What makes the first two fractions easy to compare is that they have a common denominator. In order to compare any two fractions, the first thing we will do is find a common denominator. We will convert both $\frac{6}{13}$ and $\frac{13}{30}$ to have the same denominator, $13 \times 30 = 390$.

$$\frac{6}{13} = \frac{6\times30}{13\times30} = \frac{180}{390} \text{ and } \frac{13}{30} = \frac{13\times13}{30\times13} = \frac{169}{390}.$$

Since $180 > 169$, $\frac{6}{13} > \frac{13}{30}$. In order to compare two fraction, find a common denominator and then compare the tops of the fractions.

Problem: Which is larger $\frac{11}{18}$ or $\frac{13}{20}$?

We will find a common denominator and then compare. One denominator we could use is $18 \times 20 = 360$.

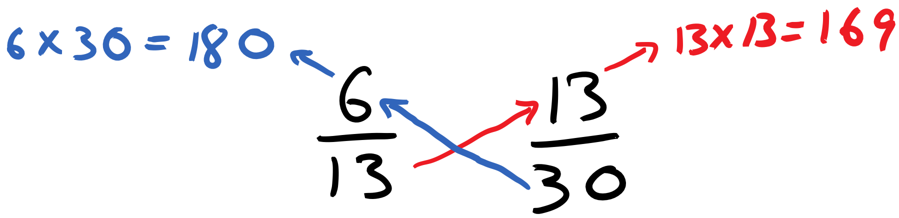
$$\frac{11}{18} = \frac{11\times20}{18\times20} = \frac{220}{360} \text{ and } \frac{13}{20} = \frac{13\times18}{20\times18} = 234/360.$$

Since $234 > 220$, $\frac{13}{20} > \frac{11}{18}$. To compare these fractions, we might also have used the smaller denominator 180.

Problem: Which is larger $\frac{11}{18}$ or $\frac{14}{30}$?

We could find a common denominator her; however, there is a way to tell which of these fractions is larger much more quickly. It happens to be that half of 18 is 9, so $\frac{11}{18}$ is greater than $\frac{1}{2}$. On the other hand, half of 30 is 15, so $\frac{14}{30}$ is less than $\frac{1}{2}$. Since $\frac{14}{30} < \frac{1}{2} < \frac{11}{18}$, it follows that $\frac{11}{18} > \frac{14}{30}$. Reasoning this way where we locate a known fraction between the two we are comparing is called using a **benchmark**.

Some people compare fractions with cross-multiplying. To use this technique, we multiply the top of each fraction by the bottom of the other and then compare. This approach for comparing $\frac{6}{13}$ and $\frac{13}{30}$ would look something like this:



Since the product 180 on the left is larger, $\frac{6}{13}$ is the larger fraction. Notice that this approach is equivalent to finding a common denominator. We simply never multiply the denominators to see what that common denominator might be.

Adding and Subtracting Fractions

It is easy to add and subtract fractions with common denominators. For example, $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$ just like 2 apples plus 1 apple is 3 apples. Therefore, to add or subtract fractions, we simply first find a common denominator and then add or subtract the numerators.

Problem: Add $\frac{7}{8}+\frac{11}{12}$. Write your answer as a completely reduced mixed number.

First we find a common denominator. Since $8\times 3=24$ and $12\times 2=24$, we use 24 as a common denominator.

$$\frac{7}{8}=\frac{7\times 3}{8\times 3}=\frac{21}{24}\text{ and }\frac{11}{12}=\frac{11\times 2}{12\times 2}=\frac{22}{24}.$$

Therefore

$$\frac{7}{8}+\frac{11}{12}=\frac{21}{24}+\frac{22}{24}=\frac{21+22}{24}=\frac{43}{24}.$$

Now, the problem asks for a mixed number answer, so we convert this improper fraction to a mixed number. Since $43\div 24=1R19$, it follows that

$$\frac{7}{8}+\frac{11}{12}=\frac{43}{24}=1\frac{19}{24}.$$

Now, if 19 and 24 had a common factor, we would reduce this final answer. As it is, we do not need to.

Problem: Subtract $\frac{1}{6}-\frac{5}{8}$. Write your answer as a completely reduce fraction.

We first find a common denominator. Since $6\times 4=24$ and $8\times 3=24$, we again use 24.

$$\frac{1}{6}=\frac{1\times 4}{6\times 4}=\frac{4}{24}\text{ and }\frac{5}{8}=\frac{5\times 3}{8\times 3}=\frac{15}{24}.$$

Therefore

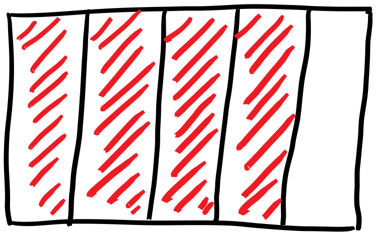
$$\frac{1}{6}-\frac{5}{8}=\frac{4}{24}-\frac{15}{24}=\frac{4-15}{24}=-\frac{9}{24}.$$

Now, since the top and bottom of this fraction are both divisible by 3, we can reduce the fraction.

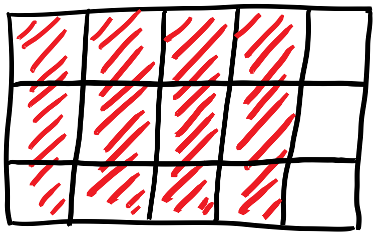
$$\frac{1}{6}-\frac{5}{8}=-\frac{9}{24}=-\frac{9\div 3}{24\div 3}=-\frac{3}{8}.$$

Multiplying Fractions

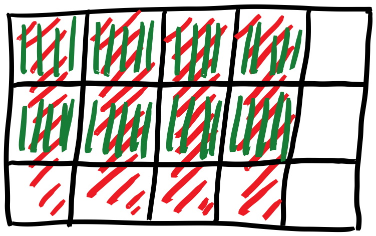
We now turn our attention to multiplying. When we introduce multiplication for whole numbers, we said that $A\times B$ is the number of objects in A groups of B objects each. We can interpret fraction multiplication such as $\frac{2}{3} \times \frac{4}{5}$ as the size of $\frac{2}{3}$ of a group of size $\frac{4}{5}$ (of a whole or unit). That is, $\frac{2}{3} \times \frac{4}{5}$ should be $\frac{2}{3}$ of $\frac{4}{5}$. We can use a bar (or rectangle) model to calculate $\frac{2}{3}$ of $\frac{4}{5}$. First, we draw a thick bar (or rectangle) representing one whole unit. We divide the bar with vertical lines into 5 equal parts and shading 4 of them red to represent $\frac{4}{5}$.



We want to shade $\frac{2}{3}$ of the red area. To do so, we divide the red area into three equal size parts with horizontal lines. Since the red area is already inside of one unit, we draw the lines all the way across to divide the entire unit into three equal size parts. Notice now that the entire unit is divided into 3 rows of 5 parts each for a total of 3×5 parts.



Now we shade $\frac{2}{3}$ of the red area green (remember that the red represented $\frac{4}{5}$).



We now have 2 rows with 4 green shaded parts each for a total of 2×4 green shaded parts. Since there are 3×5 total parts, the green now represents $\frac{2\times4}{3\times5} = \frac{8}{15}$. We now have

$$\frac{2}{3} \times \frac{4}{5} = \frac{2}{3}\ of\ \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}.$$

Let us look closely where these numbers come from. The top of the product fraction, 2×4, is the number of green parts. This is the number of green rows – which is the numerator of $\frac{2}{3}$ – times the number of red columns – which is the numerator of $\frac{4}{5}$. The top of the product fraction is thus the product of the tops of the original two fractions. The bottom of the product fraction, 3×5, is the number of rows – which is the denominator of $\frac{2}{3}$ – times the number of columns – which is the denominator of $\frac{4}{5}$. Thus the bottom of the product fraction is just the product of the bottoms of the original fractions. Therefore, to multiply two fractions, simply multiply the tops of the fractions and multiply the bottoms of the fractions.

$$\frac{A}{B} \times \frac{C}{D} = \frac{A \times C}{B \times D}.$$

Problem: Multiply $\frac{5}{12} \times \frac{8}{15}$.

We simply multiply along the top and along the bottom:

$$\frac{5}{12} \times \frac{8}{15} = \frac{5 \times 8}{12 \times 15} = \frac{40}{180}.$$

Now 40 and 180 have common factors, so we can divide to reduce them. We start with the most obvious. Both are divisible by 10.

$$\frac{5}{12} \times \frac{8}{15} = \frac{5 \times 8}{12 \times 15} = \frac{40}{180} = \frac{40 \div 10}{180 \div 10} = \frac{4}{18}.$$

Now 4 and 18 are both divisible by 2, so

$$\frac{5}{12} \times \frac{8}{15} = \frac{5 \times 8}{12 \times 15} = \frac{40}{180} = \frac{40 \div 10}{180 \div 10} = \frac{4}{18} = \frac{4 \div 2}{18 \div 2} = \frac{2}{9}.$$

We can save a little work in this arithmetic here if we factor and cancel prior to multiplying.

$$\frac{5}{12} \times \frac{8}{15} = \frac{5}{3 \times 4} \times \frac{2 \times 4}{3 \times 5} = \frac{5 \times 2 \times 4}{3 \times 4 \times \ 3 \times 5} = \frac{{\color{red}\cancel{5}} \times 2 \times {\color{red}4}}{3 \times {\color{red}4} \times \ 3 \times {\color{red}\cancel{5}}} = \frac{2}{3 \times 3} = \frac{2}{9}.$$

Notice that the numbers in red cancel.

Division of Fractions

It is a little harder to motivate fraction division from our definition. Instead we are going to use this characterization of division which we encountered earlier:

$A \div B$ is the number we multiply by B to get A.

In fraction form,

$\frac{A}{B} \div \frac{C}{D}$ is the number we multiply times $\frac{C}{D}$ to get $\frac{A}{B}$.

Using this characterization, it is easy to verify how we should divide fractions. We propose that

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C}.$$

That is, to divide one fraction by another, invert the second fraction and multiply. The fraction $\frac{D}{C}$ is called the reciprocal of $\frac{C}{D}$. To verify that $\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C}$, all we need to do is multiply $\left(\frac{A}{B} \times \frac{D}{C}\right) \times \frac{C}{D}$ and see that we get $\frac{A}{B}$. Behold:

$$\left(\frac{A}{B} \times \frac{D}{C}\right) \times \frac{C}{D} = \frac{A \times D \times C}{B \times C \times D} = \frac{A \times {\color{red}D \times C}}{B \times {\color{red}C \times D}} = \frac{A}{B}.$$

Notice how the red numbers cancel at the last step.

Problem: Divide $\frac{4}{9} \div \frac{6}{5}$.

To divide, we invert and multiply:

$$\frac{4}{9} \div \frac{6}{5} = \frac{4}{9} \times \frac{5}{6} = \frac{2 \times 2}{9} \times \frac{5}{2 \times 3} = \frac{{\color{red}2} \times 2 \times 5}{9 \times {\color{red}2} \times 3} = \frac{2 \times 5}{9 \times 3} = \frac{10}{27}.$$

Arithmetic with Fraction, Integers, and Mixed Numbers

All of the problems we have seen so far in this section have involved only fractions. We can adjust our approach to handle arithmetic between fractions, integers, and mixed numbers. First, every integer can easily be expressed as a fraction. For example, $\frac{5}{1}$ is another way of writing $5 \div 1 = 5$. To do arithmetic with fractions and integers, just replace every integer A with $\frac{A}{1}$. For example,

Problem: Multiply $\frac{2}{3} \times 5$.

$$\frac{2}{3} \times 5 = \frac{2}{3} \times \frac{5}{1} = \frac{2 \times 5}{3 \times 1} = \frac{10}{3} = 3\frac{1}{3}$$

Problem: Divide $\frac{2}{3} \div 5$.

$$\frac{2}{3} \div 5 = \frac{2}{3} \div \frac{5}{1} = \frac{2}{3} \times \frac{1}{5} = \frac{2 \times 1}{3 \times 5} = \frac{2}{15}$$

When mixed numbers, or fractions and mixed numbers, if both numbers are positive, we simply add the whole number parts and add the fraction parts separately. For example,

Problem: Add $2 + 3\frac{4}{5}$.

$$2 + 3\frac{4}{5} = (2 + 3) + \frac{4}{5} = 5 + \frac{4}{5} = 5\frac{4}{5}$$

Problem: Add $2\frac{3}{4} + 5\frac{6}{7}$.

$$\begin{aligned} 2\frac{3}{4} + 5\frac{6}{7} &= (2 + 5) + \left(\frac{3}{4} + \frac{6}{7}\right) \\ &= 7 + \left(\frac{3 \times 7}{4 \times 7} + \frac{6 \times 4}{7 \times 4}\right) \\ &= 7 + \left(\frac{21}{28} + \frac{24}{28}\right) \\ &= 7 + \frac{21 + 24}{28} \\ &= 7 + \frac{45}{28} \\ &= 7 + 1\frac{17}{28} \\ &= 8\frac{17}{28} \end{aligned}$$

If negatives are involved with the addition, or if we are subtracting, multiplying, or dividing, we convert the mixed numbers to improper fractions to perform the arithmetic. For example,

Problem: Multiply $2\frac{3}{4} \times 5\frac{6}{7}$

$$2\frac{3}{4} \times 5\frac{6}{7} = \frac{11}{4} \times \frac{41}{7} = \frac{451}{28} = 16\frac{3}{28}$$

Here, $2\frac{3}{4} \times 5\frac{6}{7}$ would better be written as $\left(2\frac{3}{4}\right) \times \left(5\frac{6}{7}\right)$ since mixed numbers are really an abbreviation of addition. What we have written is a universally accepted abuse of notation. A much less appealing alternative here is to use the distributive property:

$$\begin{aligned}
2\frac{3}{4} \times 5\frac{6}{7} &= \left(2 + \frac{3}{4}\right) \times \left(5 + \frac{6}{7}\right) \\
&= \left(2 + \frac{3}{4}\right) \times 5 + \left(2 + \frac{3}{4}\right) \times \frac{6}{7} \\
&= 2 \times 5 + \frac{3}{4} \times 5 + 2 \times \frac{6}{7} + \frac{3}{4} \times \frac{6}{7} \\
&= 2 \times 5 + \frac{3}{4} \times \frac{5}{1} + \frac{2}{1} \times \frac{6}{7} + \frac{3}{4} \times \frac{6}{7} \\
&= 2 \times 5 + \frac{3 \times 5}{4 \times 1} + \frac{2 \times 6}{1 \times 7} + \frac{3 \times 6}{4 \times 7} \\
&= 10 + \frac{15}{4} + \frac{12}{7} + \frac{18}{28} \\
&= 10 + \frac{15 \times 7}{4 \times 7} + \frac{12 \times 4}{7 \times 4} + \frac{18}{28} \\
&= 10 + \frac{105}{28} + \frac{48}{28} + \frac{18}{28} \\
&= 10 + \frac{171}{28} \\
&= 10 + 6\frac{3}{28} \\
&= 16\frac{3}{28}
\end{aligned}$$

Subtraction of mixed numbers is easy in some circumstances. If the first whole part is greater than the second, and if the first fraction is greater than the second, we can just subtract the whole parts and fraction parts.

Problem: Subtract $3\frac{4}{5} - 1\frac{2}{5}$

$$3\frac{4}{5} - 1\frac{2}{5} = (3 - 1) + \left(\frac{4}{5} - \frac{2}{5}\right) = 2 + \frac{4 - 2}{5} = 2 + \frac{2}{5} = 2\frac{2}{5}$$

Things are rarely this nice though. In most circumstances, we convert to improper fractions.

Problem: Subtract $5\frac{1}{3} - 2\frac{2}{3}$.

$$5\frac{1}{3} - 2\frac{2}{3} = \frac{16}{3} - \frac{8}{3} = \frac{16 - 8}{3} = \frac{8}{3} = 2\frac{2}{3}$$

An alternative here is to borrow from the 5 because $5\frac{1}{3} = 5 + \frac{1}{3} = 4 + 1 + \frac{1}{3} = 4 + 1\frac{1}{3} = 4 + \frac{4}{3}$. This gives

$$5\frac{1}{3} - 2\frac{2}{3} = 4 + \frac{4}{3} - 2\frac{2}{3} = (4 - 2) + \left(\frac{4}{3} - \frac{2}{3}\right) = 2 + \frac{4 - 2}{3} = 2 + \frac{2}{3} = 2\frac{2}{3}$$

Ratios and Proportions

We use the word **ratio** as a synonym for the word fraction. These statements all mean that the fraction $\frac{\# cats}{\# dogs}$ is equivalent to the fraction $\frac{3}{2}$.

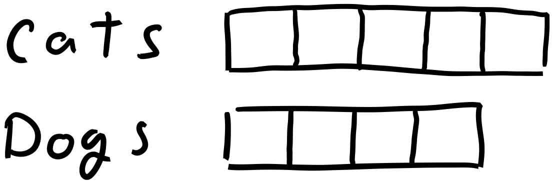
- The ratio of cats to dogs is 3 to 2.
- The ratio of cats to dogs is $\frac{3}{2}$.
- The ratio of cats to dogs is 3:2.
- For every 3 cats there are 2 dogs.
- There is a number N so that the number of cats is $3 \times N$ and the number of dogs is $2 \times N$.
- The number of cats is $\frac{3}{2}$ of the number of dogs.
- The number of dogs is $\frac{2}{3}$ of the number of cats.

Notice that the use of the colon 3:2 is just another notation for $\frac{3}{2}$. We might equate this to $3 \div 2$ without righting the horizontal bar.

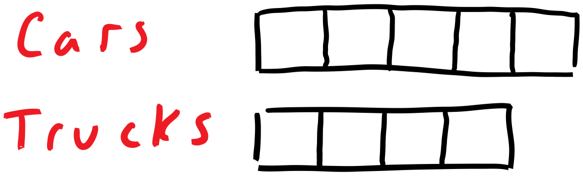
There are currently 10 pens and 6 pencils on my desk. We can describe the ratio of pens to pencils on my desk as “10 to 6” or “5 to 3” or $10{:}\,6$ or 5:3 or $\frac{10}{6}$ or $\frac{5}{3}$. Generally, the reduced versions of the ratios will be most useful because they will simplify arithmetic. Consider a recipe that calls for 3 teaspoons of cinnamon and 2 teaspoons of nutmeg. The ratio of cinnamon to nutmeg in one batch of this recipe is 3:2. If we make two batches, the ratio is 6:4, which is equivalent to 3:2. If we make 100 batches, the ratio will still be 3:2. The fact that the ratio of pens to pencils on my desk is 5:3 is almost an accident. Tomorrow, that ratio will likely be different. The ratio of cinnamon to nutmeg in this recipe is different. It is no accident that the ratio is 3:2. The amounts of cinnamon and nutmeg in this recipe are in a special relationship so that the ratio will always be 3:2. We call this a proportional relationship. Two changing quantities are in a **proportional relationship** if the ratio between the two quantities is always the same.

Problem: The ratio of cats to dogs in a certain neighborhood is the same as the ratio of cars to trucks. There are 5 cats and 4 dogs in the neighborhood. If there are 15 cars in the neighborhood, then how many trucks are there?

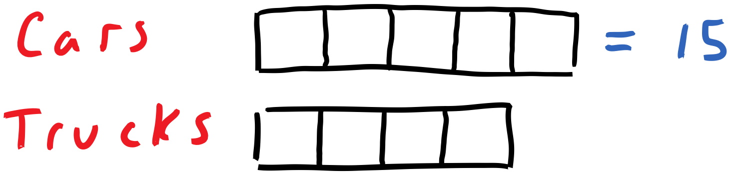
We will approach this problem three different ways, first with bar models and then with a new tool called a ratio table (or fraction table), and then with equivalent fractions. Students are frequently taught to solve these problems with some simple algebra. The algebra approach would let x be the number of trucks and would then solve the equation $\frac{5}{4} = \frac{15}{x}$. The point here is that children can solve problems such as these without algebra. Here is the bar model approach. First, we draw a bar representing cats and dogs. We break the cat bar into 5 parts and the dog bar into 4 parts that are the same size because there are 5 cats for every 4 dogs.



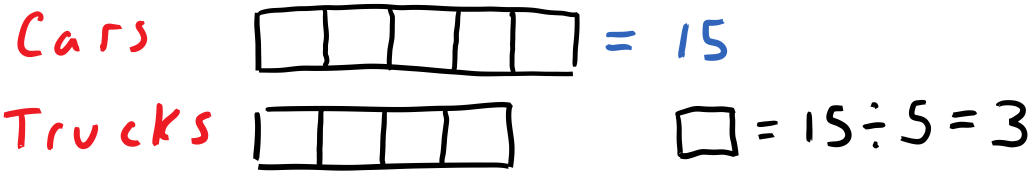
Next, since the ratio of cars to trucks is the same as the ratio of cats to dogs, we change the labels from cats and dogs to cars and trucks. The bars now indicate the proper ratio between cars and trucks. We just do not know how many cars or trucks are in each part.



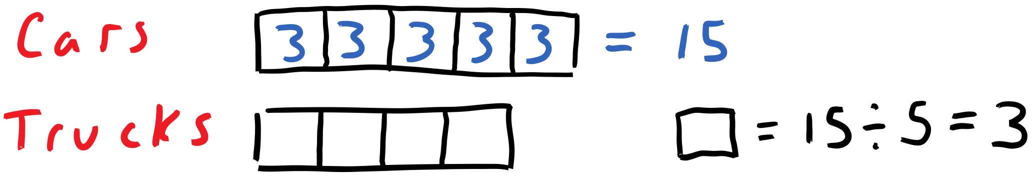
We know that there are 15 cars among the 5 equal size parts.



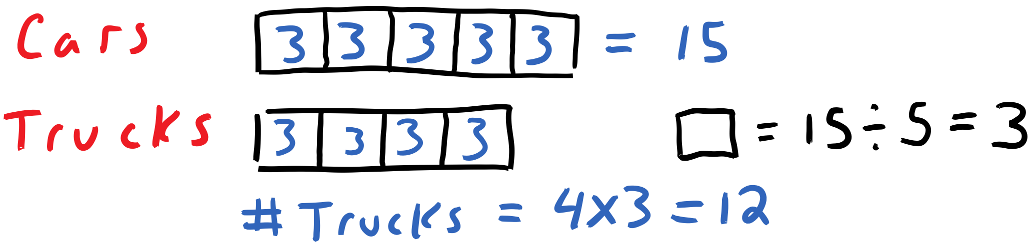
To find out how many cars are in each part, we divide $15 \div 5 = 3$.



There are 3 cars in each part.



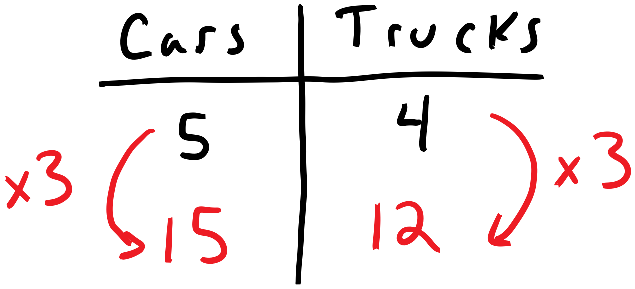
This means that there are 3 trucks in each of 4 parts, so there are a total of 4×3 trucks.



Our next approach is to use what we will call a ratio table or a fraction table. We will make a table with two columns, one for cars and one for trucks. In the table, we will place coordinated values of the numbers of cars and truck that are in the ratio of 5 to 4. We will then manipulate the columns until we see 15 in the car column. First, we know that if there are 5 cars, then there are 4 trucks.

Cars	Trucks
5	4

The way we manipulate the table is to multiply or divide both columns by the same number. This is the same process as multiplying the top and bottom of a fraction by the same number. Here, we multiply both columns by 3 because $5 \times 3 = 15$, and we have 15 cars.



The number of trucks in the neighborhood is again 12.

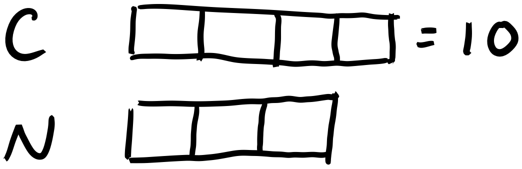
For our final approach at this problem, we will use equivalent fractions. The thought process is similar to the table approach. We set up a fraction $\frac{5}{4}$ and convert it to have a numerator of 15.

$$\frac{\#\ cars}{\#\ trucks} = \frac{5}{4} = \frac{5 \times 3}{4 \times 3} = \frac{15}{12}$$

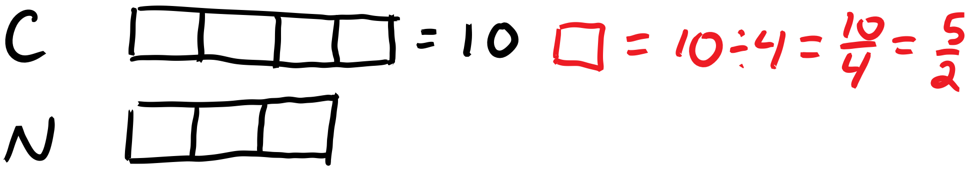
Thus, the number of trucks is still 12.

Problem: A recipe calls for 4 teaspoons of cinnamon for every 3 teaspoons of nutmeg. Sam is going to make more than one batch of this recipe. If he has 10 teaspoons of cinnamon, then how much nutmeg would he need?

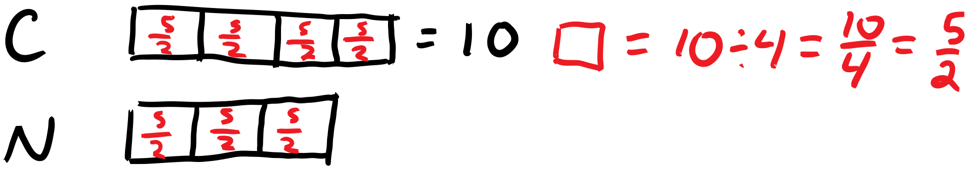
We first solve this problem with a bar model. We draw two bars showing that the ratio of cinnamon to nutmeg is 4:3. Notice that the parts here need not be teaspoons. We also indicate that there are 10 teaspoons of cinnamon.



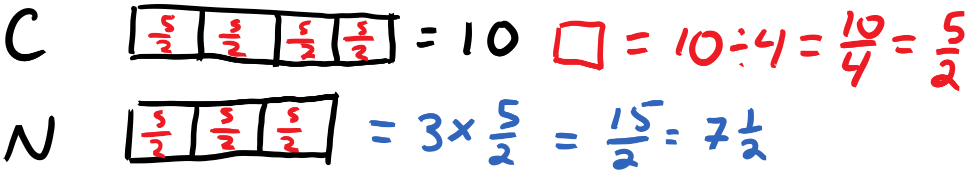
Since there are 10 teaspoons of cinnamon in 4 parts, we can divide to find out the size of each part. Notice that we end up with a fraction.



There are $\frac{5}{2}$ teaspoons in each part.



We can now multiply to find out how much nutmeg Sam used.

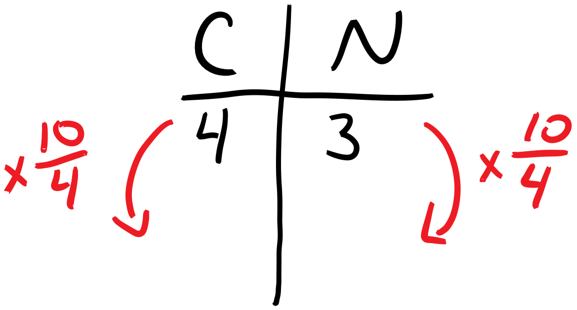


Sam used $7\frac{1}{2}$ teaspoons of nutmeg.

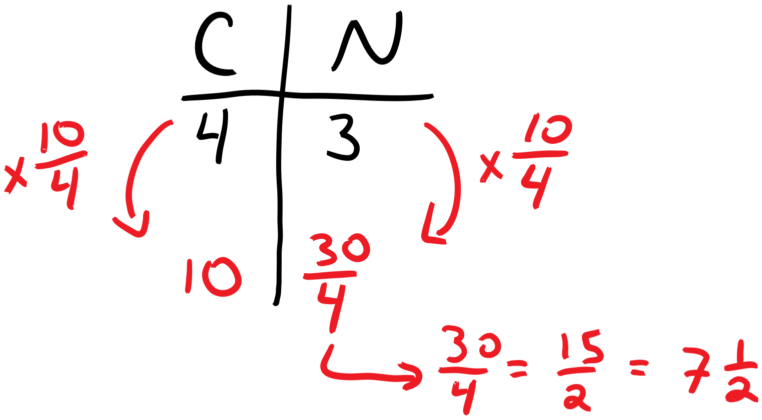
Now we approach the same problem with a table. We set up a table with columns for cinnamon and nutmeg, and we begin with a row of 4 teaspoons of cinnamon and 3 teaspoons of nutmeg. We will work through the table twice to show two different strategies.

C	N
4	3

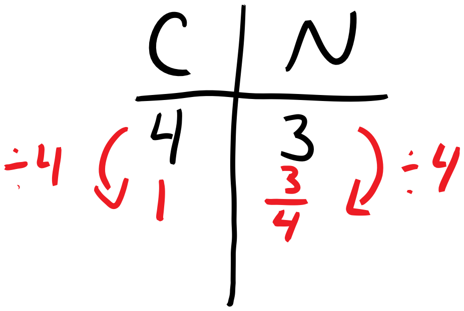
We want to manipulate the first column until we see 10. Fractions were made just for this purpose, since $4 \times \frac{10}{4} = 10$, we multiply both columns by $\frac{10}{4}$.



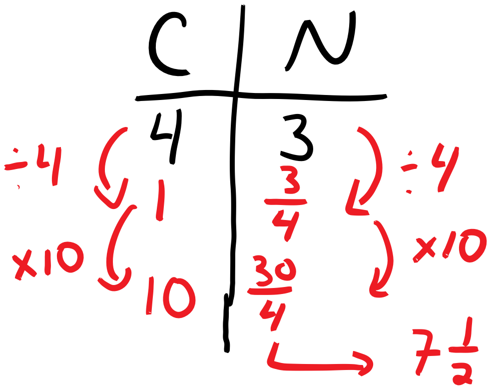
This gives us 10 in the first column (as desired) and $\frac{30}{4} = 7\frac{1}{2}$ in the second.



Sam used $7\frac{1}{2}$ teaspoons of nutmeg. This approach works fine if we are comfortable with fractions. If not, there is another approach we can use. The strategy here is to manipulate the first column until we see a 1. This is easy. Just divide by 4.



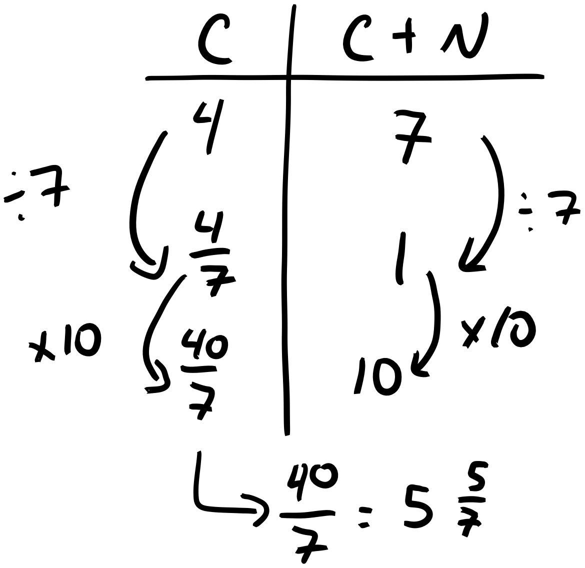
Now that we have a 1 in the first column, it is easy to manufacture a 10. Just multiply by 10.



We still arrive at a final answer of $7\frac{1}{2}$ teaspoons of nutmeg. This method is called **going through 1**. Notice in the right had column that the final answer reduces to $\frac{3}{4} \times 10$. Each number here is meaningful. The fraction $\frac{3}{4}$ is the ratio $\frac{nutmeg}{cinnamon}$. The 10 is a quantity of cinnamon, and the product is a quantity of nutmeg.

Problem: A recipe calls for 4 teaspoons of cinnamon for every 3 teaspoons of nutmeg. Sam is going to make more than one batch of this recipe. If he uses 10 teaspoons of cinnamon and nutmeg combined, then how much cinnamon did he use?

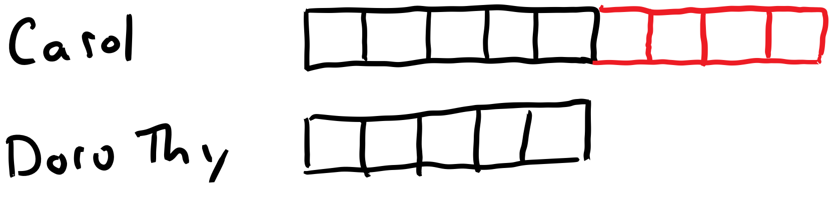
What distinguishes this problem from the earlier ones is that we are relating the quantity of cinnamon to the combined quantity of cinnamon plus nutmeg. Since the ratio of cinnamon to nutmeg is 4:3, the ratio of cinnamon to cinnamon plus nutmeg is 4:(4+3) or 4:7. We use a table to solve this problem, and we choose to go through one. We first divide by 7 to get a 1 in the column for the combined spices. We then multiply by 10.



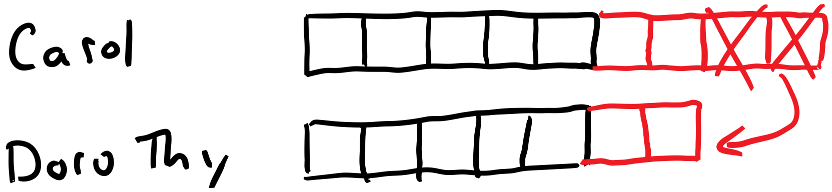
Sam used $5\frac{5}{7}$ teaspoons of cinnamon. Notice here that the final answer is $\frac{4}{7} \times 10$. The $\frac{4}{7}$ is the ratio $\frac{cinnamon}{combined}$. The 10 is a quantity of the combined, and the product is a quantity of cinnamon.

Problem: The ratio of Carol's cats to Dorothy's cats is 9 to 5. After Carol gave Dorothy 4 cats, they had the same number of cats. How many cats did Dorothy start with?

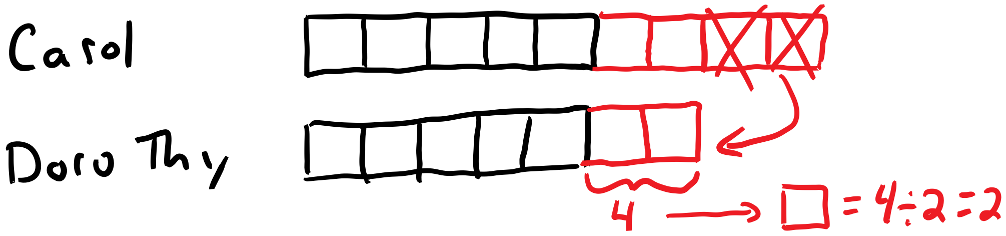
First, we draw a bar model with parts indicating the ratio of Carol's cats to Dorothy's cats is 9:5.



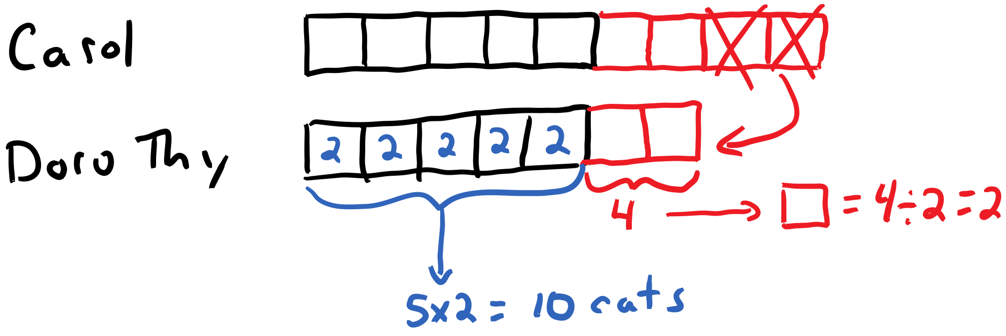
Carol's bar is 4 parts longer than Dorothy's bar. If Carol gave enough cats to Dorothy so that their numbers of cats are the same, then she had to give 2 parts of her bar to Dorothy. We cross out 2 of Carol's parts and give them to Dorothy.



Since we know that Carol gave Dorothy 4 cats, and we know that she gave Dorothy 2 parts, we can divide to see how many cats are in each part.



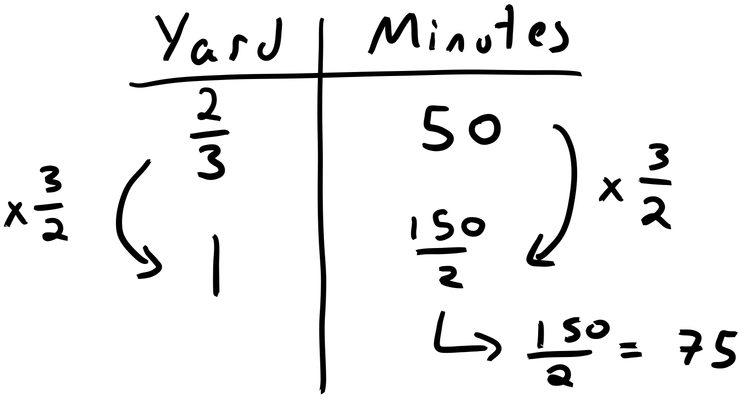
There are 2 cats in each part. We place 2 cats in each of Dorothy's initial 5 parts, and we can multiply to see how many cats she started with.



Dorothy initially had 10 cats.

Problem: Sue can mow $\frac{2}{3}$ of her yard in 50 minutes. How long will it take her to mow the entire lawn?

We will solve this problem with a table. The table will have columns for fraction of yard mowed and minutes. We simply manipulate the fraction of yard column to 1 by multiplying by $\frac{3}{2}$.



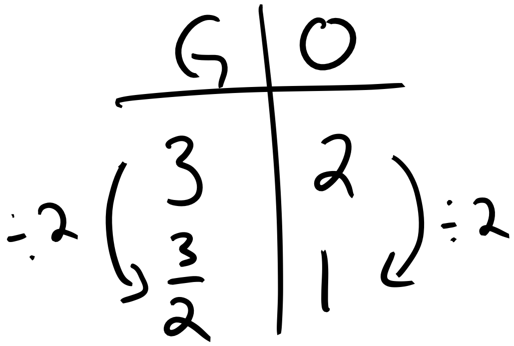
Sue can mow her yard in 75 minutes.

Unit Rates

Suppose that the ratio of grape juice to orange juice in a punch recipe is 3:2. This means that for every 3 parts grape juice, the punch contains 2 parts orange juice. Here, a part might be a cup or a liter or a gallon. Any measure will due.

Problem: Suppose that a mixture of this punch contains 1 gallon of orange juice. How much grape juice should it contain?

We solve this problem with a table. We set up a column with 3 gallons of grape juice and another column with 2 gallons of orange juice.

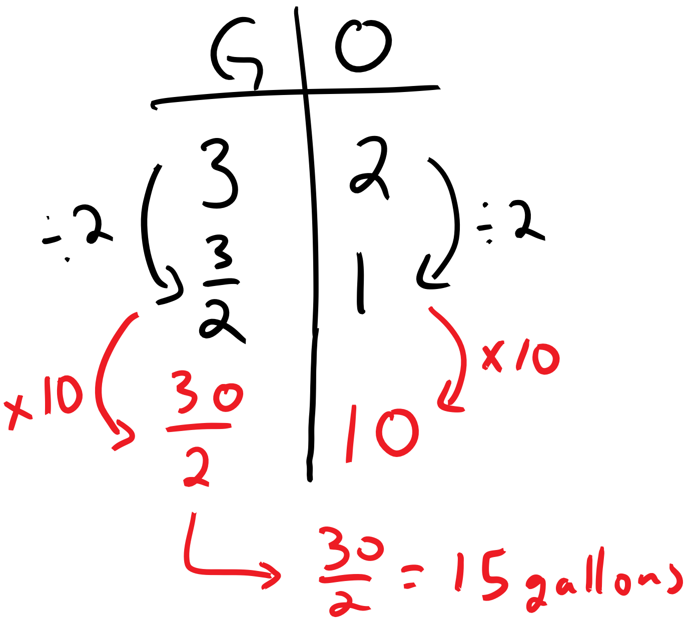


G	O
3	2
$\frac{3}{2}$	1

We divide both columns by 2 to manufacture a 1 in the orange juice column. The result is that we have $\frac{3}{2}$ gallons of grape juice in the first column. This means that there are $\frac{3}{2}$ gallons of grape juice for every gallon of orange juice. This ratio $\frac{3}{2}$ is called the **unit rate** of parts grape juice per part orange juice. This fraction is made by dividing $\frac{grape}{orange}$. The reciprocal $\frac{2}{3} = \frac{orange}{grape}$ is the unit rate of parts of orange juice per part of grape juice. For each part (gallon, cup, whatever) of grape juice, the punch contains $\frac{2}{3}$ parts of orange juice. Unit rates appear in solutions to many ratio problems.

Problem: Suppose that a mixture of this punch contains 10 gallons of orange juice. How much grape juice should it contain?

We solve this problem with a table going through 1, beginning just like we did in the last problem.



After making a 1 in the orange column, we multiply by 10 to get 10 gallons of orange juice. This gives $\frac{3}{2} \times 10 = \frac{30}{2} = 15$ gallons of grape juice. Note that the solution here is $\frac{3}{2} \times 10$. This is the unit rate of grape juice per orange juice times a quantity of orange juice.

Given a quantity of orange juice, to find the corresponding quantity of grape juice, we multiply the quantity of orange juice times the unit rate of grape juice per orange juice. Similarly, given a quantity of grape juice, to find the corresponding quantity of orange juice, we multiply the quantity of grape juice times the unit rate of orange juice per grape juice. These pseudo-equations may make it easier to recall this:

$$\frac{grape}{orange} \times orange = grape \;\; and \;\; \frac{orange}{grape} \times grape = orange.$$

Problem: Suppose that the ratio of blue fish to green fish in a certain lake is always 7 to 5. If there are 3000 green fish in the lake, then how many blue fish are there?

We will solve this problem with unit rates. We are asked for a number of blue fish and are given a number of green fish, so we want the unit rate of blue fish (what we are asked for) per green fish (what we are given). This is $\frac{7}{5}$. To calculate the number of blue fish, we multiply this unit rate times the number of green fish:

$$\#blue\ fish = \frac{blue}{green} \times (\#\ green) = \frac{7}{5} \times 3000 = 4200$$

There are 4200 blue fish in the lake. The number of blue fish and the number of green fish here are related by an equation

$$blue = \frac{7}{5} \times green.$$

In general when two quantities A and B are in a proportional relationship, then there is a number k so that $A = k \times B$. The number k is called the **constant of proportionality**. It is just the unit rate of A per B. When k is positive (which happens with most but not all real world applications), whenever A

increases, so does B, and whenever A decreases, so does B. This characterization of proportional relationships is the most commonly used characterization in upper level math classes and in science classes.

Proportional and Inversely Proportional Relationships

Suppose that a crew of 5 widget painters can paint 100 widgets in 20 hours. Suppose also that all widget painters paint at the same rate. The number of widget painters working is proportional to the number of widgets they can paint in 20 hours. If we double the number of painters, we double the number of widgets. If we triple the number of painters, we triple the number of widgets. We can make a table which displays numbers of painters and how many widgets they can paint in 20 hours.

Painters	Widgets Painted in 20 hours
5	100
10	200
15	300
20	400
25	500

Notice that every time the number of painters is increased by 5, the number of widgets is increased by 100. Notice also that we have the constant ratio

$$\frac{\#\ widgets}{\#painters} = 20.$$

Suppose now that we are interested in how long it takes a crew of workers to paint 500 widgets. We know that it takes 20 hours for 5 workers to paint 100 widgets. Therefore, it should take them 5 times as long to paint 500 widgets. That is, 5 workers take 100 hours to paint 500 widgets. If we double the number of workers, then the crew should work twice as fast and should be able to complete the 500 widgets in half the time, 50 hours. On the other hand, if we have only 1 painter, he should take 5 times as long to paint 500 widgets as a 5 painter crew. Here is a table comparing the number of painters with the time it takes to paint 500 widgets.

Painters	Hours Required to Paint 500 Widgets
1	500
2	250
5	100
10	50
20	25

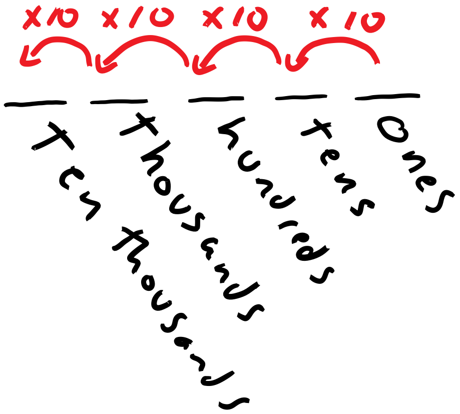
Notice here that when we multiply the number of painters by 2, the time required gets divided by 2. If we multiply painters by 5, we divide hours by 5. This is not a proportional relationship. Rather than the ratio $\frac{\#\ painters}{hours\ required}$ being constant, the product $(\#\ painters) \times (hours\ required)$ is constant:

$$(\#\ painters) \times (hours\ required) = 500.$$

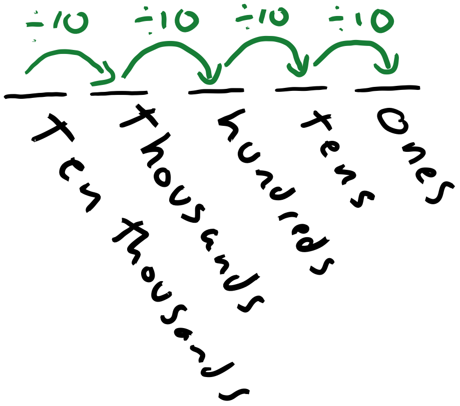
Such a relationship is an **inversely proportional relationship**.

Decimals

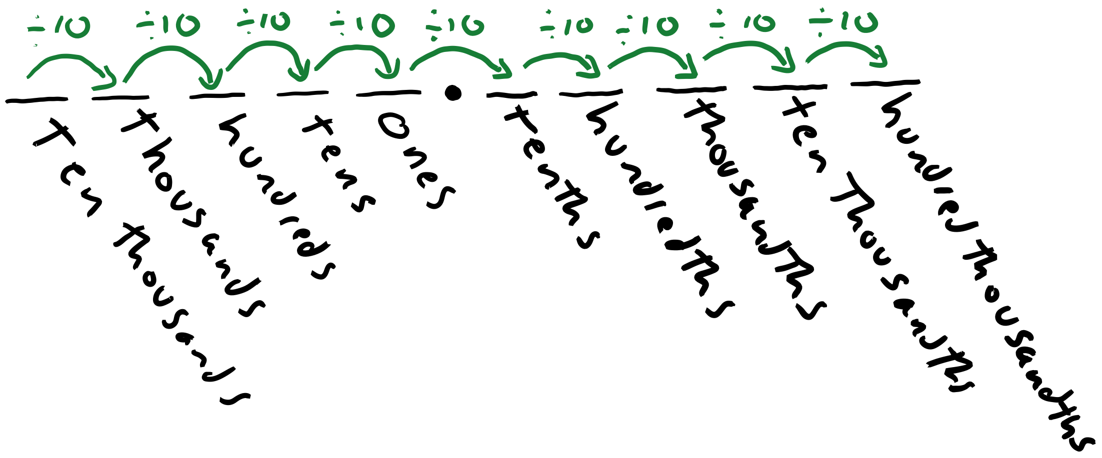
Base ten as we know it provides an efficient way to represent arbitrarily large numbers with a few symbols in such a way that algorithms for arithmetic operations are relatively simple and are based on the operations for small numbers. Fractions are useful for representing parts of number, but arithmetic with fractions can be tedious. In this section, we expand the base ten system to be able to represent fractions of integers. As we move from right to left in a base ten number, place value is multiplied by 10:



This means that to move from left to right in a base ten number, place value is divided by 10:



The idea behind decimals is simply to continue this process going to the right, adding places for tenths $\left(\frac{1}{10}\right)$, hundredths $\left(\frac{1}{100}\right)$, thousandths $\left(\frac{1}{1000}\right)$, ten thousandths $\left(\frac{1}{10000}\right)$, and so on. We place a period immediately left of the ones place for reference so that we know which place is which, and we continue the fractional place values to the right. This period is called a **decimal** or **decimal point**.



With this notation,

$$234.567 = 2 \times 100 + 3 \times 10 + 4 + \frac{5}{10} + \frac{6}{100} + \frac{7}{1000}$$

Notice that the fractional part of this number can be combined into one fraction with a common denominator

$$\frac{5}{10} + \frac{6}{100} + \frac{7}{1000} = \frac{500}{1000} + \frac{60}{1000} + \frac{7}{1000} = \frac{567}{1000}.$$

Therefore, we might read 234.567 as “two hundred thirty four and five hundred sixty seven thousandths.” We might also write $234.567 = 234\frac{567}{1000}$ so that the decimal is really just an abbreviation of the mixed number. We can read the number 3.14 as “three point one four” or as “three and fourteen hundredths.” Also, since

$$3.14 = 3 + \frac{1}{10} + \frac{4}{100} = \frac{300}{100} + \frac{10}{100} + \frac{4}{100} = \frac{314}{100}$$

we can also read 3.14 as “three hundred fourteen hundredths.” This last observation will be the basis for how we draw bundling diagrams for decimal numbers and for how we extend our arithmetic operations to decimal numbers. Notice that

$$1.230 = 1 + \frac{2}{10} + \frac{3}{100} + \frac{0}{1000} = 1 + \frac{2}{10} + \frac{3}{100} + 0 = 1 + \frac{2}{10} + \frac{3}{100} = 1.23$$

To the right of the decimal, trailing 0s on the right do not change the value of a number, just like to the left of the decimal, leading 0s to the left do not change the value of a number. A decimal number such as “twelve thousandths” could be written as .012. Notice that the 0 in the tenths place is not optional. It is necessary to place the 2 in the thousandths place. Usually, we will include a 0 in the ones place and write this number as 0.012. This leading 0 is optional. However, it is useful in helping students not to accidentally lose the decimal point when they are writing quickly.

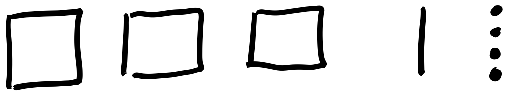
Bundling Diagrams for Decimal Numbers

To draw a base ten bundling diagram for a decimal number such as 3.14, we simply acknowledge that 3.14 can be expressed as 314 hundredths.

Problem: Draw a bundling diagram for 3.14.

We draw a bundling diagram for 314, and we add to it a key that indicates that each dot in our bundling diagram corresponds to a hundredth rather than a one.

3.14 = 314 hundredths



Key: • = 0.01

From now on, all of our bundling diagrams should include a key. If a bundling diagram does not include a key, we will assume that each dot represents a one.

Adding and Subtracting Decimal Numbers

To add or subtract decimal numbers, we follow the same tactic we use for bundling and realize each decimal number as a whole number count of tenths, or hundredths, or so forth.

Problem: Add $3.14 + 1.432$.

If we want to add 3.14 and 1.432, we first "pad" the decimals to be the same length, 3.140 and 1.432. We first express the two numbers as 3140 thousandths and 1432 thousandths. Then we ignore the thousandths and add $3140 + 1432 = 4572$ (because it does not matter if we are adding thousandths or apples or cats). Then we bring back in the thousandths, 4572 thousandths is 4.572. We can accomplish the same thing by stacking the padded numbers, adding, and then copying the decimal straight down:

$$\begin{array}{rl} & 3.140 \\ + & 1.432 \\ \hline & 4.572 \end{array}$$

We can accomplish the same thing without adding the extra 0s for padding by simply lining up the decimal points. We just have to acknowledge that nothing plus 2 is 2 in the thousandths place.

$$\begin{array}{rl} & 3.14 \\ + & 1.432 \\ \hline & 4.572 \end{array}$$

We follow the same procedure for subtracting. Pad the numbers to make them the same length (or line up the decimals), stack, subtract, and bring the decimal down. Notice in this example that the extra 0 is not options because we have to borrow to subtract.

Problem: Subtract $3.14 - 1.432$.

$$\begin{array}{r} \overset{2}{\cancel{3}}.{}^{1}1\overset{3}{\cancel{4}}{}^{1}0 \\ -1.432 \\ \hline 1.708 \end{array}$$

Multiplying Decimal Numbers

To see how to multiply decimal numbers, we first convert an example to fractions and then multiply. We multiply 1.23×3.4:

$$1.23 \times 3.4 = \frac{123}{100} \times \frac{34}{10} = \frac{123 \times 34}{100 \times 10} = \frac{123 \times 34}{1000}.$$

To multiply 1.23×3.4, then, we need to first multiply 123×34:

$$\begin{array}{r} 1\overset{1}{2}3 \\ \times\ \ 34 \\ \hline {}^{1}492 \\ 3690 \\ \hline 4182 \end{array}$$

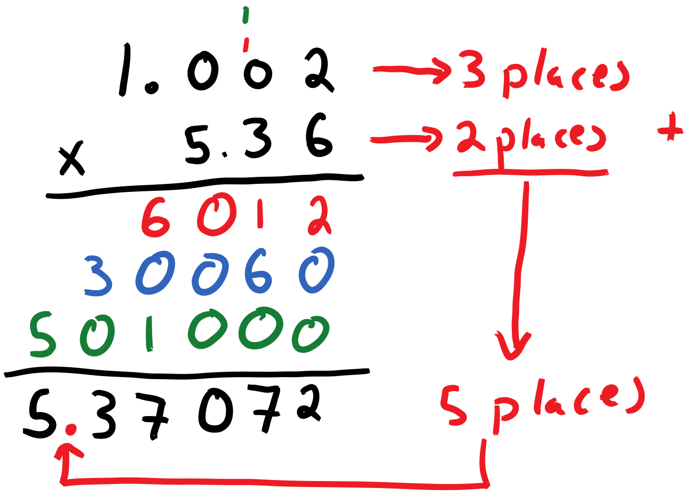
Therefore,

$$1.23 \times 3.4 = \frac{123 \times 34}{1000} = \frac{4182}{1000} = 4.182$$

Notice what we did here. We first multiplied the numbers together while ignoring the decimal points. Then, we divided by 1000. The reason we divided by 1000 is that the original two numbers were fractions over 100 and over 10. Dividing by 1000 is equivalent to dividing by 100 and then dividing by 10. Also, dividing by 1000 is equivalent to moving the decimal place three places left. These three places are the sum of the two places from 1.23 and the one place from 3.4.

This is our strategy for multiplying decimal numbers. First multiply the numbers while ignoring the decimals. Then count the total number of decimal places in the numbers being multiplied and move the decimal in the product that many places to the left.

Problem: Multiply 1.002×5.36.

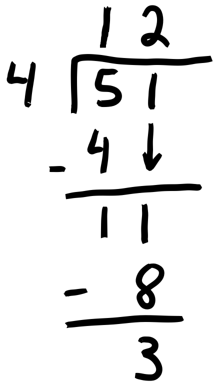


Dividing Decimal Numbers

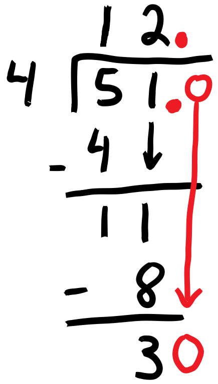
To introduce dividing decimal numbers, we first divide two whole numbers. We begin with a problem that does not look like it involves decimals.

Problem: Divide $7 \div 4$.

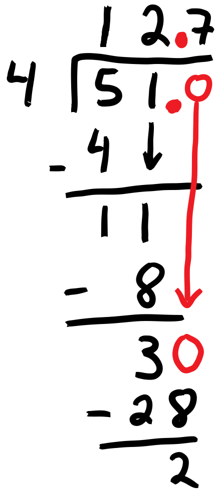
We start as we did before.



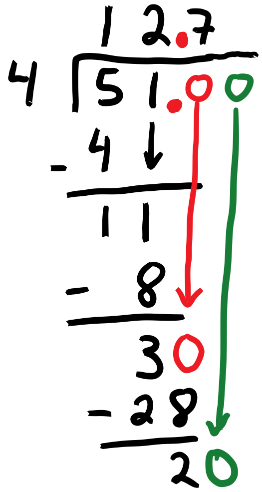
In the standard interpretation of division, we are trying to distribute 51 tallies among 4 boxes. So far, we have distributed all but 3 of the tallies. Prior to now, we would have stopped at this point and said that $51 \div 4 = 12R3$ or $51 \div 4 = 12\frac{3}{4}$. Notice that $\frac{3}{4} = \frac{75}{100}$ so $12\frac{3}{4} = 12\frac{75}{100} = 12.75$. We are going to arrive at the same decimal through the division algorithm by trying to distribute the remaining 3 tallies into boxes. We break each of these 3 remaining tallies into 10 pieces so that we have 30 tenths to distribute. In our division algorithm, this means that we add a 0 in the tenths place of 51 and bring it down next to the remainder of 3.



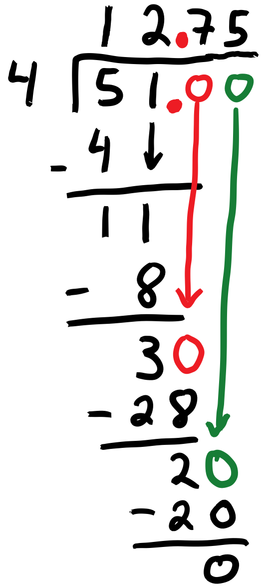
We are now dividing 30 tenths among 4 boxes. We can place 7 in each box with 2 left over. Notice that since we are distributing tenths here, we place that 7 to the right of a decimal in the quotient. The decimal in the quotient and dividend should line up.



To distribute the 2 remaining tenths among 4 boxes, we divide each into ten pieces, so that we have 20 hundredths. This amounts to adding another 0 to the dividend to bring down.



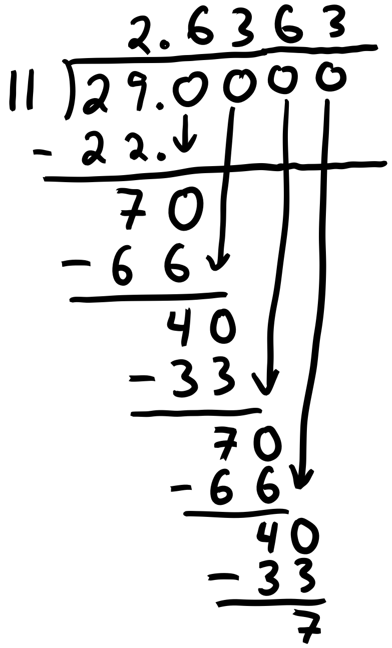
We now distribute these 20 hundredths among the 4 boxes by placing 5 in each box.



We now have that $51 \div 4 = 12.75$. This is consistent with our work before because $12.75 = 12\frac{75}{100} = 12\frac{3}{4}$. Note that dividing this way means that we may end up with a decimal quotient even when the divisor and dividend are whole numbers. For all practical purposes, to divide like this, we simply add a decimal to the dividend and add *as many 0s to the right of the decimal as needed*. We just need to be sure to keep the decimal in the dividend and quotient lined up.

Problem: Divide $29 \div 11$.

There is a problem in the statement that we add *as many 0s to the right of the decimal as needed.* The problem is determining how many decimals are needed. In the example below, we divide $29 \div 11$. We seem to get stuck in an infinite repeating pattern.



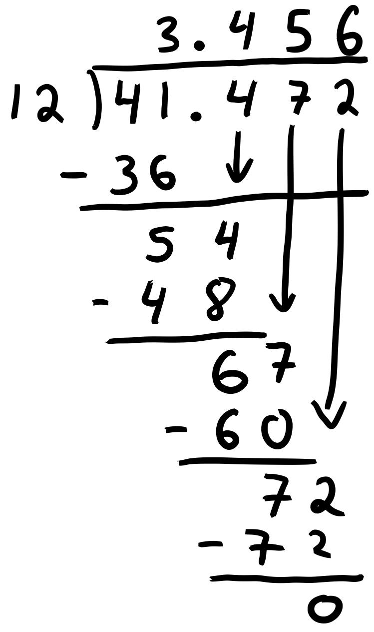
The digits in our quotient will keep alternating 6 and 3 no matter how far we divide. It happens to be that $29 \div 11$ can be written as a sum of infinitely many smaller and smaller fractions:

$$\frac{29}{11} = 2 + \frac{6}{10} + \frac{3}{100} + \frac{6}{1000} + \frac{3}{10000} + \frac{6}{100000} + \frac{3}{1000000} + \cdots$$

To indicate that a pattern in a decimal continues to repeat forever, we use a bar over the repeating digits: $29 \div 11 = 2.\overline{63}$. Some decimals repeat. Some terminate (have only a finite number of digits to the right of the decimal), and some extend forever without repeating or terminating. We will encounter this last type much later.

Problem: Divide $41.472 \div 12$.

We can just as easily divide a number involving a decimal by a whole number. Here we calculate $41.472 \div 12$:

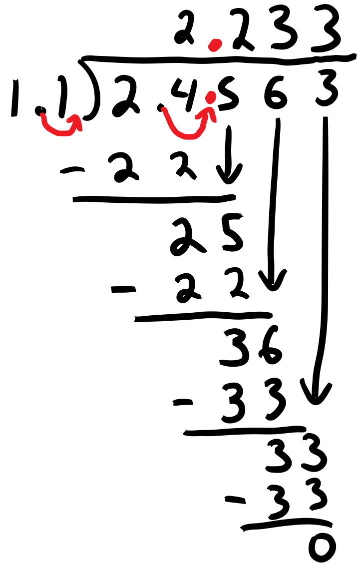


Problem: Divide $2.4563 \div 1.1$.

Finally, we are ready to address dividing a decimal number by a decimal number. We know how to divide by whole numbers. The secret is to change the problem so that the divisor (the number we are dividing by) is a whole number. Here, we will calculate $2.4563 \div 1.1$. First, notice that we can rewrite the problem this way:

$$2.4563 \div 1.1 = \frac{2.4563}{1.1} = \frac{2.4563 \times 10}{1.1 \times 10} = \frac{24.563}{11} = 24.563 \div 11$$

Thus, to divide $2.4563 \div 1.1$, we can simply move the decimal point in each number to the right until the divisor is a whole number and divide $24.563 \div 11$. Moving the decimal to the right is equivalent to multiplying the top and bottom of a fraction by 10. In the division algorithm, this looks like:



So $2.4563 \div 1.1 = 2.233$.

Converting between Fractions and Decimals

We will sometimes need to convert between fractions and decimals. To convert a fraction to a decimal, we have two options. The most common way to convert a fraction to a decimal is to use the division algorithm like we did in the last section. Sometimes, we can convert a fraction to have a denominator that is a power of ten, and then we can write it as a decimal easily. For example:

$$\frac{3}{5} = \frac{3 \times 2}{5 \times 2} = \frac{6}{10} = 0.6$$

Problem: Convert $\frac{7}{8}$ to a decimal by converting denominators.

Notice that 100 is not a multiple of 8, but 1000 is.

$$\frac{7}{8} = \frac{7 \times 125}{8 \times 125} = \frac{875}{1000} = 0.875.$$

To convert a decimal that terminates to a fraction, we use the same type of observation that we used for bundling diagrams. A number such as 0.1234 can be realized as 1234 ten thousandths or as the fraction $\frac{1234}{10000}$. Notice that the number of digits to the right of the decimal is the same as the number of 0s on the bottom of this fraction. Of course, this fraction can be reduced.

Problem: Convert 1.245 to a fraction. Write your answer as a mixed number with a completely reduce fractional part.

We first write $1.245 = 1 + 0.245$. Now we know the whole part of the mixed number and just need to convert the decimal to a fraction. Since $0.245 = \frac{245}{1000} = \frac{49}{200}$, we have that $1.245 = 1\,\frac{49}{200}$.

Rounding Decimals

We round decimals just like we rounded whole numbers. Find the digit in question. If the digit to the right is 5 or more, round up. Otherwise round down.

Problem: Round 1.246810 to the nearest thousandth.

The digit in the thousandths place here is 6 ($1.24\underline{6}810$). The number to the right of 6 is 8, which is larger than 5, so we round the 6 up to 7. The number 1.246810 rounds to 1.247.

Scientific Notation

Here are some of the place values that we use in base ten notation, now including some decimals.

1000	Thousand
100	Hundred
10	Ten
1	One
0.1	Tenth
0.01	Hundredth
0.001	Thousandth
0.0001	Ten Thousandth

If we move up this table one line, we multiply the place value by 10. This is evident in the fact that we can write the place values in the top half of the table as powers of 10.

$1000 = 10^3$	Thousand
$100 = 10^2$	Hundred
$10 = 10^1$	Ten
$1 = 10^0$	One
0.1	Tenth
0.01	Hundredth
0.001	Thousandth
0.0001	Ten Thousandth

As we move down the top half of the table, our exponent decreases by 1 each step. A logical progression would be to continue this into the decimal range and use negative exponents.

$1000 = 10^3$	Thousand
$100 = 10^2$	Hundred
$10 = 10^1$	Ten
$1 = 10^0$	One
$0.1 = 10^{-1}$	Tenth
$0.01 = 10^{-2}$	Hundredth
$0.001 = 10^{-3}$	Thousandth
$0.0001 = 10^{-4}$	Ten Thousandth

This motivates our definition of negative exponents. If A is any nonzero number and if B is a positive integer, then $A^{-B} = \frac{1}{A^B}$. With this notation, our usual rules of exponentiation such as $A^B \times A^C = A^{(B+C)}$ and $A^B \div A^C = A^{(B-C)}$ still hold as long as we avoid raising 0 to the 0th power.

Sometimes we encounter numbers that are so large or so small that they are hard to compare with each other and they are tedious to write, much less do arithmetic with. Such numbers show up frequently in science when we encounter distances between planets and starts, populations, or sizes of subatomic particles. Here are some examples:

Weight of Sun
4,383,749,000,000,000,000,000,000,000,000 pounds
Weight of Moon
161,950,000,000,000,000,000,000 pounds
Distance to Proxima Centauri
24,810,000,000,000

Population of Earth
7,550,000,000 people
Distance to Sun
94,510,000 miles
Distance to Moon
226,982 miles
Weight of Water Molecule
0.00000000000000000000000006614 pounds
Width of Water Molecule
0.0000000038 inches

We have special notation using exponents to write these numbers in a more compact way. A number is in scientific notation if it is written in the form

$$\#.\#\#\#\#\# \times 10^{\#}$$

In this notation, only one digit is allowed to the left of the decimal, and it must be nonzero. As many digits are you like are allowed to the right of the decimal. The numbers above in scientific notation are in the table below.

Weight of Sun	
4,383,749,000,000,000,000,000,000,000,000 pounds	4.383749×10^{30}
Weight of Moon	
161,950,000,000,000,000,000,000 pounds	1.6195×10^{23}
Distance to Proxima Centauri	
24,810,000,000,000	2.481×10^{13}
Population of Earth	
7,550,000,000 people	7.55×10^{9}
Distance to Sun	
94,510,000 miles	9.451×10^{7}
Distance to Moon	
226,982 miles	2.26982 × 10^5
Weight of Water Molecule	
0.00000000000000000000000006614 pounds	6.614×10^{-26}
Width of Water Molecule	
0.0000000038 inches	3.8×10^{-9}

Problem: Convert 12,340,000,000 to scientific notation.

To convert to scientific notation, we will move the decimal to the left until it is between the 1 and 2, counting our steps. We will then multiply the new number by 10 to the number of steps.

$$\underbrace{12{,}340{,}000{,}000}_{\text{10 steps}} = 1.234 \times 10^{10}$$

Problem: Convert 0.000001234 to scientific notation.

To convert this number to scientific notation, we will move the decimal to the right until it is between the 1 and 2, counting steps. We then multiply the new number by 10 to the negative of this number of steps.

$$0.\underbrace{000001}_{6\ \text{steps}}234 = 1.234 \times 10^{-6}$$

Problem: Convert 5.67×10^4 to decimal notation.

Here, we simply move the decimal 4 places to the right, inserting 0s as necessary.

$$5.67 \times 10^{4} = 5\underbrace{6700}_{4\ \text{steps}}.$$

Problem: Convert 5.67×10^{-7} to decimal notation.

Here, we simply move the decimal 7 places to the left, inserting 0s as necessary.

$$5.67 \times 10^{-7} = 0.\underbrace{0000005}_{7\ \text{steps}}67$$

Problem: Multiply $(4.23 \times 10^4) \times (5.34 \times 10^5)$. Write your answer in scientific notation.

To multiply numbers in scientific notation, we use the commutativity and associativity of multiplication.

$$\begin{aligned}(4.23 \times 10^4) \times (5.34 \times 10^5) &= (4.23 \times 5.34) \times (10^4 \times 10^5)\\ &= (4.23 \times 5.34) \times 10^{(4+5)}\\ &= (4.23 \times 5.34) \times 10^9\end{aligned}$$

Now,

$$\begin{array}{r} {}^{1}4.{}^{1}23 \\ \times\ 5.34 \\ \hline {}^{1}1{}^{1}692 \\ 12690 \\ 211500 \\ \hline 22.5882 \end{array}$$

Thus we have

$$\begin{aligned}(4.23 \times 10^4) \times (5.34 \times 10^5) &= (4.23 \times 5.34) \times 10^9 \\ &= 22.5882 \times 10^9\end{aligned}$$

It may look like we are done but we are not. Remember that in scientific notation we are only allowed one digit to the left of the decimal. Here we have 2. Therefore, we are going to move the decimal to the left one place (thereby dividing by 10) and then increase the exponent on 10 by 1 (thereby multiplying by 10 to undo the division).

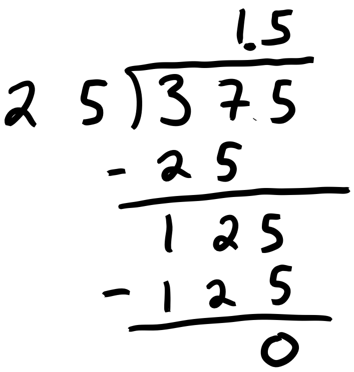
$$\begin{aligned}(4.23 \times 10^4) \times (5.34 \times 10^5) &= 22.5882 \times 10^9 \\ &= 2.25882 \times 10^{10}\end{aligned}$$

Problem: Divide $(3.75 \times 10^4) \div (2.5 \times 10^7)$. Write your answer in scientific notation.

We will again use arithmetic properties here. However, division is not associative, and we have division and multiplication mixed here. If we are sly and replace division notation with fraction notation, then our path is clearer.

$$\begin{aligned}(3.75 \times 10^4) \div (2.5 \times 10^7) &= \frac{3.75 \times 10^4}{2.5 \times 10^7} \\ &= \frac{3.75}{2.5} \times \frac{10^4}{10^7} \\ &= (3.75 \div 2.5) \times 10^{(4-7)} \\ &= (3.75 \div 2.5) \times 10^{-3} \\ &= (37.5 \div 25) \times 10^{-3}\end{aligned}$$

Now,



So

$$
\begin{aligned}
(3.75 \times 10^4) \div (2.5 \times 10^7) &= (37.5 \div 25) \times 10^{-3} \\
&= 1.5 \times 10^{-3}
\end{aligned}
$$

Problem: Add $(1.23 \times 10^5) + (3.45 \times 10^6)$. Write your answer in scientific notation.

To add numbers in scientific notation, we need to make sure the powers on 10 are the same in both numbers. Here, they are not. If we move the decimal in 1.23×10^5 left one place (dividing by 10) and increase the exponent by 1 (multiplying by 10), then we see that $1.23 \times 10^5 = 0.123 \times 10^6$. Now

$$
\begin{aligned}
(1.23 \times 10^5) + (3.45 \times 10^6) &= (0.123 \times 10^6) + (3.45 \times 10^6) \\
&= (0.123 + 3.45) \times 10^6 \\
&= 3.573 \times 10^6
\end{aligned}
$$

Since this last expression follows the guidelines for scientific notation, we are now done.

Percents

Comparing fractions and performing arithmetic with fractions can be difficult if the fractions have different denominators. It would be convenient to convert every fraction to have the same denominator so that comparisons and arithmetic would be easier. A natural choice for such a universal denominator might be something like 100. The notion of percent is an attempt to perform fraction arithmetic with denominators all equal to 100. To begin our work with percents, we make these definitions.

- "P **percent**" or $P\%$ means $\frac{P}{100}$.
- To say that "A is $P\%$ of B" means (literally) that $A = \frac{P}{100} \times B$.
    - This is equivalent to $\frac{A}{B} = \frac{P}{100}$ and to $P = \frac{A}{B} \times 100$.
- To **convert a fraction** $\frac{A}{B}$ **to a percent** is to find P so that $\frac{A}{B} = \frac{P}{100}$.

Problems involving percents can often be solved a variety of ways, including equivalent fractions, division and multiplication, ratio tables, and algebra. We will solve problems in this section without algebra.

Problem: Convert the fraction $\frac{7}{20}$ to a percent.

We will solve this problem two ways. First, we use equivalent fractions. Since $20 \times 5 = 100$, we can multiply the top and bottom of $\frac{7}{20}$ by 5 to convert to a fraction with denominator 100:

$$\frac{7}{20} = \frac{7 \times 5}{20 \times 5} = \frac{35}{100}$$

Therefore, $\frac{7}{20}$ is 35%.

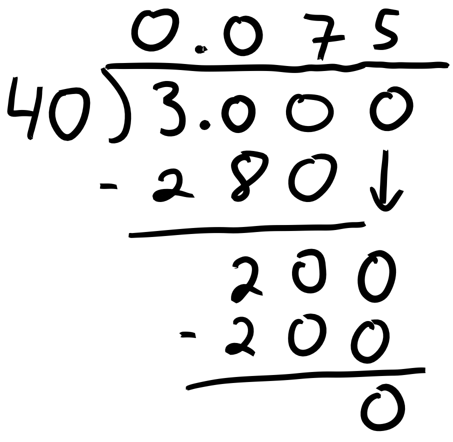
Next, we are going to solve this problem by dividing. We know that $\frac{7}{20} = P\%$ is equivalent to $P = \frac{7}{20} \times 100$. Therefore, we are going to divide to find the decimal equivalent of $\frac{7}{20}$, and then we will multiply by 100. First, we divide.

$$\begin{array}{r} 0.35 \\ 20\,\overline{)\,7.00} \\ -6\;0\downarrow \\ \hline 1\;0\;0 \\ -1\;0\;0 \\ \hline 0 \end{array}$$

Since $\frac{7}{20} = 0.35$, and since $0.35 \times 100 = 35$, then $\frac{7}{20}$ is 35%.

Problem: Convert the fraction $\frac{3}{40}$ to a percent.

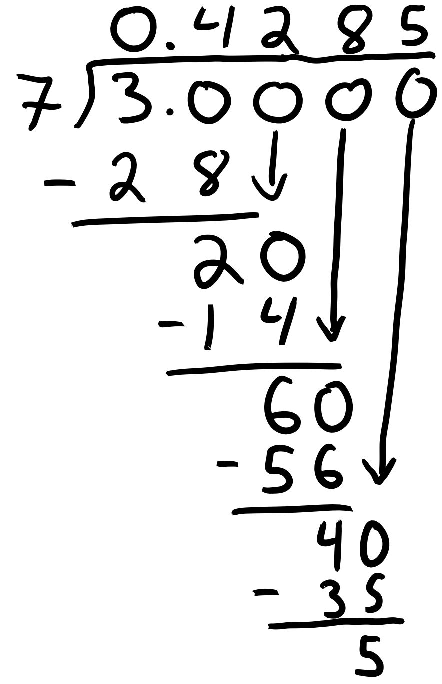
Since 100 is not a multiple of 40, we cannot use the equivalent fraction approach here. Instead, we go straight to dividing.



Since $\frac{3}{40} = 0.075$, and since $0.075 \times 100 = 7.5$, then $\frac{3}{40}$ is 7.5%.

Problem: Convert $\frac{3}{7}$ to a percent. Round to the nearest tenth of a percent.

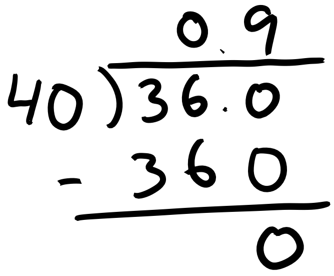
Again, 100 is not a multiple of 7, so we have to divide. The challenge this time is that when we divide by 7, the division process (the decimal) will not terminate. This is the reason for the rounding directions. We want to round the final percent to one decimal place. When we divide, the first two decimal places are the whole percent. The next (the third) is the first decimal place of the percent, so we need one more in order to round. Therefore, we will divide until we have at least 4 decimal places.



Now we know that $\frac{3}{7} = 0.4285\ldots$ (the ellipses indicate the digits we did not compute). If we multiply by 100, $\frac{3}{7} = 42.85\ldots\%$. Rounding to one decimal place gives $\frac{3}{7} \approx 42.9\%$.

Problem: What percent of 40 is 36?

To answer this question, we want to find a number P so that $36 = \frac{P}{100} \times 40$ or so that $\frac{36}{40} = \frac{P}{36}$. This just means we need to convert $\frac{36}{40}$ to a percent. We will do this three ways (because it is so much fun). First, we divide.



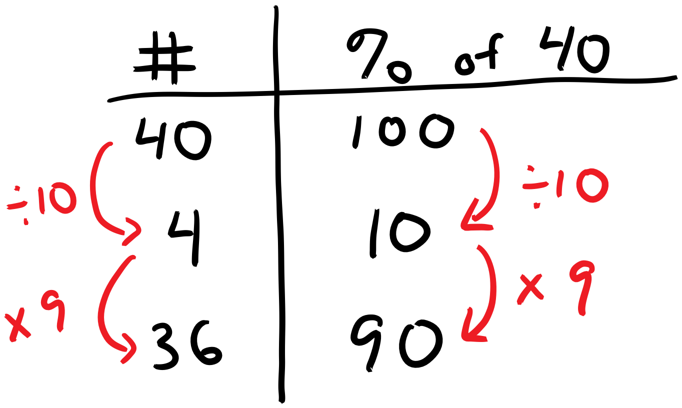
Since, $\frac{36}{40} = 0.9$, multiplying by 100 gives that $\frac{36}{40} = 90\%$. That is, 36 is 90% of 40.

Next, we will convert the fraction to have a denominator of 100. Recall above that we said we could not do this with $\frac{3}{40}$ because 100 is not a multiple of 40. We lied.

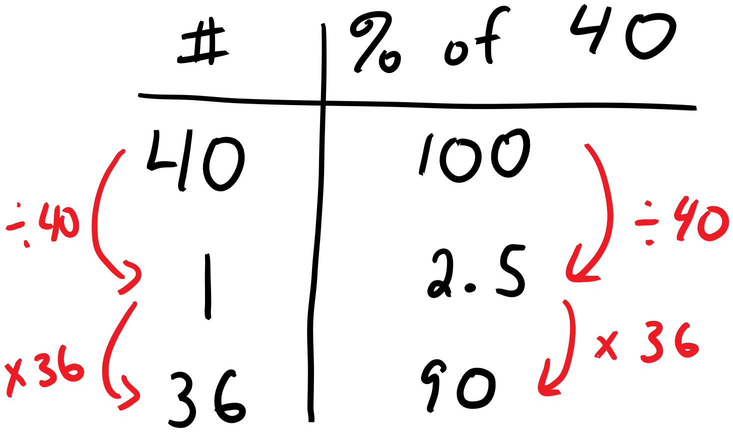
$$\frac{36}{40} = \frac{36 \div 2}{40 \div 2} = \frac{18}{20} = \frac{18 \times 5}{20 \times 5} = \frac{90}{100}$$

So 36 is 90% of 40. Note that if we had tried this same process with $\frac{3}{40}$ we would have encountered a decimal on top of our fraction. That is fine, although it may make you cringe a little.

Finally, we are going to use a ratio table. The secret here is that there is a proportional relationship between all positive numbers and the percents they are of 40. We will make a ratio table with columns for numbers and percents. We know that 40 is 100% of 40 so we start with a row relating them, and we manipulate the number column until we see 36 in it. This is slightly tricky, we need to find a path from 40 to 36 which involves multiplying and dividing. Since 40 and 36 are both multiples of 4, we first divide to get a 4 in the number column. Then we multiply to get a 36 in that column.



Again, we see that 36 corresponds to 90% of 40. There is another approach to the table we can use if we did not see the common factor of 4. We can divide by 40 to get a 1 in the number column. Then multiplying to get 36 is easy. The arithmetic for this approach is slightly harder, but the logic is easier in some sense. This approach is called going through 1.



#	% of 40
40	100
1	2.5
36	90

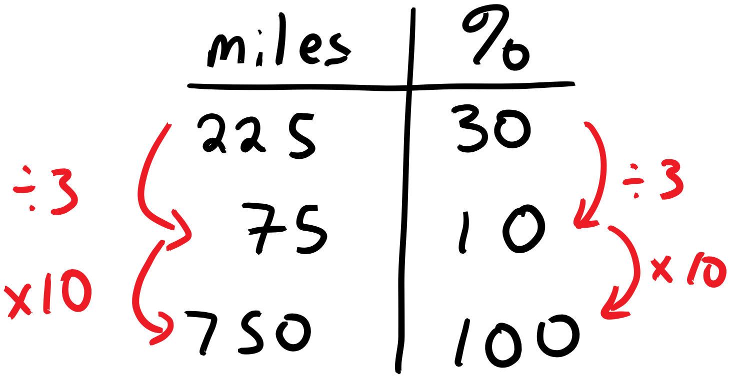
Notice that in the second column we calculated

$$100 \div 40 \times 36 = \frac{100}{40} \times 36 = \frac{100 \times 36}{40} = 100 \times \frac{36}{40}$$

So this is equivalent to our first approach at the problem.

Problem: Sue is on a road trip. She has driven 225 miles so far, which is 30% of the entire trip. How long will the entire trip be?

This is a standard percent problem in an algebra class, but we want to solve it without algebra. We are going to use a ratio table. One column will be distance in miles. The other column will be percent of the entire trip. We know that 225 miles corresponds to 30%, so that gives our first column. To find the length of the entire trip, we want to manipulate the second column to be 100%. This would be simple if 100 were a multiple of 30, but it is not. We need a path from 30% to 100% by multiplying and dividing. Since 30% and 100% are both multiples of 10%, we will go through 10%.

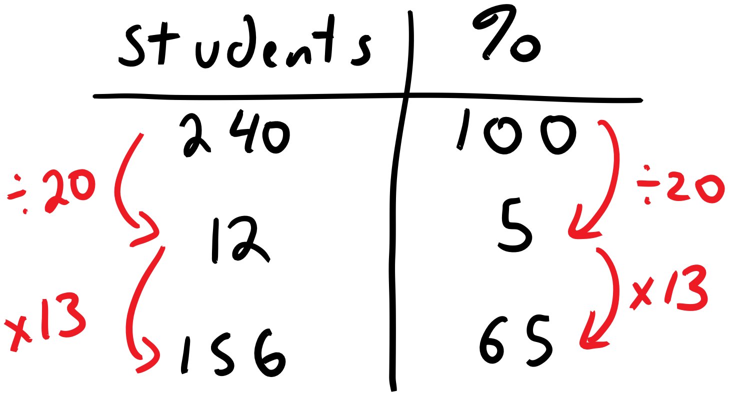


miles	%
225	30
75	10
750	100

The trip is 750 miles.

Problem: There are 240 students at a certain school. One day, 65% of them brought lunch to school. How many brought lunch?

We will solve this problem two ways. First, we will use a ratio table. We have two columns, one for the number of students, and one for percent. We know that 240 students is 100% of the students, so that gives us the first row of the table. We then need to manipulate the second column to be 65%. We go through 5% since 240, 100, and 65 are all multiples of 5.



Students	%
240	100
12	5
156	65

There were 156 students who brought their lunches.

We now work this problem again (in a much quicker way). We want to know what 65% of 240 is. Remember here that "of" means multiply. All we have to do is multiply

$$65\%\ of\ 240 = 0.65 \times 240 = 156$$

Again, we see that 156 students brought their lunch.

Percent Increase and Percent Decrease

Percents are often use when describing how much a population (of people, animals, plants, bacteria, money, anything) increases or decreases. Suppose that a population increases by 1326. Is that a large increase? If the original population was 7.3×10^9, then this is not much of a change. If the original population was 1272, then this was a significant change. To determine the significance of a change in a population, we often report the change as a percentage of the original population.

Problem: The population of a small town was 6780. The population increased by 35%. What was the new population?

To solve this problem, we are going to find 35% of 6780 and then add this to 6780. First

$$35\%\ of\ 6780 = 0.35 \times 6780 = 2373$$

Therefore, the new population is $6780 + 2373 = 9153$. Notice that we could have done this arithmetic all at once

$$6780 + 0.35 \times 6780$$

Problem: The population of a small town was 8375. After a new plant moved in, the population grew to 9213. What was the percent increase in the population? Round your answer to the nearest percent as necessary.

First, we calculate how much the population increased: $9213 - 8375 = 838$. Now we just want to know what percent 838 is of 8375. To do so, we divide: $838 \div 8375 = 0.1000597 \ldots$ Multiplying by 100 and rounding gives that the percent increase was about 10%.

Notice in this problem that we used the initial population to calculate the percent increase. Also notice that to find the percent of 8375 we divided by 8375.

Problem: The value of a piece of land in 2010 was $\$73{,}000$. In 2015, the value was $\$61{,}000$. What was the percent decrease in the value of the land? Round to the nearest percent as necessary.

We will first calculate how much the value decreased, and then we will calculate the percent of the original value. The amount of decrease was $73000 - 61000 = 12000$. We need to know what percent 12000 is of 73000, so we divide: $12000 \div 73000 = 0.16438 \dots$ Multipying by 100 and rounding gives a decrease of about 16%.

Problem: The population of a small town was 6000 in 2010. From 2010 to 2015, the population decreased by 11%. What was the population in 2015?

We will find 11% of 6000 and then subtract this from 6000. First, 11% of 6000 is

$$11\%\ of\ 6000 = 0.11 \times 6000 = 660$$

The new population was $6000 - 660 = 5340$. Notice that we could have done this arithmetic all at once

$$6000 - 0.11 \times 6000$$

Problem: The population of a small town was 6000. When a new factory moved in, the population increased by 12%. The population did not change for a while until the factory shut down. Then the population decreased by 12%. How does the new population compare with the original population?

We will calculate the new population and compare it to the original. First we will calculate the population after the initial increase of 12%:

$$6000 + 0.12 \times 6000 = 6720$$

After the factory moved in, the population was 6720. Now we will decrease the population by 12%:

$$6720 - 0.12 \times 6720 = 5913.6$$

Since we cannot have 0.6 people, we round the new population to 5914. Notice that after increasing and then decreasing by 12%, the new population is actually less than the original.

Exponents and Percent Increase and Decrease

The process we followed above to find a population after a percent increase or decrease works well for only one or two changes. If the population changes many times, the process can be tedious.

Problem: The population of a small town was 6000. Every year for ten years, the population increased by 3%. What was the new population? Round to the nearest person.

We could follow the steps we did above ten times, but that would be tedious. We want a better way. We can rewrite one increase to see a better way:

$$Population\ after\ one\ 3\%\ increase = 6000 + 0.03 \times 6000 = 6000 \times (1 + 0.03) = 6000 \times 1.03$$

Notice that to increase the population by 3% we simply multiply by 1.03. To increase the population by 3% ten times, we multiply by 1.03 ten times – or once by 1.03^{10}. Now we can calculate the new population quickly:

$$6000 \times 1.03^{10} = 8063.498 \dots$$

Since we are talking about whole people here, we round to 8064. After ten years of 3% increases, the population was 8064.

Problem: Property values in Bell County, Texas, are currently increasing at a rate of 8% per year. A particular piece of property is valued at $\$100{,}000$. If this growth rate continues for 20 years, how much will the property be worth? Round to the nearest dollar.

Following the pattern from the previous problem, to increase by 8%, we simply multiply by 1.08. To do this twenty times, we multiply by 1.08^{20}.

$$100{,}000 \times 1.08^{20} = 466095.7144$$

Rounding to the nearest dollar gives a value of $\$466{,}096$.

Expressions

In this section, we will make some of the ideas we have been using in relation to mathematical symbols more forma. A **numerical expression** is a *meaningful* string of numbers, parentheses, and operation symbols. Here are some examples of numerical expressions:

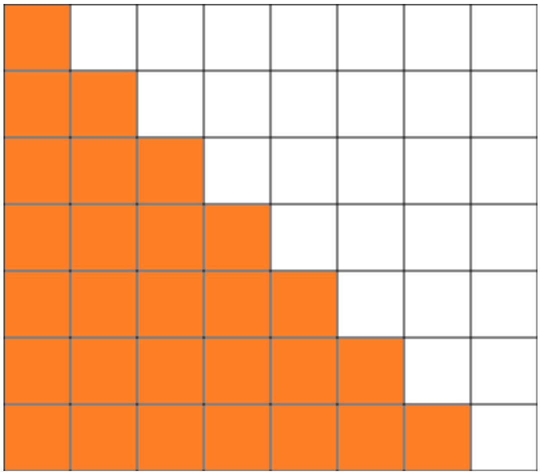
$$1+1$$
$$2\times(7+8)^{(2+4\times(9-3))}-7$$
$$2\div 3\times(4\div 9)+3\ \times(3\times 3)$$

The word meaningful here is tricky. A proper definition of meaningful would require more time than we have. It roughly means that the parentheses used correctly indicate a grouping for order of operations and that the operation symbols have the right number of arguments. These would not be meaningful strings of numbers, parentheses, and operation symbols:

$$)+(-2\ \times\div)3$$
$$2)+4)$$

To **evaluate** a numerical expression is to perform the indicated operations on the numbers in the expression. The final number calculated is the **value** of the expression.

Problem: Write a numerical expression for the number of small orange squares in this figure.



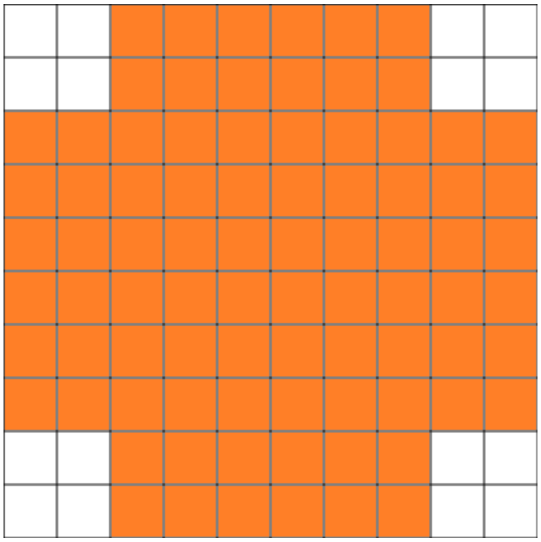
The orange squares are arranged in rows with 1 in the first row, 2 in the second, 3 in the third, all the way to 7 squares in the seventh row. We can simply add the number of orange squares in each row to get

$$1+2+3+4+5+6+7.$$

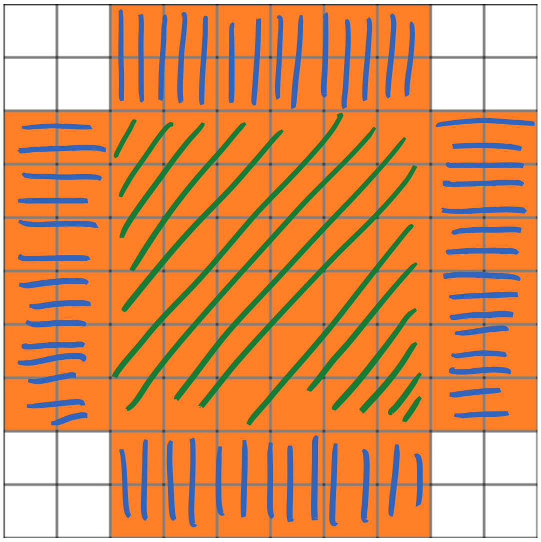
Another approach would be to note that all of the orange squares are in an array of squares with 7 rows and 8 columns. This array has 7×8 squares. Half of them are orange, so the number of orange squares is

$$\frac{1}{2}\times(7\times 8).$$

Problem: Write a numerical expression for the number of small orange squares in this figure.



One approach to this problem is to break the orange squares into smaller groups that are easier to count. Here is one way of doing that:



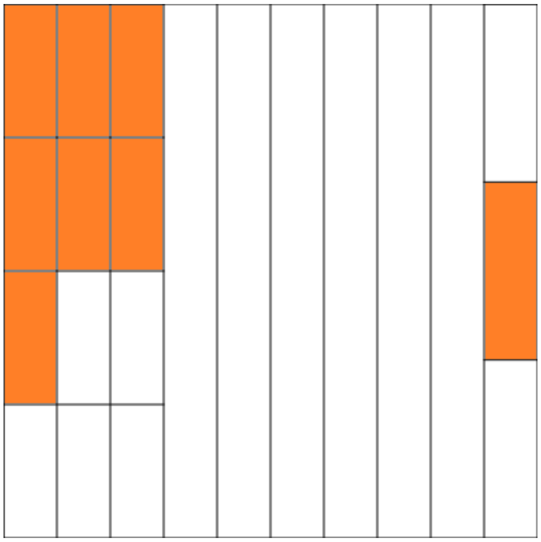
In this grouping, we have 4 groups shaded blue, each with 2×6 squares. The total number of blue squares is $4 \times 2 \times 6$. We also have 6×6 squares shaded green. The total number of orange squares is

$$4 \times 2 \times 6 + 6 \times 6.$$

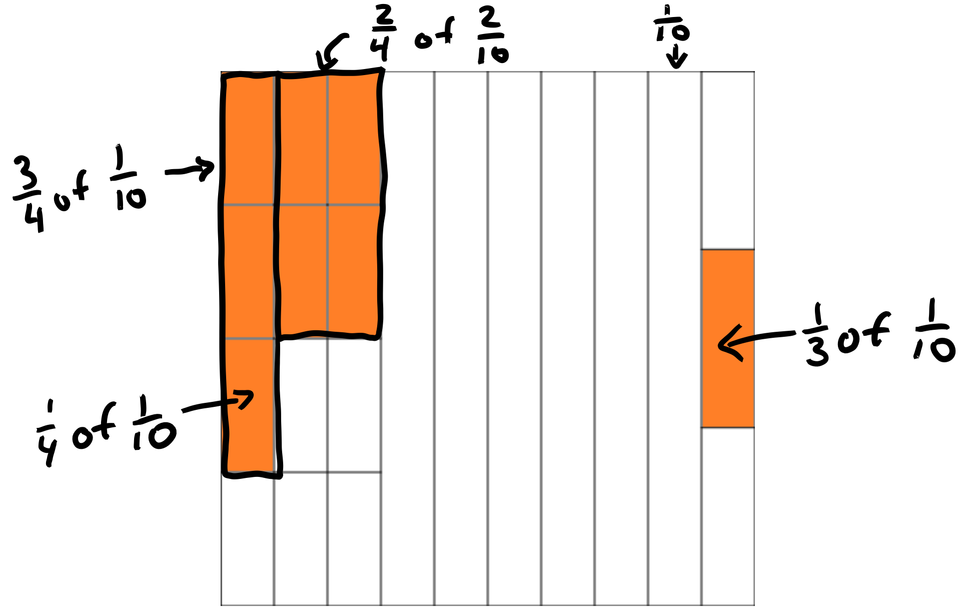
Another approach to this problem is to consider the total number of squares (orange or white) in the figure and subtract off the number of white squares. There are 10×10 total squares. There are 4 groups of white squares, each containing 2×2 squares. The number of orange squares is

$$10 \times 10 - 4 \times 2 \times 2.$$

Problem: Write a numerical expression for the fraction of this square which is shaded orange.



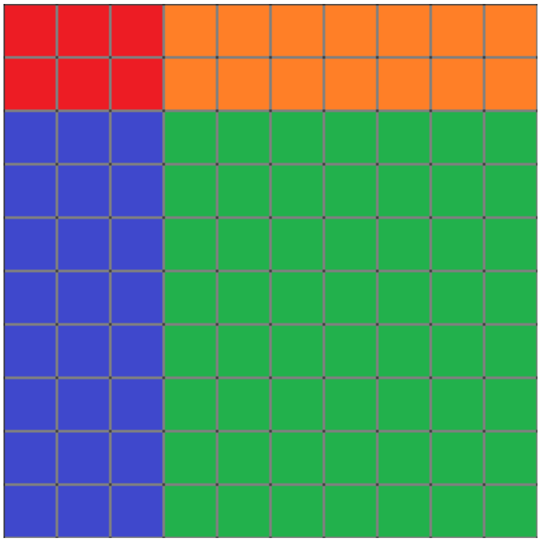
First, ignore the shading and the horizontal lines. The long vertical lines divide the shape into 10 equal size parts, so each long vertical part is $\frac{1}{10}$ of the figure. On the far right, horizontal lines divide one of these parts into three equal size parts, one of which is shaded orange. This part is $\frac{1}{3}$ of one of the parts which is $\frac{1}{10}$. Therefore, this orange part is $\frac{1}{3} \times \frac{1}{10}$ of the whole. Now, on the far left, horizontal lines divide the tenths into quarters. Three of the quarters making the first tenth are shaded orange, so this tall orange strip is $\frac{3}{4}$ of $\frac{1}{10}$ or $\frac{3}{4} \times \frac{1}{10}$. The next block of orange is $\frac{2}{4}$ of $\frac{2}{10}$ or $\frac{2}{4} \times \frac{2}{10}$.



The total shaded area now is

$$\frac{3}{4} \times \frac{1}{10} + \frac{2}{4} \times \frac{2}{10} + \frac{1}{3} \times \frac{1}{10}.$$

Problem: Write a numerical expression for the total number of small squares in this diagram.

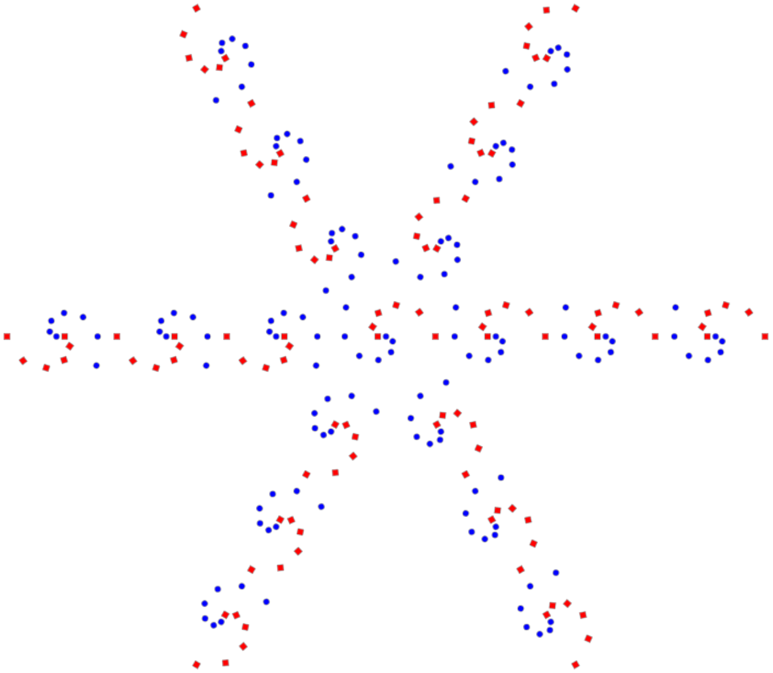


We could focus on the colors. There are 2×3 red squares. There are 2×7 orange squares. There are 8×3 blue squares, and there are 8×7 green squares. This gives a total of

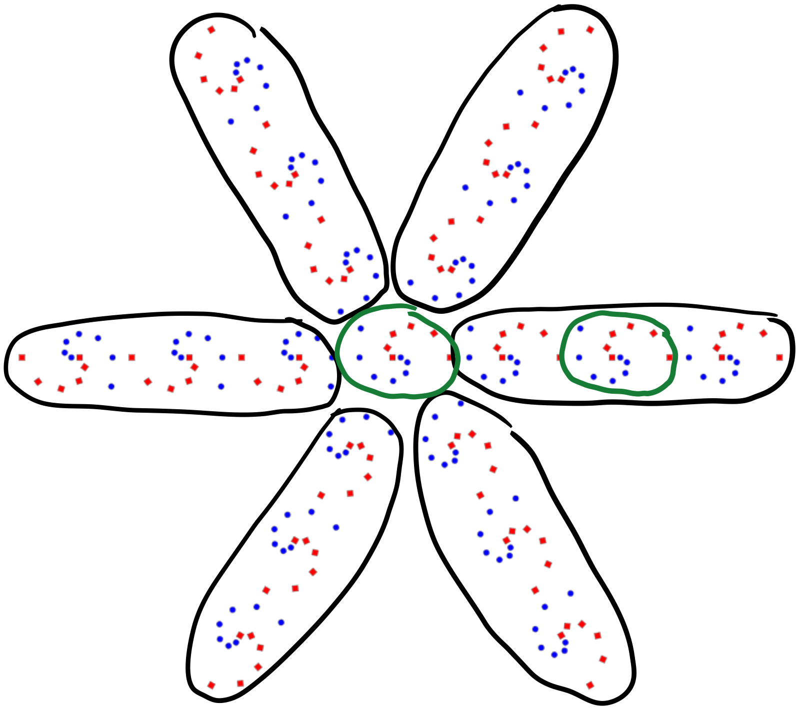
$$2 \times 3 + 2 \times 7 + 8 \times 3 + 8 \times 7.$$

Alternatively, we might notice that there are 10 rows of 10 squares for a total of 10×10 squares.

Problem: Write an expression for the number of red squares in this figure.



There are 6 arms in this figure circled below in black. In each arm, there are 3 spiral patterns, a couple of which are circled in green below. In addition, there is an extra spiral pattern in the middle of the figure. Each spiral pattern contains 6 red squares.



Therefore, there are a total of

$$6 \times 3 \times 6 + 6$$

red squares.

Variables and Notation for Multiplication

A **variable** is a symbol such as a letter or a box that is used to represent a number. We will be using variables for much of the rest of our work. Variables are frequently used to represent unknown numbers for example, students early on see problems such as, "What number goes in the box if $2 + \square = 5$?" Here, the box is a variable. The bars we have been using to solve problems with bar models can be thought of as variables. Usually, we will use letters for variables, and the most common letter we will use as a variable is x. This means that we might sometimes treat x like a number and combine it with other symbols using arithmetic operations such as $x + 2$ or $3 \times (x \div 4)$. This is rather unfortunate since our symbol for multiplication × looks a lot like an x. For this reason, we now adopt new notation for multiplication. We will use a single dot for multiplication. For example, for 3×2 we will now write $3 \cdot 2$. When variables are involved, we will frequently use juxtaposition to indicate multiplication. Rather than $3 \times x$ or $3 \cdot x$ we will simply write $3x$. An expression such as xyz will mean a product $x \cdot y \cdot z$. We will also use juxtaposition when a number is multiplied times an expression in parentheses. For example, instead of $3 \times (2 + x)$ or $3 \cdot (2 + x)$ we will often write $3(2 + x)$. We should be careful with this convention. Although it is legal, you should never write $3 \cdot 2$ as $3(2)$. This is technically legal, but using notation like this will make students more likely to make certain errors in pre-calculus and calculus.

Algebraic Expressions

An **algebraic expression** is a meaningful string of numbers, *variables*, parentheses, and operation symbols. What distinguishes this definition from the definition of numerical expression is the use of variables. Notice that every numerical expression is also an algebraic expression. Here are some algebraic expressions:

$$x^2 + 3x + 4$$

$$\pi r^2 h$$

$$\frac{xy - yx}{x + y}$$

Substitution

To **substitute** a value for a variable in an expression is to replace every occurrence of that variable in the expression with that value. For example, if we substitute 2 for x in the expression $x^2 + 3x + 4$, then we arrive at $2^2 + 3 \cdot 2 + 4 = 14$.

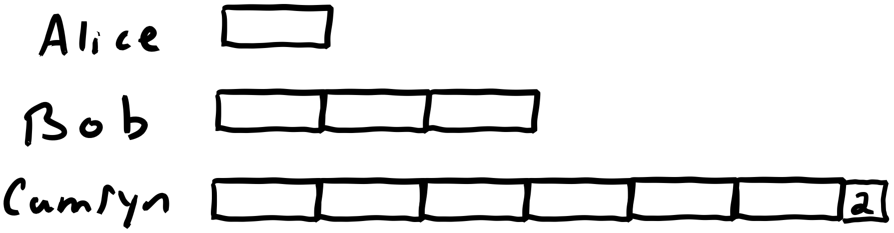
Equivalent Expressions

Two expressions are equivalent if their values are the same no matter what numbers are substituted for the variables in the expressions. For example, $x + x$ and $2x$ are equivalent. Two expressions are equivalent if and only if one of them can be changed using our properties of arithmetic to look like the other. The problems below are all of the form, “Write an expression for…” There are many different ways to work each of these problems. Any two legal ways to work one of these will result in equivalent expressions.

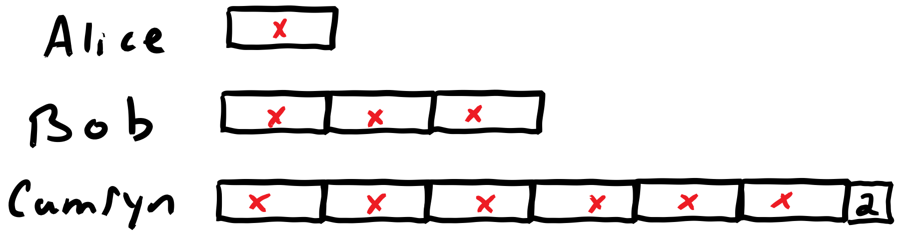
Problems

Problem: Bob has three times as many toy cars as Alice. Camryn has 2 more than twice as many toy cars as Bob. Let x be the number of cars that Alice has and write an expression for the total number of toy cars the three have together.

We will build an expression by first drawing a bar model of the situation. We draw a bar for the number of cars that Alice has. The bar for the number of Bob’s cars is three times as long as Alice’s bar. The bar for Camryn’s number of cars is twice as long as Bob’s bar plus an additional 2 cars.



Since we were directed to let x be the number of cars that Alice has, we place an x in each of the bars that are the same size as Alice’s bar.



We can now see from the diagram that there are a total of 10 bars of size x and 2 more for a total of $10x + 2$ cars.

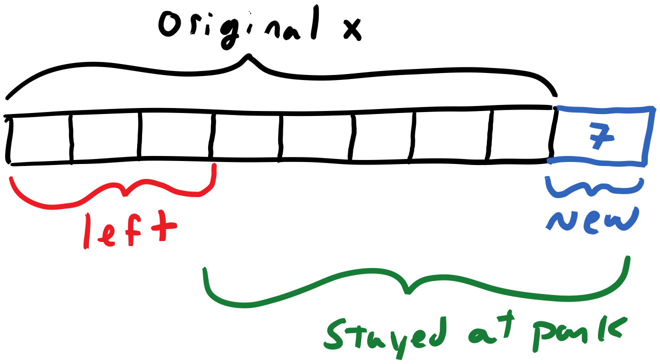
If we did not want to draw the bar model here, we could approach the problem this way. If Alice has x cars, then, since Bob has three times as many cars as Alice, then Bob has $3x$ cars. Since Camryn has 2 more than twice as many cars as Bob, Camryn has $2 + 2 \cdot 3x$ cars. Together, they have

$$x + 3x + 2 + 2 \cdot 3x = 10x + 2$$

cars.

Problem: There were x children in the park. First, $\frac{3}{8}$ of the children left. Then 7 more children arrived. Write an expression for the number of children in the park at that time.

We begin by drawing a bar model off of which we will be able to read our answer. We draw a bar for the original number x of children in the park. We divide this bar into 8 parts since the question refers to $\frac{3}{8}$ of these children. We mark off $\frac{3}{8}$ of the boxes as leaving and then add 7 more children to the bar.

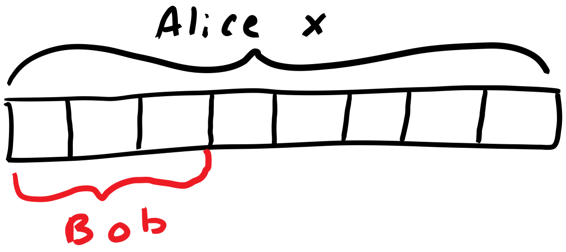


Notice that we do not know how the box with 7 new children should compare with the other 8 boxes. At this point we have five of the original 8 boxes and 7 more children. Since each of the original boxes was $\frac{1}{8}$ of x, that gives $5 \cdot \left(\frac{1}{8} \cdot x\right) + 7 = \frac{5}{8}x + 7$ children.

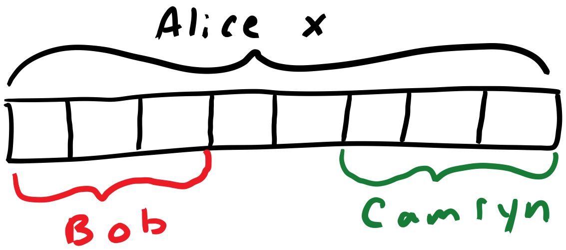
We can approach this problem without the bar model. At first, there were x children in the park. Then $\frac{3}{8}$ of the children left. If $\frac{3}{8}$ of the children left, then $\frac{5}{8}$ of the children (or $\frac{5}{8}x$) stayed. When 7 more arrived, the number of children was up to $\frac{5}{8}x + 7$.

Problem: Alice made some cookies. She gave $\frac{3}{8}$ of her cookies to Bob and $\frac{3}{5}$ of the remainder to Camryn. Camryn gave $\frac{1}{2}$ of his cookies to Doug. Let x be the number of cookies that Alice made and write an expression for the number of cookies that were given to Doug.

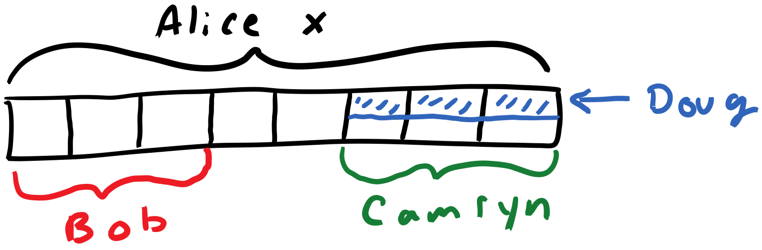
We again start with a bar model. We draw a bar for the number x of cookies that Alice made. We divide this bar into 8 equal parts since she gave Bob $\frac{3}{8}$ of the cookies.



Now Alice gives $\frac{3}{5}$ of the remaining boxes to Camryn. Luckily, Alice has 5 boxes left over, so this means she gives 3 boxes to Camryn.



Camryn now gives half of his boxes to Doug, so we divide Camryn's boxes in half.

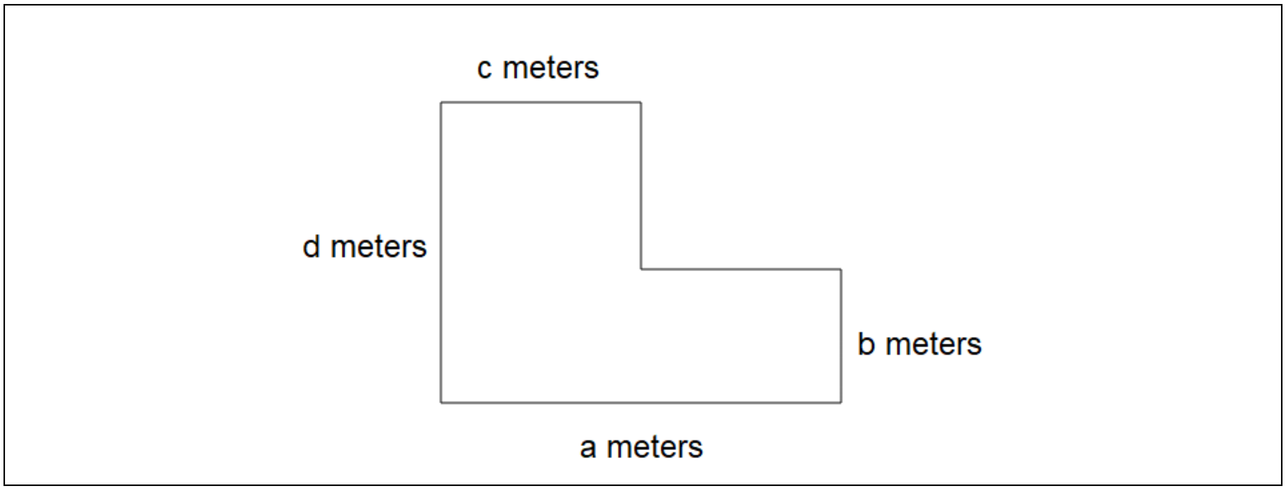


Doug now gets three small blue boxes of cookies. Each small blue box is $\frac{1}{2}$ of one of the larger black boxes. Each black box is $\frac{1}{8}$ of the total number x of cookies that Alice made. Therefore, Doug receives $3 \cdot \frac{1}{2} \cdot \frac{1}{8} \cdot x = \frac{3}{16}x$ cookies.

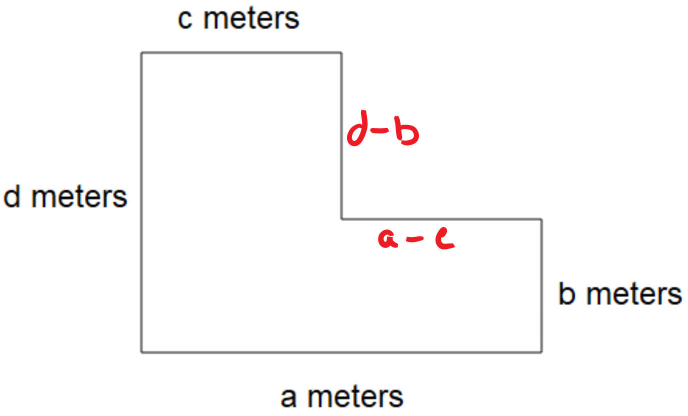
Here is an approach without a bar model. Alice made x cookies. She gave Bob $\frac{3}{8}x$ cookies, leaving her with $x - \frac{3}{8}x = \frac{5}{8}x$ cookies. Now she gave Camryn $\frac{3}{5}$ of these, so Alice gave Camryn $\frac{3}{5} \cdot \frac{5}{8}x = \frac{3}{8}x$ cookies. Now Camryn gave half of these, or $\frac{1}{2} \cdot \frac{3}{8}x = \frac{3}{16}x$ cookies, to Doug.

The next two examples refer to perimeter and area. We have not talked about these topics, but what we need to know is not much. The perimeter of a shape is the distance around the shape. To find it, we can add up the lengths of all of the sides of the shape. As for area, the figures below are composed of rectangles, we need to know how to find the area of a rectangle. The area of a rectangle is length times width. We also need to know that the opposite sides of a rectangle have the same length.

Problem: Write an expression for the perimeter of this figure.



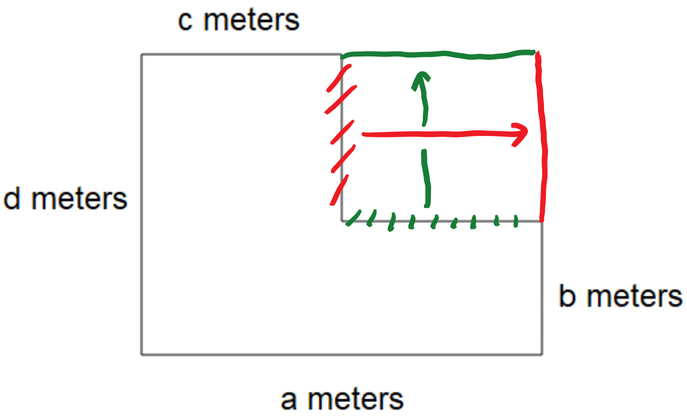
We will do this twice. For perimeter, we simply need to add the lengths of all of the sides of the shape. Unfortunately, there are two sides which are not labeled. We need to find them first. The distance all the way across the figure horizontally is $a\ meters$. The top side of the figure is $c\ meters$. If we subtract these, we get the length of the unlabeled horizontal piece, $a - c\ mters$. Similarly, the unlabeled vertical piece is $d - b\ meters$.



Now we can find the perimeter by adding

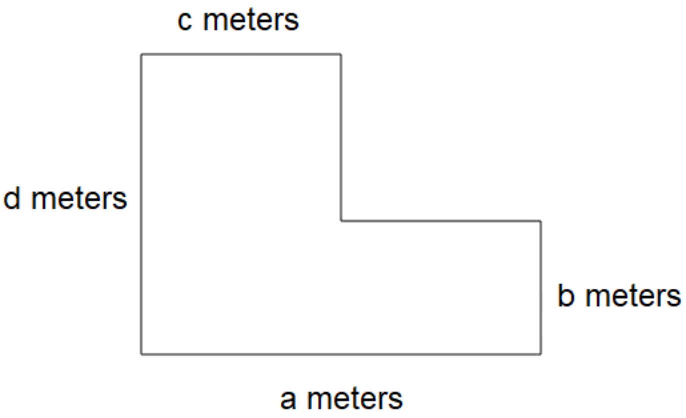
$$\begin{aligned} a + b + (a - c) + (d - b) + c + d &= a + (a - c) + c + d + (d - b) + b \\ &= (a + a) + (-c + c) + (d + d) + (-b + b) \\ &= 2a + 2d \end{aligned}$$

Here is another approach. We can move the edges marked red and green below to new positions to make a rectangle. Since the opposite sides of a rectangle have the same length, the top of this rectangle has a length of $a\ meters$ and the right side has a length of $d\ meters$.

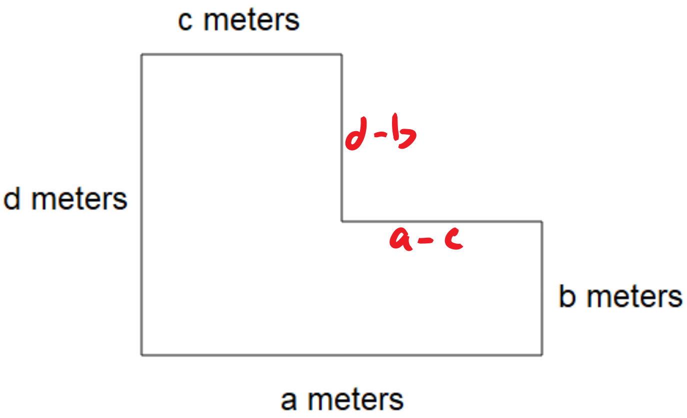


The total perimeter is $a + a + d + d = 2a + 2d\ meters$.

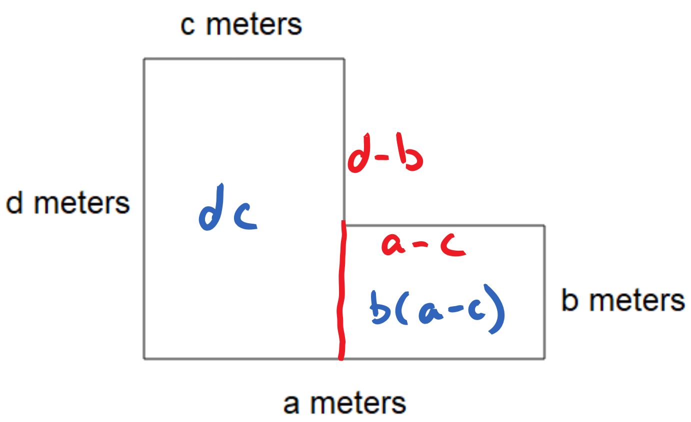
Problem: Write an expression for the area of this figure.



We will solve this problem twice, once adding areas and once subtracting. First, we are going to need to know the lengths of the two unlabeled sides that we found above.

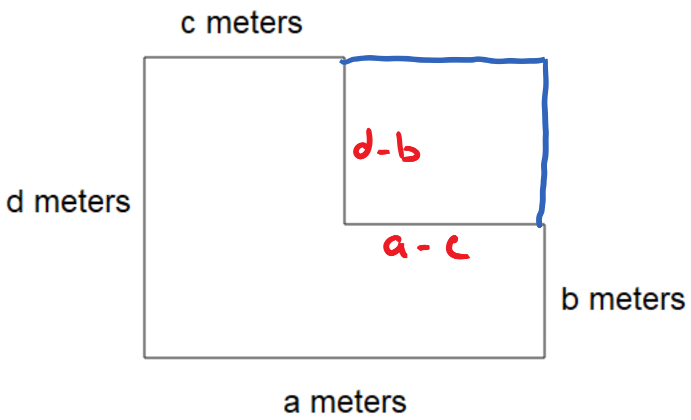


Next, we will divide the original shape into two rectangles.



The area of the left side is dc square meters. The area of the right side is $b(a-c)$ square meters for a total area of $dc + b(a-c)$ square meters.

We now approach this problem with subtraction. We can imagine that this shape was constructed by cutting a small rectangle out of the corner of a larger rectangle like we have indicated in the picture below.



The area of the large rectangle is ad square meters. The area of the small rectangle is $(a-c)(d-b)$ square meters. The area of the left over area after we cut out the small rectangle is

$$ad-(a-c)(d-b)$$

Equations

An **equation** is a statement that two expressions are equal. For example,

$$1 + 1 = 2$$
$$A = \pi r^2$$
$$x^2 + y^2 = 1$$

Equations have many uses. We can use equations to

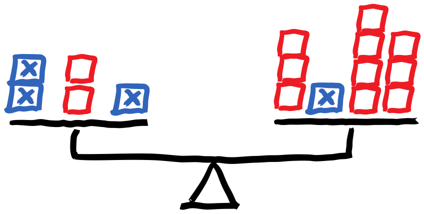
- *Show Calculations*: We have used equal signs repeated throughout these notes to show the steps in a computation. For example,
$$\begin{aligned} 2 + 3 \cdot (7 + 8)^2 &= 2 + 3 \cdot 15^2 \\ &= 2 + 3 \cdot 225 \\ &= 2 + 675 \\ &= 677 \end{aligned}$$
Notice that this is really an abbreviation of four different, but related, equations.
$$\begin{aligned} 2 + 3 \cdot (7 + 8)^2 &= 2 + 3 \cdot 15^2 \\ 2 + 3 \cdot 15^2 &= 2 + 3 \cdot 225 \\ 2 + 3 \cdot 225 &= 2 + 675 \\ 2 + 675 &= 677 \end{aligned}$$

- *State Identities*: An identity is an equation that is true no matter what values are substituted for the variables involved. We have seen several examples of identities. One example is the distributive property of multiplication.
$$A \cdot (B + C) = A \cdot B + A \cdot C$$
- *Describe Relationships between Variables*: We might use an equation to describe how variables in a problem are related. For example, if A is the area of a circle and if R is the radius of a circle, then A and R are related by the equation
$$A = \pi R^2$$
An equation such as this where one side is a single variable and the other is an expression involving only other variables is often called a **formula**. Formulas tell us how to calculate certain values. For example, this formula tells us how to calculate the area of a circle given its radius. Another example of an equation that describes a relationship between variables is
$$x^2 + y^2 = 1$$
This equation gives the relationship between x and y when the point (x, y) is on the unit circle.
- *Solve Problems*: If an equation involves variables, then a **solution** to that equation is a set of values for those variables which, when substituted into the equation, make the equation true. For example, the values $x = 3$ and $y = 4$ give a solution to the equation $x^2 + y^2 = 25$ because $3^2 + 4^2 = 25$ is true. Finding solutions to an equation is called **solving** the equation. One of the ways in which mathematics has been most influential on society is through the problem solving tools that mathematics has to offer. One of the most powerful problem solving tools we get from mathematics is the ability to translate real world problems into equations whose solutions provide solutions to the real world problems.

Solving Equations

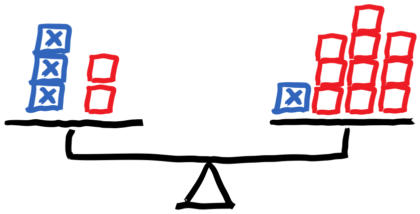
To motivate how we solve equations, it is helpful first to imagine the equation as a scale balance where the expressions on each side of the balance "weigh" the same amount. Here is an equation and a representation of the equation as a scale balance. On the scale, each blue block with an x represents x, and each red block represents a 1.

$$2x+2+x = 3+x+7$$



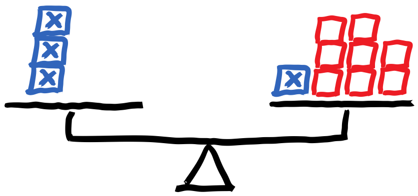
The objective is to figure out how much the blue blocks should weigh to make the scale balance. We do so by moving blocks around on the scale and by taking blocks off or putting blocks on the scale until one pan of the scale holds only x and the other pan holds only red blocks. Whatever we do to the scale, if we put blocks on or take blocks off, we have to do the same thing to both sides to keep the scale in balance. Our first step might be to clean up both sides of the equation by replacing the expressions with simpler expressions. On the left side of the equation, we can add the $2x$ and the x. On the right side of the equation, we can add the 3 and 7. We call this **collecting like terms**. On the scale, this amounts to rearranging each pan on the balance to get the blue blocks next to each other and to get the red blocks next to each other.

$$\begin{array}{c} 2x+2+x = 3+x+7 \\ \downarrow \\ 3x+2 = x+10 \end{array}$$



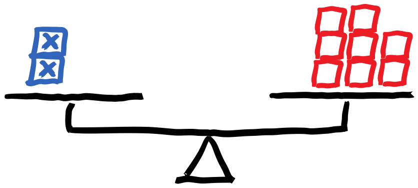
Now we will try to remove blocks until we have just one x on the left pan and just red blocks on the right. First, we remove 2 red blocks from both pans. Algebraically, this means that we subtract 2 from both sides of the equation.

$$\begin{array}{rcl} 3x+2 & = & x+10 \\ -2 & & -2 \\ \hline 3x & = & x+8 \end{array}$$



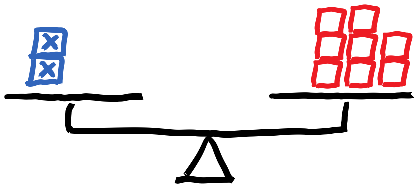
We do not want any xs on the right pan, so we next remove one x from both pans. In the equation, this means that we subtract x from both sides of the equation.

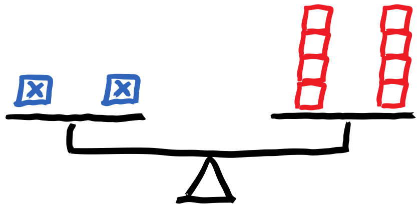
$$\begin{array}{rcl} 3x & = & x + 8 \\ -x & & -x \\ \hline 2x & = & 8 \end{array}$$

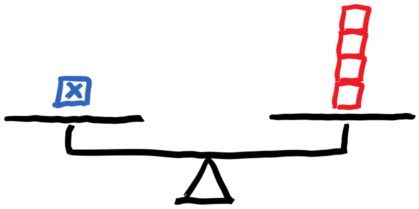


We now have 2 blue blocks on the left pan. We divide those blue blocks into two equal piles. Then we divide the red blocks into two equal piles. Finally, we remove half of the blocks from each pan. This will keep the scale balanced. On the equation side, this means that we divide both sides of the equation by 2.

$$\begin{array}{rcl} \frac{2x}{2} & = & \frac{8}{2} \\ \hline x & = & 4 \end{array}$$







We now have the solution $x = 4$. To make sure we did not make a mistake, we should check our answer. Remember that the original equation was $2x + 2 + x = 3 + x + 7$. If we substitute $x = 4$ in the left side of this equation we get $2 \cdot 4 + 2 + 4 = 14$. If we substitute $x = 4$ into the right side of the equation we get $3 + 4 + 7 = 14$. Thus, if $x = 4$, then the equation $2x + 2 + x = 3 + x + 7$ is true.

Problem: Solve the equation $\frac{2x+3}{2x+4} = \frac{x-3}{x}$.

This will likely be the worst equation we have to solve in this class.

$$\frac{2x+3}{2x+4} = \frac{x-3}{x}$$

Large fractions like this can make equations hard. We want to eliminate them. To begin with we multiply both sides of the equation by x.

$$x \cdot \frac{2x+3}{2x+4} = \frac{x-3}{x} \cdot x$$

On the right side of the equation, the denominator cancels.

$$x \cdot \frac{2x+3}{2x+4} = \frac{x-3}{\cancel{x}} \cdot \cancel{x}$$

We are down to one fraction.

$$\frac{x(2x+3)}{2x+4} = x-3$$

Now we multiply both sides of the equation by the other denominator $(2x+4)$.

$$(2x+4) \cdot \frac{x(2x+3)}{2x+4} = (x-3) \cdot (2x+4)$$

This cancels the denominator on the left, $2x+4$.

$$\cancel{(2x+4)} \cdot \frac{x(2x+3)}{\cancel{2x+4}} = (x-3) \cdot (2x+4)$$

This eliminates the other fraction, and we are down to an equation with no fractions.

$$x(2x+3) = (x-3) \cdot (2x+4)$$

Now we have some multiplying to do. We distribute the x on the right. Notice that $x \cdot 2x = 2 \cdot x \cdot x =$ $2x$^2 and $x \cdot 3 = 3x$.

$$2x^2 + 3x = (x-3) \cdot (2x+4)$$

Now we begin distributing on the right. First, we distribute $(x-3)$.

$$2x^2 + 3x = (x-3) \cdot 2x + (x-3) \cdot 4$$

Next, we distribute the $2x$ and the 4. Notice that $-3 \cdot 2x = (-3 \cdot 2)x = -6x$.

$$2x^2 + 3x = 2x^2 - 6x + 4x - 12$$

In the middle of the right had side, we have $-6x+4x=(-6+4)x=-2x$.

$$2x^2+3x=2x^2-2x-12$$

We now begin the process of trying to move every x to the left and everything else to the right. We start with $2x^2$ because it is the scariest thing here. We subtract $2x^2$ from both sides of the equation to eliminate it from the right.

$$\begin{array}{rcl} 2x^2+3x & = & 2x^2-2x-12 \\ -2x^2 \quad & & -2x^2 \qquad \\ \hline 3x & = & -2x-12 \end{array}$$

Luckily for us, all of the terms with x^2 magically vanish. Now we want to eliminate the $-2x$ from the right. We do so by adding $2x$ to both sides of the equation since $-2x+2x=0$.

$$\begin{array}{rcl} 3x & = & -2x-12 \\ +2x & & +2x \qquad \\ \hline 5x & = & -12 \end{array}$$

To isolate x, we now just divide by 5.

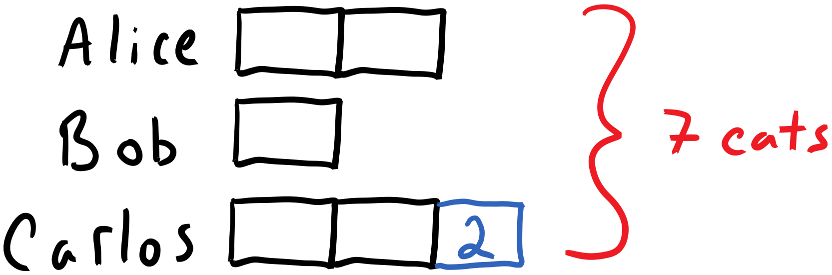
$$\frac{5x}{5} = -\frac{12}{5}$$

We now have a solution of $x=-\frac{5}{12}$.

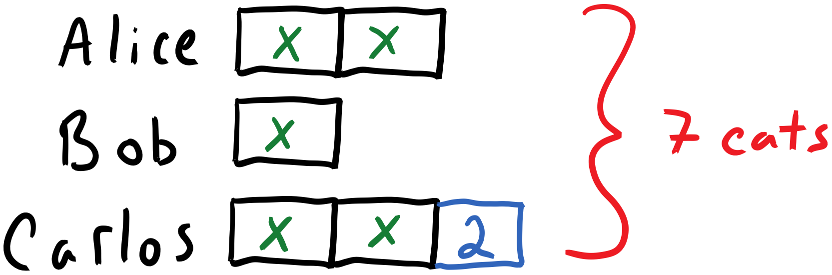
$$x=-\frac{12}{5}$$

Problem: Alice has twice as many cats as Bob. Carlos has 2 more cats than Alice. Together, the three have 7 cats. How many cats does Alice have?

We will draw a bar model for this question. Then we will select a variable and turn the bar model into an equation to solve. First, since Alice has twice as many cats as Bob, we draw a bar for the number of Bob's cats and draw a bar twice as long for the number of Alice's cats. Then Carlos's bar should be the same as Alice's with an additional two cats added on. Notice that we do not know how the box for the 2 additional cats relates in size to the original boxes. We then indicate that all of the bars add up to 7 cats.



When we look at this bar model, we see that every bar is made up mostly of bars the same size as Bob's bar. Therefore, we let this size be x. If we let x be the number cats that Bob has, then we can fill in an x in every box which is the same size as Bob's bar.



Now we know that all of the bars add to 7 cats. There are 5 boxes containing x cats and 1 box with 2 cats. Together this gives $5x + 2$. This must be equal to 7, so our equation is

$$5x + 2 = 7$$

Subtracting 2 gives

$$5x = 5$$

Dividing by 5 now gives

$$x = 1.$$

This means that Bob has $x = 1$ cat. Now we need to be sure to answer the question. The question asks how many cats Alice has. Since Alice has twice as many cats as Bob, and since Bob has 1 cat, Alice has 2 cats.

Proportional and Inversely Proportional Relationships

Using equations, we can now more formally define what it means for two quantities to be in a proportional or inversely proportional relationship. Two quantities A and B are **proportionally related**, or A is proportional to B if there is a number k so that $A = kB$. When proportional relationships occur in nature, the quantities A and B are usually positive, so k is also. For this reason, many books insist that k be positive. From a theoretical point of view. This is not necessary. On the other hand, a quantity A is **inversely proportional** to a quantity B if there is a number k so that $AB = k$.

Sequences

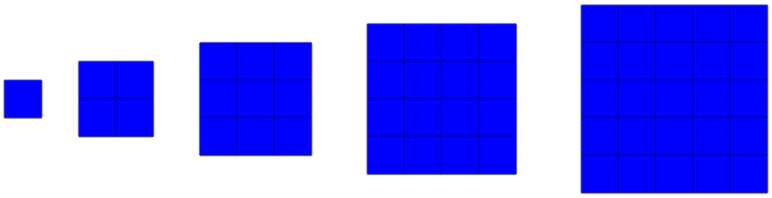
A **sequence** is an infinite ordered list of objects. Here are some examples of sequences, in each of these examples, there is a pattern which is assumed to continue.

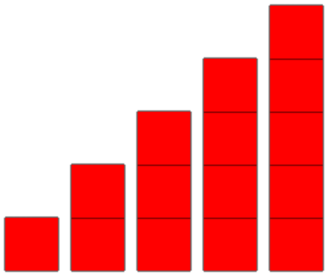
$$3, 7, 11, 15, 19, 23, 27, \ldots$$

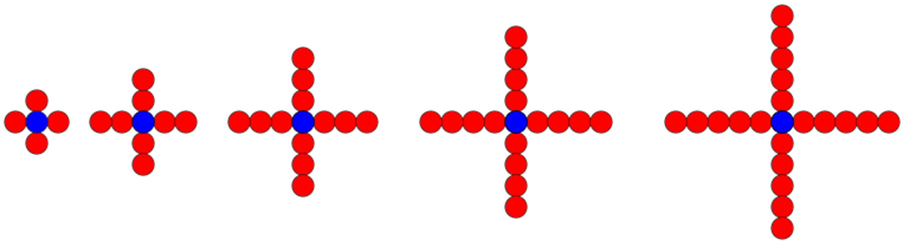
$$3, 6, 12, 24, 48, 96, 192, \ldots$$

A, A, A, A, A, A, A, A, A, A,…

A, B, A, B, B, A, B, B, B, A, B, B, B, B,…







Each object in a sequence is called a term of the **sequence**. The terms of a sequence are numbered beginning with 1. In this sequence

$$A, B, C, D, E, F, G, A, B, C, D, E, F, G, \ldots$$

The first term is A. The second term is B. The third is C, and so on.

Problem: Assume that the pattern in this sequence continues forever. What are the next 5 terms?

A, B, A, B, B, A, B, B, B, A, B,…

This sequence contains blocks of Bs that seem to be growing. One B, two Bs, three Bs, and so on. The sequence seems to be cut off at the beginning of a block of four Bs. The sequence should continue this way

A, B, A, B, B, A, B, B, B, A, B, B, B, B, A, B, B, B, B, B, A…

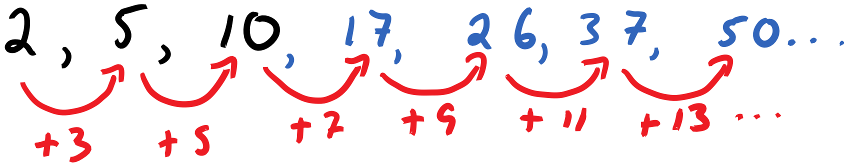
Problem: Give consider the beginning of the sequence below. Give 2 different ways of continuing the sequence.

$$2, 5, 10, \ldots$$

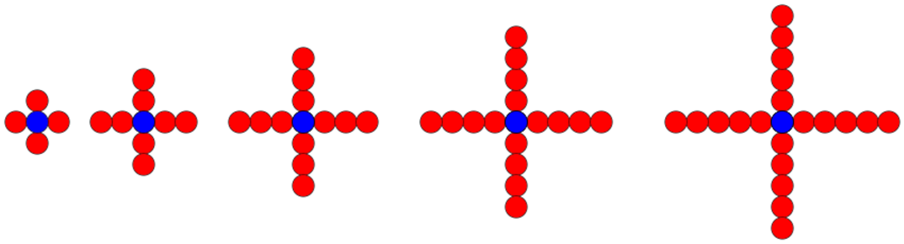
The point to this exercise is that questions such as this have no good answer. As long as we can defend our perceived pattern, we are correct. We might notice that 2 is one more than 1^2, that 5 is one more than 2^2, and that 10 is one more than 3^2. The N^{th} term of this sequence seems to be $N^2 + 1$. The next few terms are

$$2, 5, 10, 17, 26, 37, 50 \ldots$$

Here is another approach. From the first term to the second, the sequence increases by $5 - 2 = 3$. From the second term to the third, the sequence increase by $10 - 5 = 5$. Perhaps the sequence keeps increasing by odd numbers 7, 9, 11, 13… If so, the next few terms are



Problem: Consider the sequence of shapes below. Give at least three different sequences of numbers based on this sequence.



We could look at the sequence whose terms are the number of dots in each figure. This sequence would be

$$5, 9, 13, 17, 21, 25, \ldots$$

We could also look at the sequence whose terms are the number of red dots in each figure. This sequence is

$$4, 8, 12, 16, 20, 24, \ldots$$

We could look at the sequence whose terms are the number of red dots in one “arm” of the figure. This is

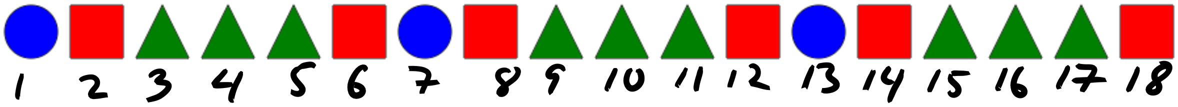
$$1, 2, 3, 4, 5, 6, 7, \ldots$$

It is much less interesting, but we could even consider the sequence whose terms are the number of blue dots in each figure.

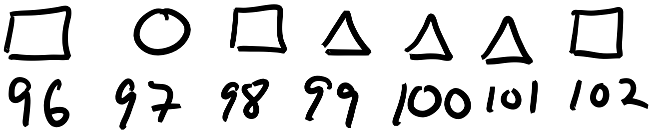
$$1, 1, 1, 1, 1, 1, 1, 1, \ldots$$

Problem: Consider the sequence of shapes below. What is the 100th term in the sequence?

First, we number the terms of the sequence to get some idea of how things are repeating.



Notice that there are 6 shapes that are repeating, that over every multiple of 6 is a square, and that this square ends one copy of the repeated pattern. If we find the largest multiple of 6 less than 100, then we know the shape above it is a square, and we start a new copy of the pattern at the next term. To find the largest multiple of 6 less than 100, we first divide $100 \div 6 = 16R4$. This means that the pattern repeats 16 full times, finishing a full pattern at $16 \times 6 = 96$. At term 97, the pattern starts over again.



If we start the pattern over at 97, we can see that the 100th term is a triangle.

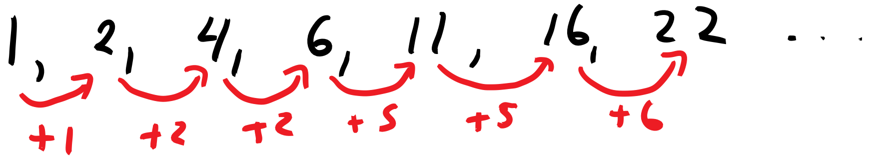
Problem: Consider the sequence of shapes below. How many triangles are in the first 100 terms of the sequence?

In the last exercise, we saw that in the first 100 terms of this sequence, the pattern of six shapes repeats 16 full times, and then there are 4 more shapes. In each repetition of the six-shape patter, there are three triangles. Therefore, there are $16 \times 3 = 48$ triangles in the first 96 terms. In the next 4 terms, there are 2 more triangles, for a total of 50 triangles in the first 100 terms.

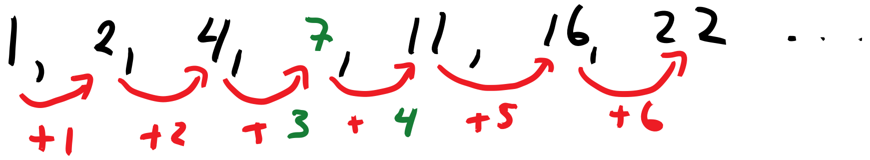
Problem: There is an error in the sequence below. Find and fix it.

$$1, 2, 4, 6, 11, 16, 22, \ldots$$

What error we perceive here depends on what pattern we think we see. With so few terms, there may be many possible patterns. We begin by looking at differences between adjacent terms.

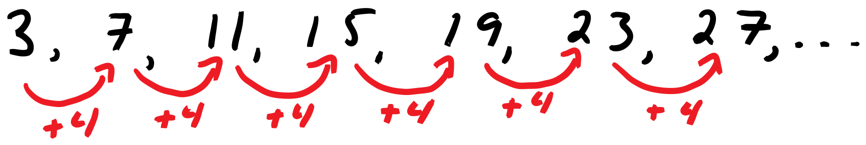


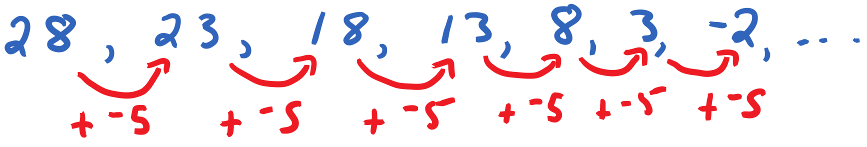
The differences start increasing 1 and then 2, and they finish increasing at the end 5 and then 6. This pattern is not consistent in the middle. If we change the fourth term from 6 to 7, the differences will increase steadily.



Arithmetic Sequences

In some of the previous examples, we found it useful to look at the difference between consecutive terms in a sequence. An **arithmetic sequence** is one in which the differences between each term and the term before it are all equal. For example, these are arithmetic sequences.





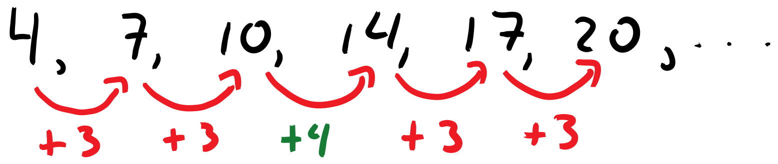
Note, $28 + {}^{-}5 = 28 - 5$

In the first example, we add 4 between terms. In the second, we add -5 between terms. Notice, that we can also think of this as subtracting 5. In an arithmetic sequence, we either add the same amount or subtract the same amount to get from one term to the next.

Problem: Explain why this is not an arithmetic sequence:

$$4, 7, 10, 14, 17, 20, 24, 27, 30 \ldots$$

Consider the differences between consecutive terms.

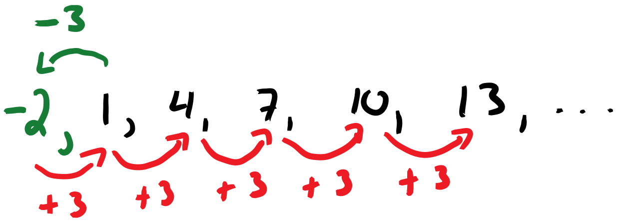


Between the first and second term, the difference is 3. Between the third and fourth term, the difference is 4. Since these differences are not the same, this is not an arithmetic sequence.

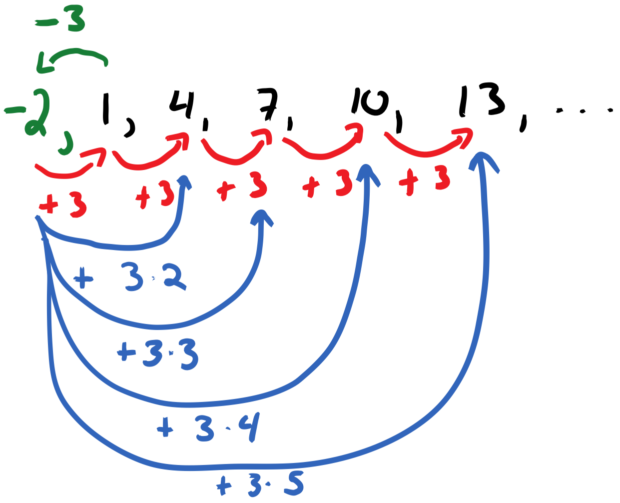
Problem: Find an expression for the N^{th} term of this arithmetic sequence.

$$1, 4, 7, 10, 13, 16, 19, 22, 25, \ldots$$

Notice that the common difference between the terms here is 3. Since we add 3 repeatedly to make terms of this sequence, a good first guess at an expression for the N^{th} term is $3N$. However, if we calculate $3 \cdot 1, 3 \cdot 2, 3 \cdot 3, 3 \cdot 4, \ldots$ we get $3, 6, 9, 12, 15, \ldots$ which is not correct. The problem is that we need to be mindful of what we are adding 3 to. To answer this question, we have to imagine a 0^{th} term that came before the first term. This 0^{th} term is something we add 3 to in order to get 1. The number we add 3 to in order to get 1 is $1 - 3 = -2$.



To get to the first term, we start at -2 and add 3. To get to the second term, we start at -2 and add 3 twice, or we add $3 \cdot 2$. To get to the third term, we start at -2 and add 3 three times, or we add $3 \cdot 3$. To get to the fourth term, we add $3 \cdot 4$ to -2.



This pattern continues. To get to the N^{th} term, we start at -2 and add 3 a total of N times, or we add $3N$. That is

$$N^{th} term = -2 + 3N$$

Here, we backed up one step from the first term to find the 0^{th} term. The N^{th} term is then given by

$$N^{th}\ term = (0^{th}\ term) + (common\ difference)N$$

Problem: Find an expression for the N^{th} term of this arithmetic sequence.

$$7, 2, -3, -8, -13, -18, -23, \ldots$$

We follow the process we derived in the last example. First, note that the common difference is -5. That is, we can either think that we are adding -5 at each step or that we are subtracting 5. To find the 0^{th} term, we subtract the common difference from the first term:

$$0^{th}\ term = 7 - -5 = 12$$

Our N^{th} term is now

$$N^{th}\ term = (0^{th}\ term) + (common\ difference)N = 12 + \ -5N = 12 - 5N$$

Problem: The acorns in a certain park are so good that squirrels from the surrounding area are moving in. Every week, 8 new squirrels move into the park. If there are 122 squirrels in the park now, how many squirrels will there be in one year? The park can only support 250 squirrels. How long will it be before there are too many squirrels in the park?

At the end of the first month, there will be 130 squirrels in the park. At the end of the second month, there will be 138 squirrels. At the end of the third month, there will be 146 squirrels. The number of squirrels is an arithmetic sequence with common difference 8 and 0^{th} term 122. The number of squirrels after N months is $122 + 8N$. At the end of one year, 12 months will have gone by, and there will be $122 + 8 \cdot 12 = 218$ squirrels.

To find out when the squirrel population will reach its limit, we solve the equation

$$122 + 8N = 250.$$

Subtracting 122 gives

$$8N = 128.$$

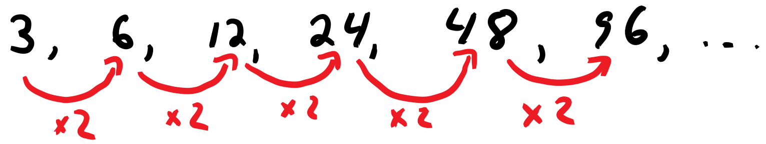
Dividing by 8 now gives

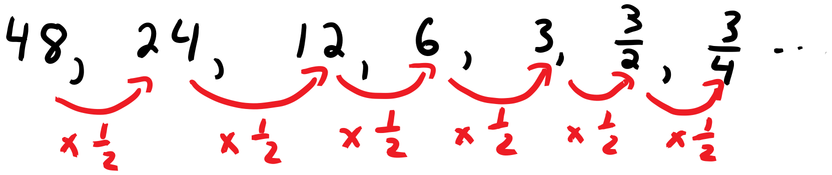
$$N = 16.$$

There will be 250 squirrels at the end of 16 months. The population will be too large during the 17th month.

Geometric Sequences

Instead of looking at the difference between consecutive terms of a sequence, we can look at the quotients. If the quotients of each term in a sequence with the term before it are all equal then we call the sequence a **geometric sequence**. Here are two geometric sequences.





Notice in the second sequence instead of multiplying by $\frac{1}{2}$ we could also divide by 2. In any geometric sequence, to get from one term to the next we either always multiply by the same number or always divide by the same number.

Problem: Explain why this is not a geometric sequence.

$$40, 20, 4, 2, 1, \frac{1}{2}\ldots$$

We consider the quotients between consecutive terms.

$$\begin{tikzcd}
40, \arrow[r, bend right, "\times\frac{1}{2}"'] & 20, \arrow[r, bend right, "\times\frac{1}{5}"'] & 4, \arrow[r, bend right, "\times\frac{1}{2}"'] & 2, \arrow[r, bend right, "\times\frac{1}{2}"'] & 1, \arrow[r, bend right, "\times\frac{1}{2}"'] & \frac{1}{2}, & \cdots
\end{tikzcd}$$

The quotient between the first two terms is $\frac{20}{40} = \frac{1}{2}$ while the quotient between the next two terms is $\frac{4}{20} = \frac{1}{5}$. Since these two quotients are not the same, this is not a geometric sequence.

Problem: Find an expression for the N^{th} term of this sequence.

$$3, 15, 75, 375, 1875, \ldots$$

We will approach this problem like we did the similar problems for arithmetic sequences. We will first find the common quotients (instead of difference), and then find the 0^{th} term. Then, to find the N^{th} term, we will multiply the 0^{th} term by the common quotients N times.

$$\begin{tikzcd}
\frac{3}{5} & 3, \arrow[l, bend right, "\div 5"'] \arrow[r, bend right, "\times 5"'] & 15, \arrow[r, bend right, "\times 5"'] & 75, \arrow[r, bend right, "\times 5"'] & 375, \arrow[r, bend right, "\times 5"'] & 1875 \ldots
\end{tikzcd}$$

The common quotient is 5. To find the 0^{th} term, we divide (rather than subtract) the first term by 5 to get $\frac{3}{5}$. Now, to get the first term, we multiply $\frac{3}{5}$ by 5. To get the second term, we multiply $\frac{3}{5}$ by 5 twice, or we multiply by 5^2. To get the third term, we multiply by 5 three times, or we multiply by 5^3. We continue this process so that to get the N^{th} term we multiply $\frac{3}{5}$ by $5\hat{}N$.

$$N^{th}\ term = \frac{3}{5} \cdot 5^N.$$

Problem: Find an expression for the N^{th} term of this sequence.

$$486, 162, 54, 18, 6, 2, \frac{2}{3}, \frac{2}{9}, \ldots$$

We work this one like the last problem. The common quotient is $\frac{1}{2}$. To find the 0^{th} term we divide 486 by $\frac{1}{2}$ to get $486 \div \frac{1}{2} = 486 \times 2 = 972$. The N^{th} term is now

$$N^{th}\ term = 972 \cdot \left(\frac{1}{2}\right)^{N}.$$

Notice that we can also write this as

$$N^{th}\ term = \frac{972}{2^N}.$$

Problem: Suppose that Clara bought a house for \$100,000 in 1990 and that inflation caused the value of the house to increase by 2.5% every year. How much will the house be worth in 2020? Round your answer to the nearest dollar.

Every year that goes by, the value of the house increases by 2.5%. Recall that to increase a number by 2.5%, we multiply by 1.025. Since we are multiplying by the same number to get from year to year, the values of the house form a geometric sequence. At the end of the first year, the value of the house is $\$100{,}000 \cdot 1.025$. At the end of the second year, the value is $\$100{,}000 \cdot 1.025^2$. At the end of the N^{th} year, the value is

$$value\ after\ N\ years = \$100{,}000 \cdot 1.025^N$$

We want the value after $N = 2020 - 1990 = 30$ years. This is

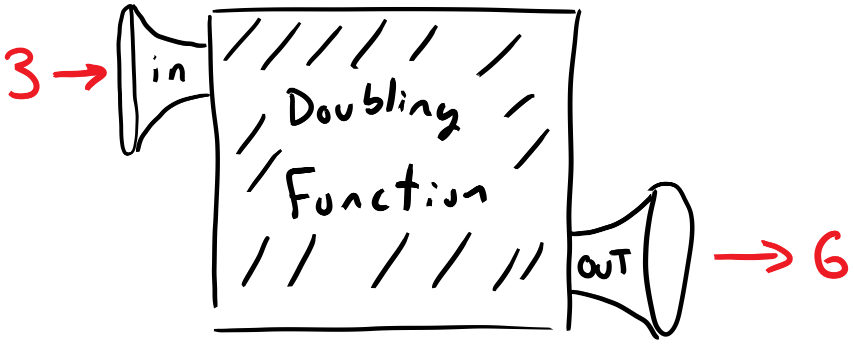
$$\$100{,}000 \cdot 1.025^{30} = \$209{,}756.7579$$

After 30 years, Clara's house will be worth about \$209,757.

Functions

In this section, we introduce the idea of a function. This one idea extends several that we have already seen – operations, proportional relationships, algebraic expressions, and sequences to name a few. The idea of a function is critical to mathematics and science and is seen by some to be the most important notion in mathematics.

A **function** can be thought of as a rule which assigns a unique output to each allowable input. The set of allowable inputs is the **domain** of the function. The set of outputs is the **range** of the function. A function is often thought of as a machine or black box. The machine has a place where inputs to into the machine and a place where outputs come out of the machine.



The key characteristic of this machine is that if the same input is fed to the machine more than once, the same output comes out. This is the meaning of the word unique in the definition of function. Here are some examples of functions:

The counting number doubling function: This function inputs a counting number and outputs twice that number. For example, if 3 is input into the function, the output would be 6.

The distance to the Moon function: This function inputs a time and date and outputs the distance from the Earth to the Moon. For example, if 5:31 P.M. CST on July 11, 2019 is input into this function, then the function should output 237,487 miles (according to wolframalpha.com).

Sequences are functions: Every sequence is a function whose inputs are the counting numbers and whose outputs are the terms of the sequence. When 1 is input to the sequence, the first term is the output. The output on 2 is the second term, and so on. Actually, the technical definition of sequence is that a sequence is a function whose domain is the set of counting numbers.

Distance function: Alice went on a road trip to Nebraska that lasted 16 hours. This gives rise to a distance function. An input to the function is a time between 0 and 16 hours. The corresponding output Alice's distance from home at that point in time.

The grape punch function: A punch recipe calls for two parts orange juice for three parts grape juice. This gives rise to grape function. Inputs to this function are numbers of cups of orange juice. Outputs are the corresponding numbers of cups of grape juice. For example, if 2 is input into the function, then the output is 3. If 4 is input into the function, then the output is 6. If 5 is input into the function, then the output is $7\frac{1}{2}$.

Mrs. Smith's class has six students – Alice, Bob, Camryn, Doug, Eve, and Frank. They are divided into pairs to study together. The pairs are Alice and Bob, Camryn and Doug, and Eve and Frank. Here are some functions based on Mrs. Smith's class:

The study buddy function: Allowable inputs are the students in the class. The output for any student is that student's study buddy. For example, if the input is Alice, the output is Bob.

The height function: This function inputs a student and outputs that student's height on the first day of class.

The science grade function: This function inputs a student and outputs that student's final grade in science class.

The sunrise function: Inputs to this function are the date. Outputs are the time that the sun rose on that date in Belton, Texas.

The plant growth function: Mrs. Smith's class planted a tree. Every day at the beginning of class, they measured the height of the tree. This gives rise to a function. The input is the number of days since planting. The output is the height when they measured it.

Tables for Functions

We can sometimes show some or all of a function using a table. Here are some examples:

Doubling Function	
Input: Counting Numbers	Output: Double the Input
1	2
2	4
3	6
4	8
5	10
6	12

Grape Punch Function	
Input: Cups of Orange Juice	Output: Cups of Grape Juice
0	0
1	1 ½
2	3
3	4 ½
4	6
5	7 ½

Study Buddy Function	
Input: Student from Mrs. Smith's Class	Output: The Input's Study Buddy
Alice	Bob
Bob	Alice
Camryn	Doug
Doug	Camryn
Eve	Frank
Frank	Eve

Height Function	
Input: Student from Mrs. Smith's Class	Output: The Input's Height the First Day of Class (in inches)
Alice	57
Bob	56
Camryn	56
Doug	58
Eve	60
Frank	60

Notice that for the Height and Study Buddy functions we can list the entire function. For the Doubling function and the Grape Punch function, we can only list some of the inputs and corresponding outputs.

Problem: Explain why this does not define a function: The inputs to this function are the heights 57, 56, 58, and 60. The output is the student from Mrs. Smith's class who has that height.

A function is required to have a single, unique output for each input. If we input 56 into this supposed function, we do not know if the output should be Bob or Camryn. Some inputs have more than one output.

Problem: Explain why this does not define a function: Inputs to this function are integers. The output for any given input is 2 divided by the input.

A function must have an output for every input. Since 0 is an integer, we would need an output corresponding to 0. However, we cannot divide 2 by 0. Since 0 cannot be assigned an output, this is not a function.

Equations for Functions

When the inputs and outputs of a function are numbers, we can sometimes (but not always) write equations which completely determine the function. To do so, we first select two variables, one to represent an input to the function and one to represent the corresponding output. Then, we write an equation where one side of the equation is the output variable and the other side is an expression involving the input variable which describes how to compute the appropriate output.

Problem: Write an equation for the doubling function.

We elect to use the variable x to represent an input and the variable y to represent an output. To double an input x, we simply multiply by 2, so our equation that describes how to compute an output y given an input x is $y = 2x$.

When there is no semantic reason to select other variables, the most commonly used variables for functions are x (for inputs) and y (for outputs); however, if possible, it is helpful to select variables in a meaningful way. For example, if a variable represents time, we may use t, and if a variable represents cups of flour, we might use C or F.

Problem: Write an equation for the grape punch function.

Recall that inputs for this function are numbers of cups of orange juice, and outputs are corresponding numbers of cups of grape juice if the juices are mixed in a ratio of 2:3. We first select variables. For output, cups of grape juice, we choose to use G. For input, cups of orange juice, we could use O. However, an O might be mistaken for 0. Therefore, we choose to use R. To write our equation, we use unit rates. We know that to convert an amount of orange juice to an amount of grape juice, we should multiply by the unit rate of cups of grape juice per cups of orange juice. This is the fraction $\frac{3}{2}$. Therefore, to compute G from R, we multiply by $\frac{3}{2}$. This gives the equation $G = \frac{3}{2}R$.

Problem: Alice has an online coffee business. She sells coffee by the pound, and she does not sell partial pounds. That is, a customer may order 1 or 12 pounds, but not $7\frac{1}{2}$ pounds. Customers pay six dollars per pound, and they pay a four dollar transaction fee for each order. Write an equation that describes the cost function whose input is the number of pounds of coffee in an order and whose output is the cost of the order.

We first define our variables. We will use P for the input, pounds of coffee. We will use C for the output, cost of order. To find our equation, we will first compute some corresponding values of P and C. If we order $P = 1$ pound of coffee, then we will pay \$6 plus the \$4 transaction fee for a total of $C = 6 + 4 = 10$ dollars. If we order $P = 2$ pounds of coffee, then we will pay \$6 for each of the two pounds, or $6 \cdot 2$ dollars, plus the transaction fee for a total of $C = 6 \cdot 2 + 4 = 16$ dollars. If we order $P = 3$ pounds of coffee, then we will pay \$6 for each of the three pounds, or $6 \cdot 3$ dollars, plus the transaction fee for a total of $C = 6 \cdot 3 + 4 = 22$ dollars. Here is a table with some values of our variables.

Input: Pounds of Coffee (P)	Output: Cost of Order (C)
1	$6 + 4 = 10$
2	$6 \cdot 2 + 4 = 16$
3	$6 \cdot 3 + 4 = 22$
4	$6 \cdot 4 + 4 = 28$
5	$6 \cdot 5 + 4 = 34$
6	$6 \cdot 6 + 4 = 40$
7	$6 \cdot 7 + 4 = 46$

There is a pattern here in the column for C. Each entry is 6 times the input plus the transaction fee. Therefore, we have the equation $C = 6 \cdot P + 4$.

The domain for this function is important. As we have worked the problem, our domain is the set of counting numbers 1, 2, 3… One could ask to use the domain of whole numbers, 0, 1, 2 , 3, 4… since it is possible to order 0 pounds of coffee by not ordering anything. In this case, our equation does not quite work because if $P = 0$ then $C = 6 \cdot 0 + 4 = 4$. Alice cannot collect $4 from everyone who orders nothing from her, so we have a problem. The solution is to use a *piecewise defined function*:

$$C = \begin{cases} 0 & P = 0 \\ 6 \cdot P + 4 & P > 0 \end{cases}$$

Another approach to coming up with the equation for this function is to notice that the values of C form an arithmetic sequence. The right-hand-side of our equation is simply the expression for the P^{th} term of the sequence, which can be found in the same way we performed that task in the sequence section.

Problem: Suppose that a function is given by $y = 2x^2 + 2x + 2$. Find the outputs of the function for the inputs $x = 0, 1, 2, \frac{1}{2}$.

If $x = 0$ then

$$y = 2 \cdot 0^2 + 2 \cdot 0 + 2 = 2 \cdot 0 + 2 \cdot 0 + 2 = 2.$$

If $x = 1$ then

$$y = 2 \cdot 1^2 + 2 \cdot 1 + 2 = 2 \cdot 1 + 2 \cdot 1 + 2 = 2 + 2 + 2 = 6.$$

If $x = 2$ then

$$y = 2 \cdot 2^2 + 2 \cdot 2 + 2 = 2 \cdot 4 + 2 \cdot 2 + 2 = 8 + 4 + 2 = 14.$$

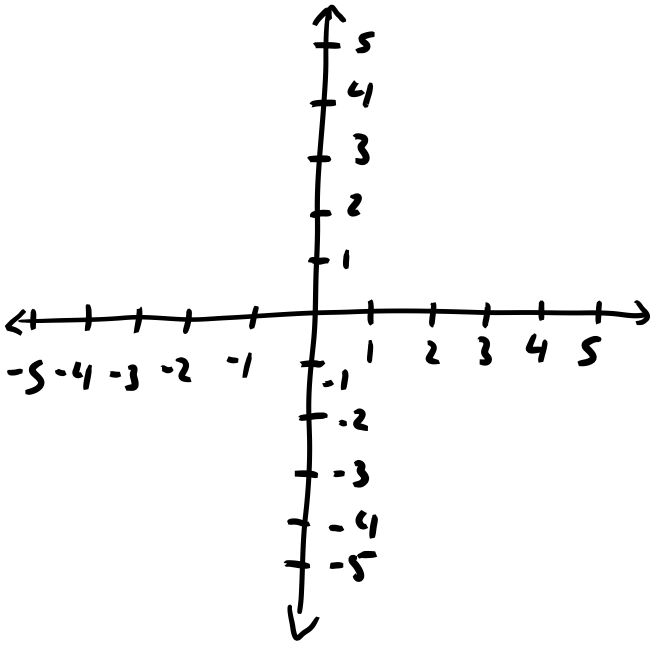
If $x = \frac{1}{2}$ then

$$y = 2 \cdot \left(\frac{1}{2}\right)^2 + 2 \cdot \frac{1}{2} + 2 = 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 2 = \frac{2}{4} + 1 + 2 = \frac{1}{2} + 3 = 3\frac{1}{2}.$$

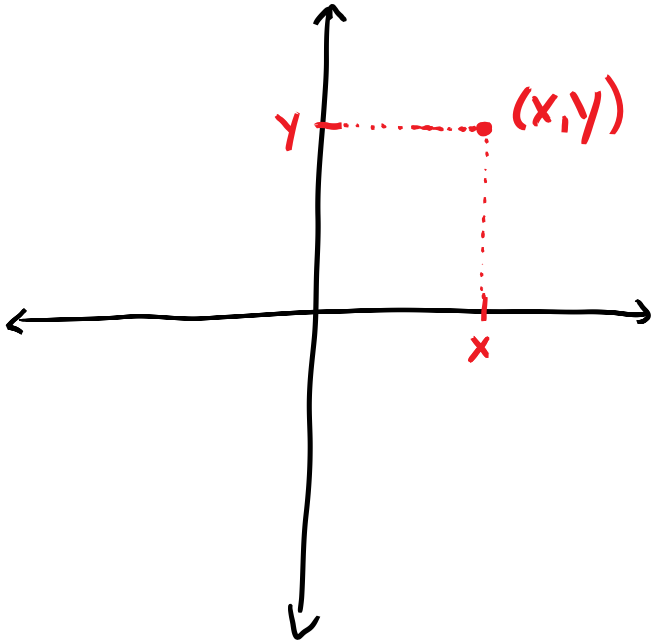
Graphing

The coordinate plane or Cartesian plane was introduced by Renee Descarte in 1637. It is arguably one of the most influential inventions humans have made. Not only did the coordinate plane provide science with crucial tools for visualization, the coordinate plane allowed the unification of geometry and algebra and prepared the way for the invention of Calculus later in the 17^{th} century by Newton and Leibniz. Calculus then provided the tools necessary for significant advancement in science and technology. Many of the ideas behind Calculus were apparent to Archimedes around 250 B.C., but it was not until the Cartesian plane allowed the algebraic study of geometric concepts that the field could flourish.

The plane is an infinitely large surface on which we can draw. The Cartesian coordinate system or rectangular coordinate system or (for our purposes) coordinate system for the plane is a method of identifying points on the plane using pairs of numbers. First, two number lines are drawn on the plane, one horizontal and one vertical. The lines are placed so that they intersect each other at 0. On the horizontal line, positive numbers are to the right, and negative numbers are to the left. On the vertical line, positive numbers are upward, and negative numbers are downward. Each of these two lines is called an axis (plural is axes).



With these two axes, any point on the plane can now be located. Given a point on the plane, we can draw a vertical line through the point and note where it intersects the horizontal axis. We can also draw a horizontal line through the point and not where it intersects the vertical axis. These two numbers are called the coordinates of the point.

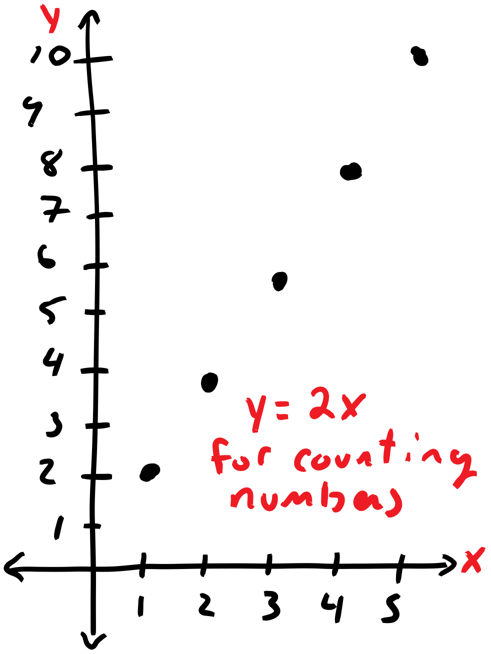


We write the coordinates of a point as an ordered pair of numbers (x, y). The first number (here called x) is the coordinate from the horizontal axis. The second number (here called y) is the number from the vertical axis. The use of x for the horizontal coordinate and y for the vertical coordinate is so pervasive that many students will simply call the horizontal axis the x-axis and the vertical axis the y-axis. In particular problems, the names of these variables and axes may vary. The point $(0,0)$ where the axes cross is called the **origin**.

By the graph of a function whose domain and range are both sets of numbers we mean the set of all points (x, y) where y is the output of the function assigned to the input x. To graph a function is to draw the plane with the points on (or in) the graph of the function drawn (or indicated).

Problem: Graph the counting number doubling function.

The counting number doubling function is given by the equation $y = 2x$ where x is assumed to be a counting number. Because the domain of this function is the counting numbers, it is easy to list the points which are on the graph of this function: $(1,2), (2,4), (3,6), (4,8), (5,10)$ … This includes infinitely many points, so we cannot draw them all, but we can graph enough to see a pattern.



Notice here that since we have described our function as $y = 2x$ with x being the input and y being the output, we labeled the horizontal axis x and the vertical axis y.

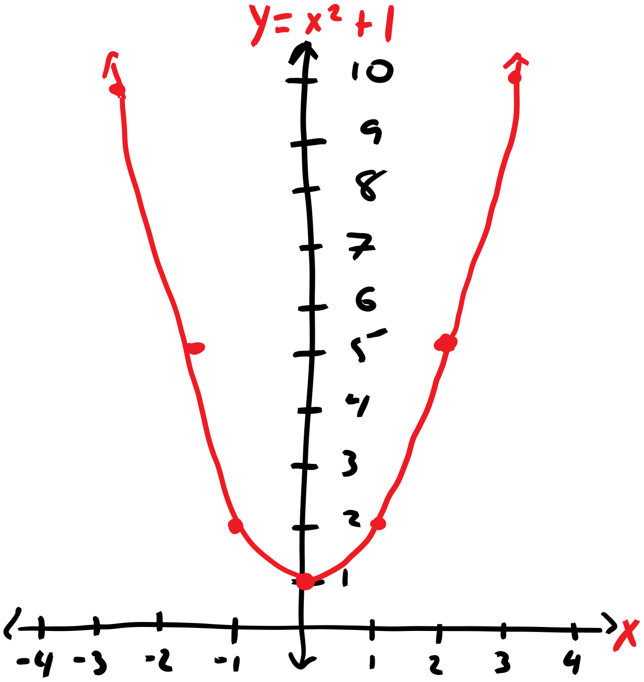
Suppose that we have an equation involving the variables x and y. A point (a, b) is a solution to the equation if the substitutions $x = a$ and $y = b$ make the equation true. For example, the point $(2,3)$ is a solution to the equation $4x + 5y = 23$ because the equation $4 \cdot 2 + 5 \cdot 3 = 23$ is true. The graph of an equation is the set of all points which are solutions to the equation. To graph an equation is to draw a coordinate plane with the solutions to the equation drawn (or indicated). Generally, we cannot draw all of the points which are solutions, but we can draw a representation that shows a general pattern.

Problem: Graph the equation $y - x^2 = 1$.

To graph $y - x^2 = 1$, we will find several points which are solutions to the equation. To do so, we could plug in some numbers for y and solve for corresponding values of x, and we could plug in some numbers for x and solve for corresponding values of y. We can simplify things a bit here by solving for y in the equation prior to doing any substitutions. This gives $y = x^2 + 1$. Now, no matter what we plug in for x, we can compute a value for y without having to do any algebraic manipulations. This problems works so nicely because y here is a function of x. Now we substitution several values for x and calculate the corresponding values of y. Each pair of numbers gives a point.

x	$y = x^2 + 1$	Point
-3	10	(-3,10)
-2	5	(-2,5)
-1	2	(-1,2)
0	1	(0,1)
1	2	(1,2)
2	5	(2,5)
3	10	(3,10)

We now plot these points. Since we have only substituted values for x which are integers, and since there are numbers in between the integers, we connect the points with a smooth curve.

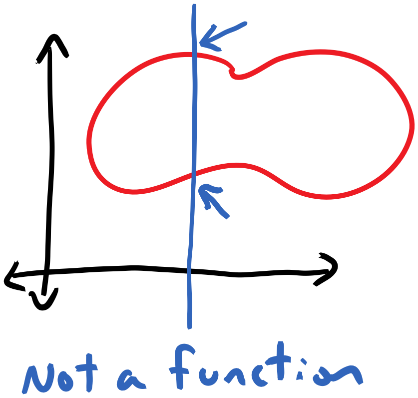


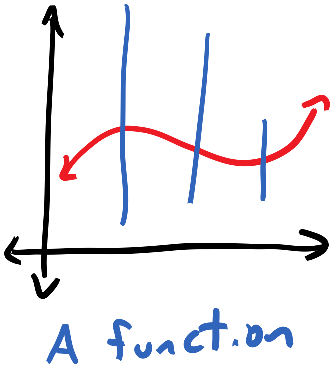
Notice that we include arrows on the ends of the graph since this graph should continue beyond inputs of 4 and -4.

The Vertical Line Test and the Definition of a Function

The first coordinate of a point on the graph of a function is an input to the function, and the second is the corresponding output. Our description of a function is a rule which assigns to every input a unique output. Therefore, no two points on the graph of a function can have the same first coordinate since two points with the same first coordinate would indicate two different outputs associated with the same input. This idea gives an easy test to see if a graph is the graph of a function or not. The set of all points on the plane with a common, fixed first coordinate is a vertical line. That no two points on the graph of a function can have the same first coordinate means that no two points on the graph of a function can lie on the same vertical line. This lead to the vertical line test:

Vertical Line Test: A graph is the graph of a function if no vertical line touches the graph at more than one point.





A vertical line intersects the graph on the left twice; therefore, the graph on the left is not the graph of a function. No vertical line intersects the graph on the right more than once. The graph on the right is a function.

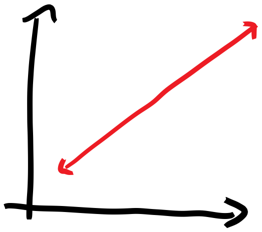
We began this section on functions with this statement: A **function** can be thought of as a rule which assigns a unique output to each allowable input. This is an awful definition. What is a rule? What does it mean to assign? What is an input? What is an output? A more formal approach to functions uses what we have seen in graphs to define a function. A function from one set to another set is a collection of ordered pairs so that

- The first coordinate of each ordered pair comes from the first set.
- The second coordinate of each ordered pair comes from the second set.
- For each element in the first set, there is exactly one ordered pair with that element as the first coordinate.

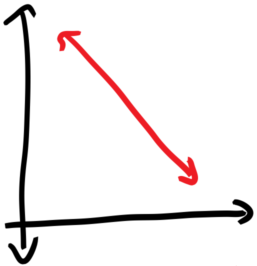
This definition of function essentially declares that a function is a graph which passes the vertical line test.

Functions of Time and Rates of Change

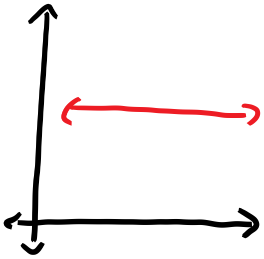
It is frequent that we encounter functions of time. A person's height is a function of time because that person cannot be two different heights at any given time. If a person goes for a walk from home, then the person's distance from home is a function of time because that person cannot be in two different places at the same time. How these functions change over time is reflected in the shape of their graphs. The direction of the slope of a graph tells us if the values are increasing or decreasing.



Increasing

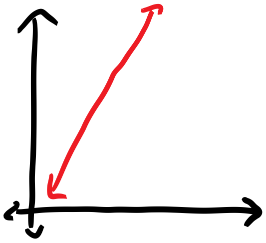


Decreasing

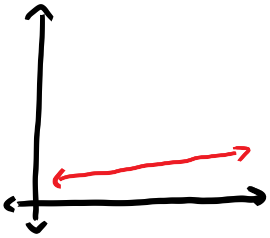


Constant
(Not changing)

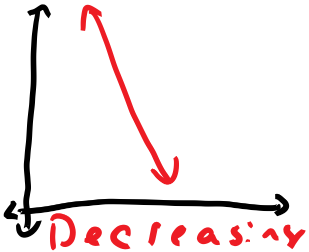
How quickly a function is changing over time is reflected in the steepness of the function.



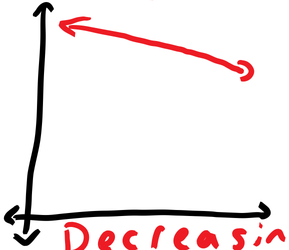
Increasing
Rapidly



Increasing
Slowly

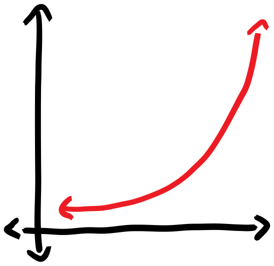


Decreasing
Rapidly

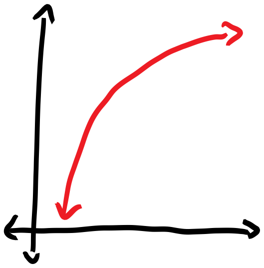


Decreasing
Slowly

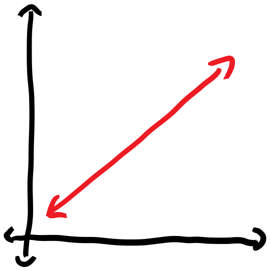
If the slope of a function changes, then we can tell if its change is speeding up or slowing down.



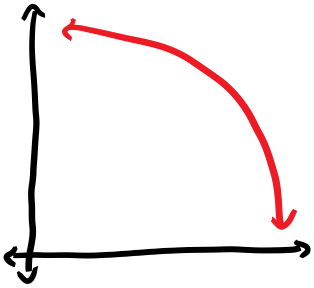
Increasing more and more rapidly



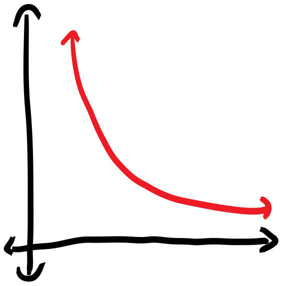
Increasing more and more slowly



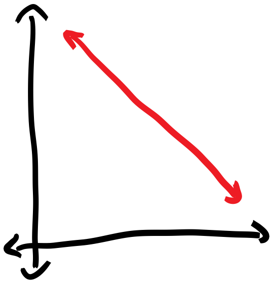
Increasing at a constant rate



Decreasing more and more rapidly



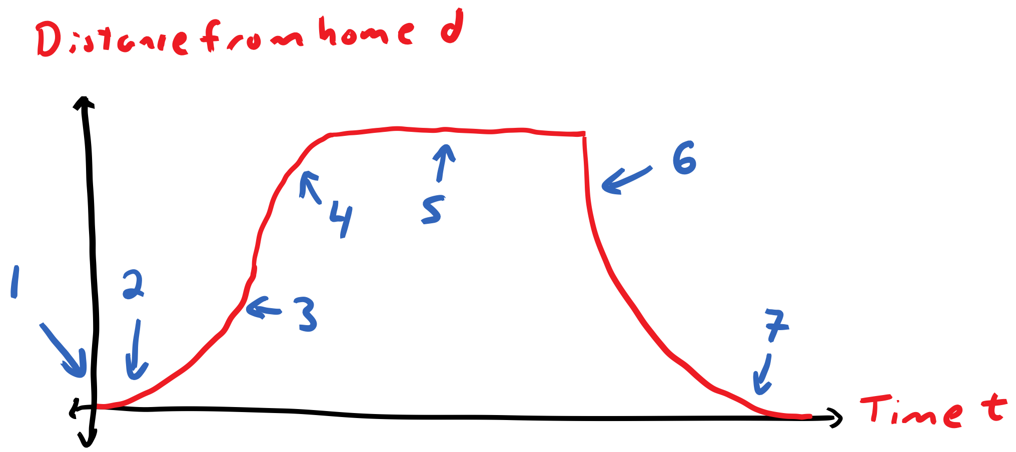
Decreasing more and more slowly



Decreasing at a constant rate

Problem: Bob went for a walk. This gives rise to a function whose input t is the time since Bob's walk started and whose output d is the Bob's distance from home. A graph of this distance function is below. Discuss Bob's walk in terms of the graph.

There are several important features of this graph. We number them below and then explain the significance of each number.

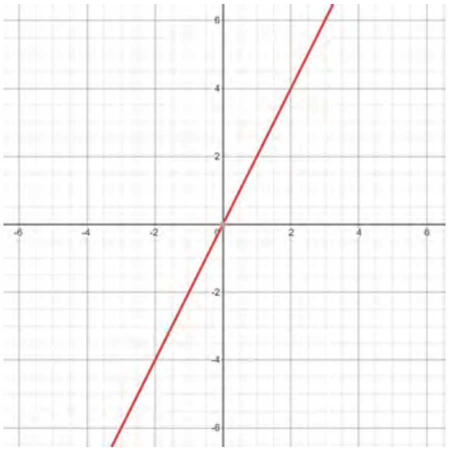


1. When $t = 0$, Bob’s distance d from home is also 0. This means that Bob’s walk started at home.
2. Initially, d is increasing so Bob is walking away from home. At 2, the graph is not very steep, so Bob is walking away from home slowly.
3. Bob continues walking away from home, but the graph gets steeper. This means he is walking away from home more and more quickly.
4. After a while the graph gets less steep, even though it is still increasing. Bob is still walking away from home, but he is slowing down.
5. For a period, the graph is horizontal. This means that Bob’s distance from home is not changing. He has stopped, and he remains in the same location. (Technically, Bob could be walking along a circle centered at his house during this time. His distance from home may not be changing, but his position might be.)
6. Bob starts walking home. Initially, he is moving very quickly toward home.
7. As Bob gets closer and closer to home he slows down.

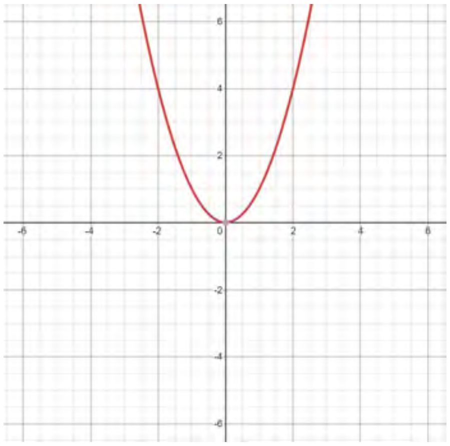
Some Common Graphs

Here are graphs of some common equations.

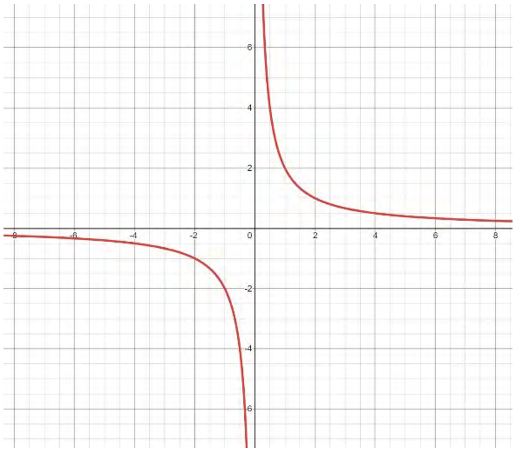
The proportional relationship $y = 2x$.



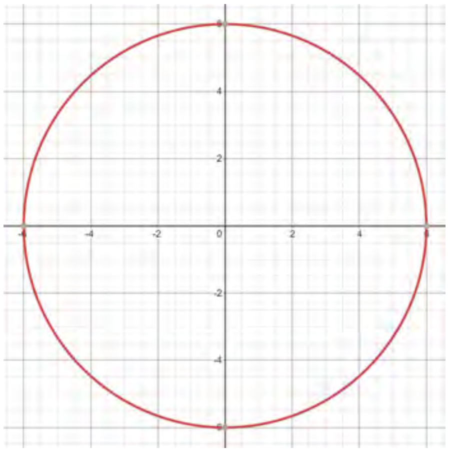
The parabola $y = x^2$.



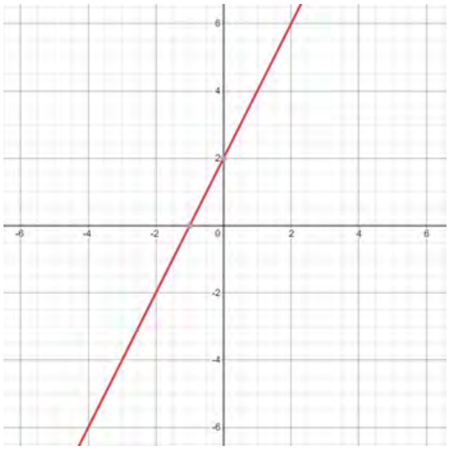
The inversely proportional relationship $xy = 2$.



The circle $x^2 + y^2 = 36$.

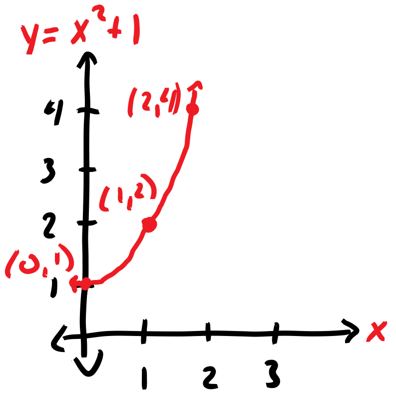


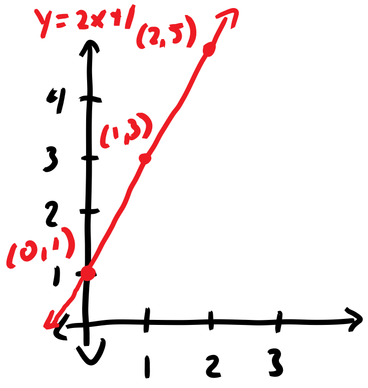
The line $y = 2x + 1$.



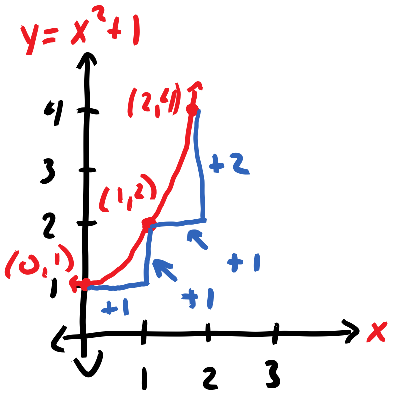
Linear Functions

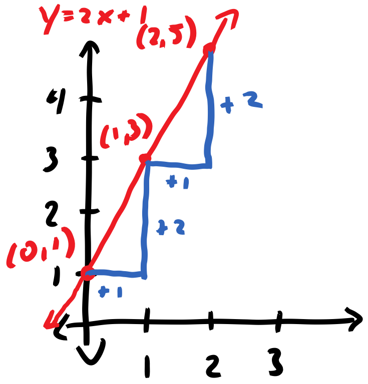
Consider these two graphs of the functions $y = x^2 + 1$ and $y = 2x + 1$.





We would like to consider how much function values change when inputs (x) change by 1. On the graph of $y = x^2 + 1$, as x changes from 0 to 1, y changes by 1 from 1 to 2. On the same graph, as x changes from 1 to 2, y changes by 2 from 2 to 4. Two changes by 1 in x correspond to changes by different amounts in y.





Now consider the graph of $y = 2x + 1$. When x changes from 0 to 1, y changes by 2. When x changes from 1 to 2, y again changes by 2. In fact, any time that x changes by 1 on this graph, y will change by 2.

Problem: Calculate the difference between $2(x + 1) + 1$ and $2x + 1$. What is the significance of this difference for the function $y = 2x + 1$?

The difference is $(2(x + 1) + 1) - (2x + 1) = 2x + 2 + 1 - 2x - 1 = 2$. The significance is that if the input to $y = 2x + 1$ is changed by 1 (from x to $x + 1$) then the output changes by 2.

The fact that values of $y = 2x + 1$ change by 2 whenever x changes by 1 is one of the characteristic features of lines. A linear function is a function which can be described by an equation of the form $y = mx + b$ where m and b are fixed numbers. For example, these equations all describe linear functions

$$y = 2x + 1$$

$$y = \frac{3}{2}x + 5$$
$$A = 7B + 12$$
$$T = 3N + 8$$
$$y = 3x$$
$$y = x$$
$$y = 7$$
$$y = 0$$

Notice in the last four examples that m or b is 0 (or both). Also notice that we are not restricted to using x and y as variables.

Problem: Consider the function which inputs a counting number N and outputs the N^{th} term T of this arithmetic sequence.

2, 5, 8, 11, 14, 17…

Find an equation that describes T in terms of N. Is this a linear function?

We already know how to find an expression for the N^{th} term of an arithmetic sequence. First, we note that the common difference between terms is 3. Then, we subtract 3 from the first term to find that the 0^{th} term is -1. Now, the N^{th} term is given by $-1 + 3N$. We can now write an equation for T.

$$T = -1 + 3N$$

This can be rewritten as $T = 3N - 1$. This is in the form $y = mx + b$, so this function is linear. In fact, every arithmetic sequence gives rise to a linear function in this way. Note here that the value of m (here, 3) is the amount that the sequence changes from term to term. Also note that the value of b (here, -1) is the 0^{th} term.

Problem: Consider the function given by the equation $T = 3N + 5$. There is a sequence whose N^{th} term is the value of T at N. Show that this is an arithmetic sequence.

The difference between any two adjacent terms is

$$(3(N + 1) + 5) - (3N + 5) = 3N + 3 + 5 - 3N - 5 = 3.$$

Since the difference between any two consecutive terms is 3, this is an arithmetic sequence. Note that the common difference between terms, or the amount that the sequence changes from term to term, is the same as the coefficient of N.

Problem: A certain recipe includes cinnamon and nutmeg in a ratio of 5 to 4. Let C be the number of teaspoons of cinnamon in a batch of this recipe, and let N be the number of teaspoons of nutmeg. Write an equation relating C and N.

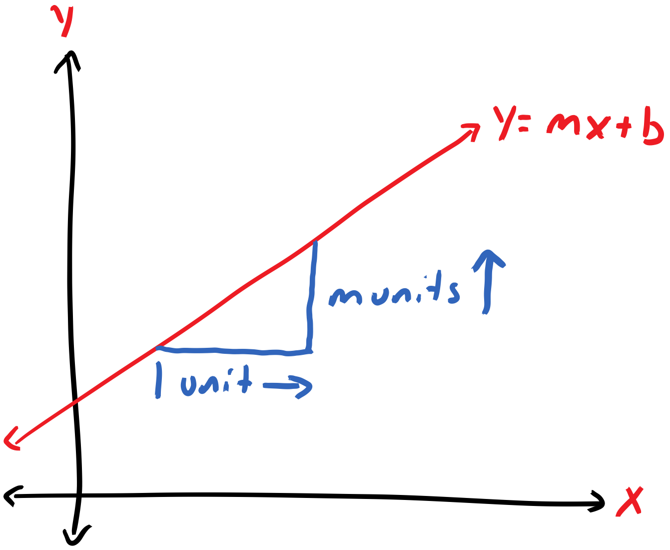
We can use unit rates here to relate C and N. If we multiply any quantity N of nutmeg by the unit rate of parts of cinnamon per part of nutmeg, we will arrive at the corresponding quantity C of cinnamon. Therefore, we have the equation $C = \frac{5}{4}N$. Notice that this is a linear function. In fact, every proportional relationship yields a linear function. Also note that if $N = 0$, then $C = 0$, so the graph of this linear function passes through the origin. The graph of every proportional relationship is a line that passes through the origin.

Problem: What is the value of the function described by $y = mx + b$ when $x = 0$?

If we substitute $x = 0$ into $y = mx + b$, we get $y = m \cdot 0 + b = b$. This means that the graph of $y = mx + b$ passes through the point $(0, b)$. Since this point is on the y-axis, we call it the y-intercept of $y = mx + b$. In general, the **y-intercept** of a graph is the point where the graph intersects the y-axis (if such a point exists). For a line $y = mx + b$, the y-intercept is the point $(0, b)$. Some books will also use the term y-intercept to refer to the value b. Notice that if we identify our linear function $y = mx + b$ with an arithmetic sequence, then b is the 0^{th} term of the sequence.

Problem: Calculate the difference between $m(x + 1) + b$ and $mx + b$. What is the significance of this difference for the linear function $y = mx + b$.

First, the difference is $(m(x + 1) + b) - (mx + b) = mx + m + b - mx - b = m$. What this means is that if we change an x-value on this graph by one unity, then the y-value will change by m. This number m, called the **slope** of the line, is a measure of how steep the line $y = mx + b$ is. If m is large, then y value change a lot when x changes by 1, and the line is steep. If m is small, then y changes by less when x changes by 1, and the line is less steep. If $m = 0$, then y does not change, so the line is horizontal. If $m > 0$, then the change in y is positive, or upward. In terminology from the last section, the line is increasing. If $m < 0$, then the change in y is negative, or downward. In this case, the line is decreasing.

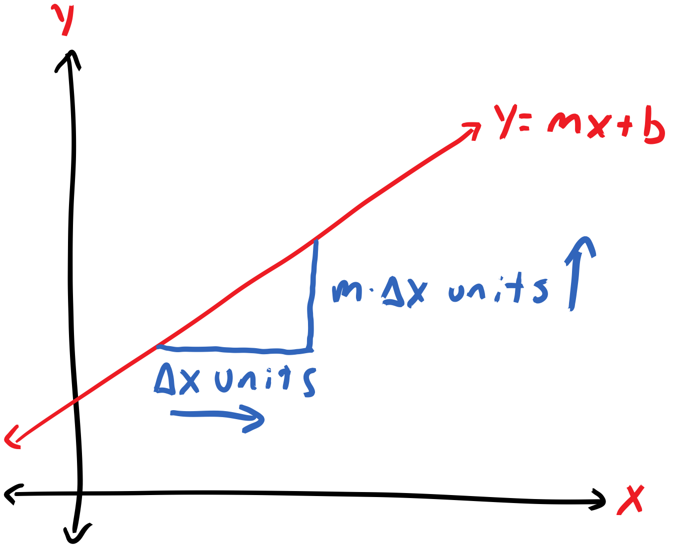


Problem: Suppose that (x_1, y_1) and (x_2, y_2) are points on the graph of $y = mx + b$. Calculate the quotient $\frac{y_1 - y_2}{x_1 - x_2}$.

Since (x_1, y_1) is on the graph of $y = mx + b$, we know that $y_1 = mx_1 + b$. Similarly, $y_2 = mx_2 + b$. Therefore

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{(mx_1 + b) - (mx_2 + b)}{x_1 - x_2} = \frac{mx_1 + b - mx_2 - b}{x_1 - x_2} = \frac{mx_1 - mx_2}{x_1 - x_2} = \frac{m(x_1 - x_2)}{x_1 - x_2} = m.$$

Note here that no matter what two points we use on the line $y = mx + b$, this quotient will give us the slope m of the line. The difference $y_1 - y_2$ is often denoted Δy where the symbol Δ (a capital Greek Delta) is read as "change in." Thus $\Delta y = y_1 - y_2$ means change in y. Similarly, $\Delta x = x_1 - x_2$ is the change in x. We can now write the calculations above more succinctly as $\frac{\Delta y}{\Delta x} = m$ or as $\Delta y = m\,\Delta x$. This equation relates change in y to change in x on the line $y = mx + b$. If x changes by an amount (Δx), then y changes by m times that amount ($m\,\Delta x$). Using terminology from earlier sections, the change in y is proportional to the change in x.



Problem: Below is a table of some values of x and y for a particular function. Explain why the function is not a linear function.

$Input: x$	$Output: y$
1	2
2	5
3	9

From the point $(1,2)$ to the point $(2,5)$ on this function, x changes by 1 unit, and y changes by 3 units. If this is a line, then whenever x changes by 1, y should change by 3. However, from the point $(2,5)$ to the point $(3,9)$ x changes by 1 but y changes by 4. This is not a linear function.

Problem: Below is a table of some values of x and y for a particular function. Explain why the function is not a linear function.

$Input: x$	$Output: y$
1	2
3	5
7	9

This problem is only slightly more complicated than the last because our x-values are more than 1 unit apart. We will calculate the slope of the line between the first two points $(1,2)$ and $(3,5)$, and we will calculate the slope of the line between the second two points $(3,5)$ and $(7,9)$. The slope from $x = 1$ to $x = 3$ is

$$\frac{1-3}{2-5} = \frac{-2}{-3} = \frac{2}{3}$$

The slope from $x = 3$ to $x = 7$ is

$$\frac{3-7}{5-9} = \frac{-4}{-4} = 1$$

Since these slopes are not the same, this function is not linear.

Problem: The table below shows some values for a linear function. Fill in the rest of the values and write an equation for the function.

x	y
0	
1	
2	5
4	
8	23
16	
32	

If we find the slope of the function, we can then easily step back from the value at $x = 2$ to the value at $x = 1$ and the value at $x = 0$. Once we have the function value at $x = 0$, we can write the equation and then use the equation to fill in the other empty blanks. We will find the slope in two ways (just because it is so much fun). First, we know the function values at $x = 2$ and at $x = 8$. This is $8 - 2 = 6$ "steps" along the line horizontally. The question is how large each step should be vertically. In the column for y, we see that these steps add up to a distance of $23 - 5 = 18$. If steps add up to 18, then one step (which is the slope we are looking for) should be $18 \div 6 = 3$. Now we will find the slope explicitly using the formula from above.

$$slope = \frac{change\ in\ y}{change\ in\ x} = \frac{23-5}{8-2} = \frac{18}{6} = 3.$$

Notice that the two approaches use the same arithmetic and arrive at the same place, but the first is much less algebraic in nature.

Now that we know the slope is 3, we can subtract this from the function value at $x = 2$ to get a function value of 2 at $x = 1$. If we subtract again, we get a function value of -1 at $x = 0$. This is the y-coordinate of the y-intercept. We know have a slope of $m = 3$ and a y-intecept of $b = -1$, so an equation for this line is $y = 3x - 1$. Using this equation, we can fill in the rest of the table below.

x	y
0	-1
1	2
2	5

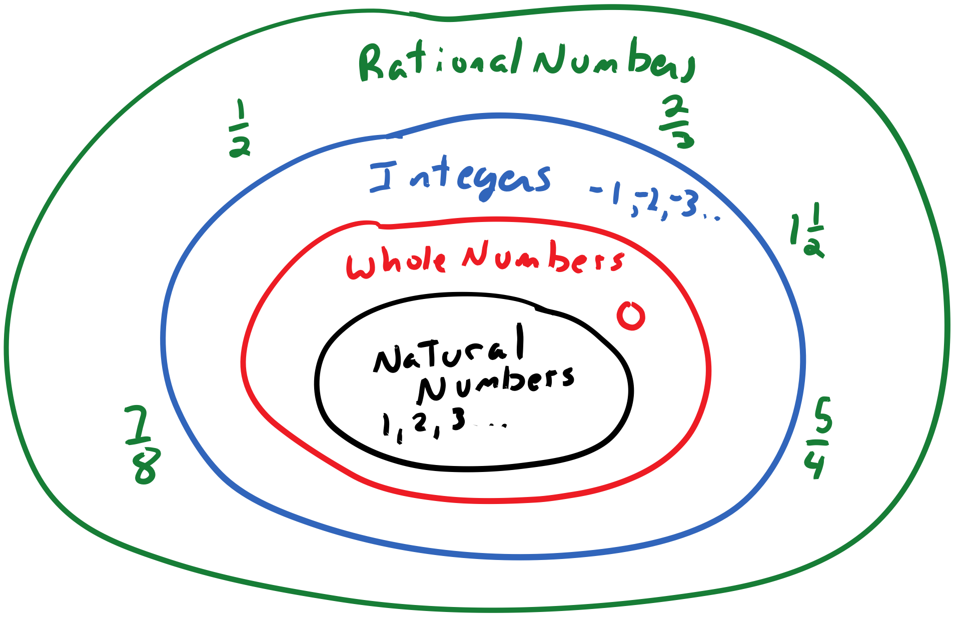
4	11
8	23
16	47
32	95

Odd and Even Numbers

In our studies so far, we have encountered these number systems:

- The **counting numbers**: These are the names of the sizes of nonempty sets of objects. We denote them as 1, 2, 3, 4… The counting numbers were historically called the **natural numbers** by mathematicians.
- The **whole numbers**: These are the counting numbers along with zero. They include 0, 1, 2, 3, 4, 5… Most of our arithmetic algorithms were developed first at the level of the whole numbers.
- The **integers**: These numbers consist of the counting numbers, zero, and negatives of the counting numbers. We can list them as …-4, -3, -2, -1, 0, 1, 2, 3, 4, 5… We could also list them as 0, 1, -1, 2, -3, 3, -3, 4, -4…
- The **rational numbers**: These are all numbers which can be expressed as fractions of integers. They include all integers along with numbers like $\frac{5}{7}$, $-\frac{4}{5}$, $2\frac{1}{3}$, and $\frac{7}{3}$.

We can draw a picture called a Venn diagram which illustrates how these number systems are related to each other:



In this diagram, ovals (or other areas) represent sets of numbers. Everything in the black oval at the center is a natural number. Everything in the red oval (including the natural numbers) is a whole number. Everything in the blue oval (including the red and black ovals) is an integer, and everything in the green oval is a rational number. Some example numbers are placed in each oval. Notice that the only number in the whole numbers which is not a natural number is 0.

The field of mathematics which studies the integers is known as **number theory**. Number theory is an ideal environment to discuss higher mathematics for the first time. The objects with which it is concerned – integers (and usually positive ones) – are things that we encounter daily, and the concepts it addresses are simple enough to explain to middle school students. However, many simple sounding problems in number theory are quite deep and require significant mathematical tools to solve.

Even and Odd Numbers

We begin our brief foray into number theory by address in the notions of even and odd whole numbers. What does it mean for a whole number to be even? Given a pile of beans (or any type of object) how can we easily tell if we have an even or an odd number of beans? Here are two possible answers to this question:

1. We can try to divide the pile of beans into two equal size piles. A simple but slow way to do this is to take two beans out of the pile at a time and place one on our left and one on our right. This creates two piles which will always have the same number of beans. At the end of the process, we will either have no beans left over – so we have *evenly* divided them into two piles – or we will have one bean left over. If no beans are left over, we have an even number of beans. If one bean is left over, then we have an odd number of beans.
2. We could also take two beans at a time out of our pile and create several piles of two beans. At the end of this process, we will either have no beans left over – so we have *evenly* divided the pile into piles of two – or we will have one bean left over. Again, if we have no beans left over, then we have an even number of beans. If we have one bean left over, then we have an odd number of beans.

Suppose that the number of beans we have here is N. Let k be the number of beans in each pile with approach number 1 above. In the even case, we have two groups of beans with k beans in each group. This means that we have $N = 2k$ beans. In the odd case, we have $N = 2k + 1$ beans. In the odd case, we would have k groups of 2 beans. This would imply $N = k \cdot 2$ or $N = k \cdot 2 + 1$, but these expressions are equivalent to the ones from the first approach because multiplication is commutative. These approaches to trying to determine if we have an odd or an even number of beans lead to the following possible definitions of even and odd:

Possible definitions of even:

1. A whole number N is **even** if N objects can be placed into two equal sized groups with none left over.
2. A whole number N is **even** if N objects can be placed into groups of size two with none left over.
3. A whole number N is **even** if there is a whole number k so that $N = 2k$.

Possible definitions of odd:

1. A whole number N is **odd** if N objects can be placed into two equal sized groups with *one* left over.
2. A whole number N is **odd** if N objects can be placed into groups of size two with *one* left over.
3. A whole number N is **odd** if there is a whole number k so that $N = 2k + 1$.

The fact that these possible definitions are all equivalent follows from the definition of multiplication and the fact that multiplication is commutative.

You might have noticed that we use *whole numbers* in our definitions. However, if we have a bunch of beans, then is the number of beans we have not a *counting number*? The only difference is that 0 is a whole number. If we have 0 beans, then we can place those beans into two piles each with 0 beans and have no beans left over. Thus, 0 should also be even.

The third (more algebraic) possible definition in each list has at least two benefits over the first two possibilities. First, this definition can be used with integers rather than just whole numbers. We cannot really talk about having a group of -17 beans. Second, we will see below that the third definition gives rise to simple proofs about odd and even numbers.

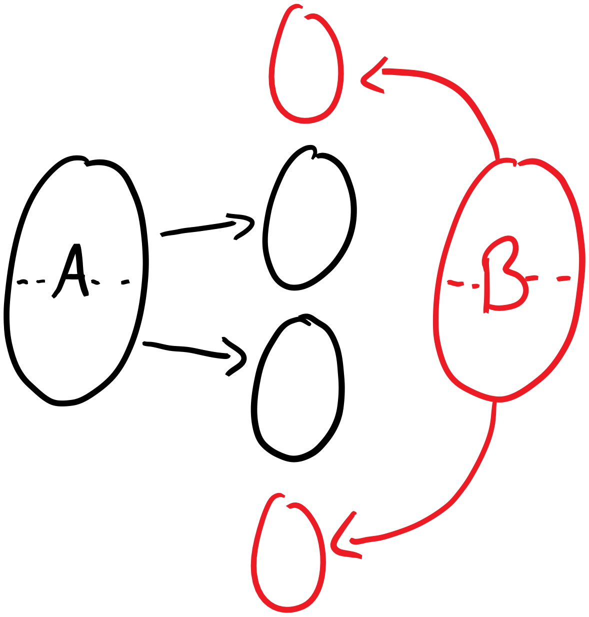
Arithmetic with Odd and Even Numbers

There are many types of mathematics and mathematicians that perform many types of functions. However, there is one thing that all mathematicians do: they write proofs. We will talk more about proof writing in our geometry section later, but we will see a few basic examples here. Proof writing has been taught to non-mathematicians for millennia in order to develop skills for clear reasoning and communication. At this point we have (three possible) definitions of even and odd. Mathematicians use definitions as outlines for writing proofs.

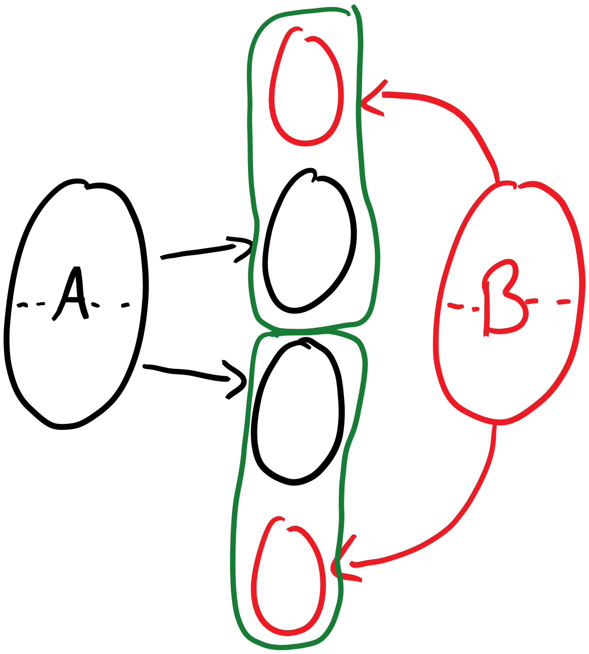
Problem: Suppose that A and B are even whole numbers. Prove that $A + B$ is also even.

We will address this problem three ways, following each of the three possible ways to define even. First, we use the two-equal-piles definition. Suppose that A and B are even whole numbers and that we have one pile of A beans and another pile of B beans. We will draw "blob pictures" to represent piles of beans.

Since A and B are both even, we can break each pile into two equal size piles:



Now, if we combine of the piles from the A beans with one of the piles from the B beans, we will have two equal size piles:



We have divided all $A + B$ beans into two equal size piles, so $A + B$ must be even.

Now we will approach the same argument using the piles-of-two definition of even. Suppose that A and B are even whole numbers, and suppose that we have a pile of A beans and a pile of B beans. Since A is even, we can place the pile of A beans into piles of two with no beans left over. Since the B is even, we can place the pile of B beans into piles of two with no beans left over. Having done so, we have placed all of the beans – all $A + B$ of them – into piles of two with none left over. Therefore, $A + B$ must be even.

Finally, we use the algebraic definition. Suppose that A and B are even whole numbers. There are whole numbers k and l so that $A = 2k$ and $B = 2l$. Then $A + B = 2k + 2l = 2(k + l)$, so $A + B$ is even.

Notice how the algebraic approach is most compact. Also, if we replace “whole number” with “integer” then we would have an argument that works for integers as well as whole numbers. The first two approaches cannot do this. On the other hand, the algebraic approach has the disadvantage that it requires algebra. Probably, the piles-of-two approach is the simplest argument that could be explained to a group of students with no algebra background.

Problem: Show that a whole number is even exactly if its ones digit is even.

To approach this problem, we are going to use the piles-of-two definition along with base ten bundling. Suppose that we have A beans. Place the beans into base ten bundles. This means we have piles of 10, 100, 1000, and so on along with a pile of beans equal to the ones place of A. Place each of the large piles (those of size 100, 1000, 10000…) into piles of 10. Now we have all of our beans in piles of 10 along with one pile equal to the ones place of A. Now place each pile of 10 into 5 piles of two. To finish placing our beans into piles of two, we must place the ones pile into piles of two. Thus, we can place all A beans into piles of two exactly if we can place a pile equal to the ones digit of A into piles of two. Therefore, A is even exactly if the ones digit of A is even.

Another Characterization of Odd

Our last problem gives us an easy way to check to see if any number is odd or even. By trial and error placing beans in piles, we can see that 0, 2, 4, 6, and 8 are even while 1, 3, 5, 7, and 9 are odd. Thus, a whole number is even if it ends in 0, 2, 4, 6, or, 8, and it is odd if it ends in 1, 3, 5, 7, or 9. This gives the following nifty fact.

Nifty Fact: A whole number is odd if and only if it is not even.

The words “if and only if” here mean that if a number is odd then it is not even, and if a number is not even then it is odd. This nifty fact is something we have been aware of since our early days of arithmetic, but only because some told us it was true. Actually proving it is true requires a bit of work.

Divisibility

Suppose that A, B, and C are counting numbers so that $A \times B = C$. Then we say that C is a **multiple** of A and B and that C is **divisible** by both A and B. Also, we say that A and B are **factors** or **divisors** of C and that A and B both **divide** C. For emphasis, we may also say that C is **evenly divisible** by both A and B or that A and B **divide** C **evenly**. Here are slightly more explicit definitions of these terms one at a time:

- A counting number C is a **multiple** of a counting number A if there is a counting number B so that $A \times B = C$.
- A counting number C is **divisible** by a counting number A if there is a counting number B so that $A \times B = C$.
- A counting number A is a **factor** or **divisor** of a counting number C if there is a counting number B so that $A \times B = C$.
- A counting number A **divides** a counting number C if there is a counting number B so that $A \times B = C$.

For example, since $4 \times 6 = 24$ we can make all of these statements:

- 24 is a multiple of 4.
- 24 is a multiple of 6.
- 24 is divisible by 4.
- 24 is divisible by 6.
- 4 is a factor of 24.
- 6 is a factor of 24.
- 4 divides 24.
- 6 divides 24.

Note: If $A \times B = C$ then $B = C \div A$. Therefore, to see if a counting number A is a factor of a counting number C, we will try to divide C by A and see if we get a counting number B as a quotient.

Problem: Find all of the factors of 48.

The approach that we will take to this problem is called brute force. We will try dividing 48 by 1, 2, 3… and see which numbers are factors. Luckily, we will discover the factors in pairs, and we will be able to exploit a pattern to limit how much work we do.

$48 \div 1 = 48$ so $1 \times 48 = 48$ and thus **1** and **48** are factors of 48.
$48 \div 2 = 24$ so $2 \times 24 = 48$ and thus **2** and **24** are factors of 48.
$48 \div 3 = 16$ so $3 \times 16 = 48$ and thus **3** and **16** are factors of 48.
$48 \div 4 = 12$ so $4 \times 12 = 48$ and thus **4** and **12** are factors of 48.
$48 \div 5 = 9.6$ and since 9.6 is not a counting number, 5 is not a factor of 48.
$48 \div 6 = 8$ so $6 \times 8 = 48$ and thus **6** and **8** are factors of 48.
$48 \div 7$ is not a counting number, so 7 is not a factor of 48.
$48 \div 8 = 6$ so $8 \times 6 = 48$ and thus **8** and **6** are factors of 48.

Notice the factors here written in bold. They form two columns. The factors in the left hand column are increasing. The factors in the right hand column are decreasing. Any new factor in the left hand column would be greater than 8, but we have already encountered 8 in the right hand column. We know that we have all factors greater than 8, so there can be no new factors in the left hand column. In the right column, any new factor would be less than 6. However, we have already found all of the factors less than 6 in the left column, so there can be no new factors in the right column either. Once we encounter a repetition in this process, we can stop. The factors of 48 are: 1, 2, 3, 4, 6, 8, 12, 16, 24, and 48.

Problem: Pencils come in packages of 48. Lana bought one package of pencils and gave them to her students. She gave every student the same number of pencils and had no pencils left over. How many students might Lana have?

Suppose that Lana has A students and that each student gets B pencils. This means that Lana has A groups of B pencils adding up to 48. That is, $A \times B = 48$. Therefore, the number of students is a factor of 48. This means that the number of students that Lana has is one of the numbers 1, 2, 3, 4, 6, 8, 12, 16, 24, or 48.

Problem: Erasers come in packages of 12. Sam bough several packages of erasers and gave one eraser to each of his students. Every student received exactly one eraser, and there were no erasers left over. How many students might Sam have?

In this problem, the number of students will have to be a multiple of 12. The possible numbers of students are 12, 24, 36, 48…

Problem: Agatha has 48 marbles which she is going to place into groups that are all the same size. How many different ways can she do this?

If Agatha uses A groups and places B marbles in each group, then $A \times B = 48$. This means that her options are:

1 group of 48
2 groups of 24
3 groups of 16
4 groups of 12
6 groups of 8
8 groups of 6
12 groups of 4
16 groups of 3
24 groups of 2
48 groups of 1

Thus, there are 10 ways that Agatha can group her marbles.

Problem: How many rectangles are there whose length and width are a counting number of inches and whose area is 48 square inches.

The length and width should be counting numbers whose product is 48. Our possibilities then are:

1 inch by 48 inches
2 inches by 24 inches
3 inches by 16 inches
4 inches by 12 inches
6 inches by 8 inches
8 inches by 6 inches
12 inches by 4 inches
16 inches by 3 inches
24 inches by 2 inches
48 inches by 1 inch

However, a rectangle that measures 4 inches by 12 inches is *the same as* a rectangle that measures 12 inches by 4 inches. Therefore, our possibilities are actually only these:

1 inch by 48 inches
2 inches by 24 inches
3 inches by 16 inches
4 inches by 12 inches
6 inches by 8 inches

Thus there are five such rectangles.

Whole numbers and Integers

We have stated the definitions of factor and multiple using counting numbers (1,2,3 …) because most of our applications will have to do with counting numbers. However, we could just as easily define these concepts for whole numbers (0,1,2,3, …) or for integers $(\ldots -3, -2, -1, 0, 1, 2, 3, \ldots)$. Note that if we do so, then 0 is a multiple of every integer since $A \times 0 = 0$. Also note that if an integer C is a multiple of an integer A, then so is the opposite $-C$.

Problem: Clarence thinks that 8 is a multiple of 10 because $10 \times \frac{4}{5} = 8$. Is he correct?

Clarence is not correct. We only talk about multiples and factors in the realm of integers. The number $\frac{4}{5}$ is not an integer, and there is no integer by which we can multiply 10 and get 8.

Odd and Even Numbers

Earlier we offered three possible definitions of an even whole number:

1. A whole number N is **even** if N objects can be placed into two equal sized groups with none left over.
2. A whole number N is **even** if N objects can be placed into groups of size two with none left over.
3. A whole number N is **even** if there is a whole number k so that $N = 2k$.

The last option was the only option that can be applied to all integers. This version should make it clear that an integer is even if and only if that integer is a multiple of 2 (or is divisible by 2). In particular, this implies that 0 is even.

Divisibility Tests

It is helpful if we can look at a whole number and tell quickly what small whole numbers it is divisible by. For example, we have seen that we can tell if a whole number is divisible by 2 (is even) by looking at its ones digit. We have quick tests for divisibility for each whole number up to 10. We will state the tests here, give examples using the tests, and then explain why some of the tests work later.

Number	Divisibility Test
2	A number is divisible by 2 if its last digit is 0, 2, 4, 6, or 8.
3	A number is divisible by 3 if the sum of its digits is divisible by 3.
4	A number is divisible by 4 if the number formed by its last two digits is divisible by 4.
5	A number is divisible by 5 if its last digit is 0 or 5.
6	A number is divisible by 6 if it is divisible by both 2 and 3.
7	To test a number by divisibility by 7, remove the last digit from the number and subtract twice this digit from the number formed by the remaining digits. Repeat this process until the number is small enough to work with. Either the new number and the original number are both divisible by 7 or neither is.
8	A number is divisible by 8 if the number formed by its last three digits is divisible by 8.
9	A number is divisible by 3 if the sum of its digits is divisible by 9.
10	A number is divisible by 10 if its last digit is 0.

Problem: Test this number for divisibility by the numbers 2 through 10: 321,465,987,312.

For divisibility by 3 and 9, we will need the sum of the digits:

$$3 + 2 + 1 + 4 + 5 + 6 + 9 + 8 + 7 + 3 + 1 + 2 = 51$$

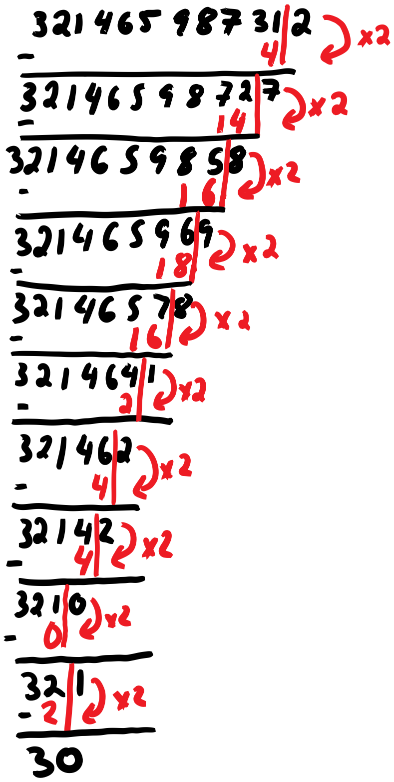
Now we can look at each divisibility test.

Number	Divisible?	Reason
2	Yes	The last digit is 2.
3	Yes	The sum of the digits is 51, and 51 is a multiple of 3.
4	Yes	Last two digits form the number 12, which is a multiple of 4.
5	No	The last digit is 2, not 0 or 5.
6	Yes	51 is divisible by 2 and by 3.
7	No	See below.
8	Yes	Last three digits are 312 and 312 is a multiple of 8. $(8 \times 39 = 12)$
9	No	The sum of the digits is 51, and 51 is not a multiple of 9.
10	No	The last digit is 2, not 0.

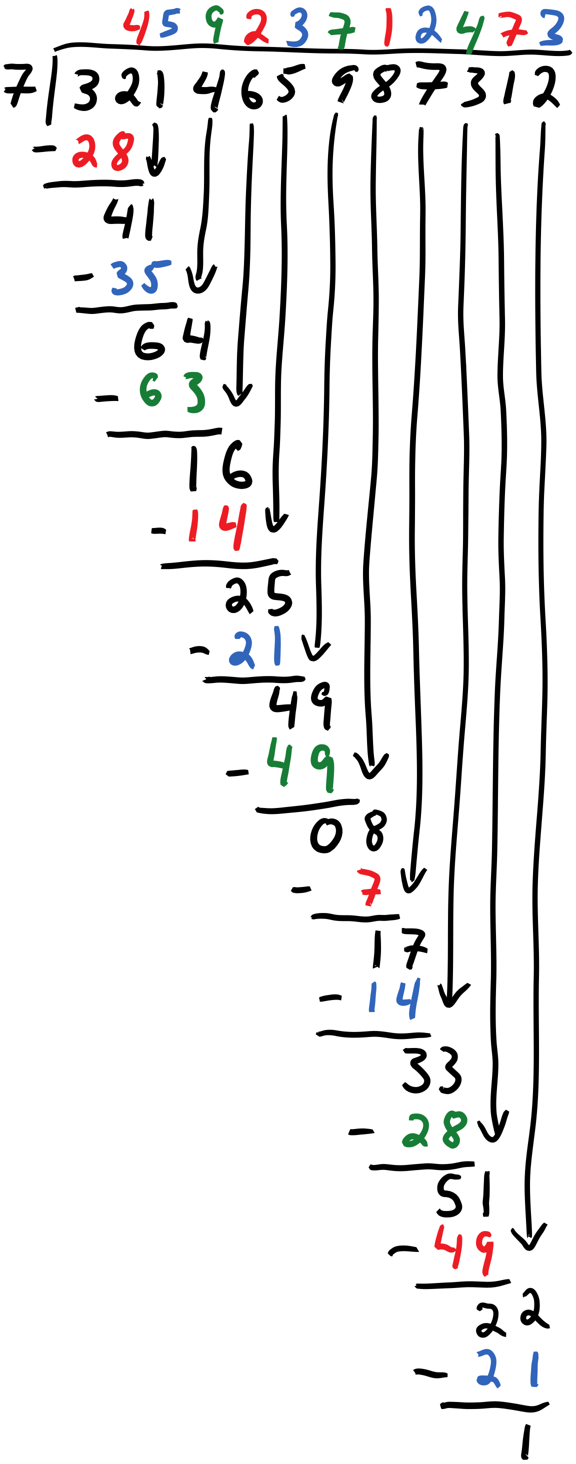
The test for divisibility by 7 is a bit more complicated, so we do it here by itself. First, we cut the last digit off of the number, double this last digit, and subtract it from the remaining digits:

$$\begin{array}{rl} 321\,465\,987\,31 & |\,2 \;\; \curvearrowright \times 2 \\ - \qquad\qquad\qquad 4 & | \\ \hline 321465987\,27 & \end{array}$$

If the first number was too large to see clearly if it is divisibility by 7, then this probably is too. Therefore, we repeat that process over and over until we have a small number:



Since 30 is not a multiple of 7, the original number is not a multiple of 7 either. This test for divisibility by 7 is somewhat tedious. For comparison, here we simply divide the number by 7 so that you can see the difference in the amount of writing:



Since we have a remainder of 1, the number is not divisible by 7. This is about the same amount of work as the divisibility test. However, at each stage of the divisibility test, all we did was double and subtract. At each stage of the division algorithm, we divide, multiply, and subtract. The test is just as long but a little simpler.

Explanations of Divisibility Tests

For divisibility by 2, 4, 5, 8, and 10, we look at the last 1, 2, or 3 digits of a number. The explanations for how to do this are all similar, and they all look like the explanation we saw for why we can determine if a number is even simply by looking at the last digit. The explanations for 2, 5, and 10 are almost verbatim of each other (just changing numbers). The explanations for 4 and 8 are very similar. We will offer an explanation for the test for divisibility by 4.

Suppose that we have a large pile of beans and that we want to know if the number of beans is a multiple of 4. To see if the number of beans is a multiple of 4, we are going to try to place the beans into piles of 4 (since a number is a multiple of 4 if and only if that number of beans can be placed into piles of 4 with none left over).

- First, place the beans into base ten bundles. That is, we have bundles of size 10, 100, 1000, 10000… and a pile for the ones place.
- Break each of the bundles of size 100, 1000, 10000, 100000… into piles of 100 and put the pile of 10 and the ones place into one pile. We now have piles of 100, and we have a pile with a number of beans equal to the number formed by the last two digits of our original number.
- Break each pile of 100 into 25 piles of 4. We now have piles of 4 beans and one pile of beans equal to the number formed by the last two digits.
- We can finish placing the beans into piles of 4 exactly if we can place the remaining pile (equal to the last two digits) into piles of 4.

Thus, the original number is a multiple of 4 exactly if the number formed by the last two digits is a multiple of 4. The explanations for 2, 5, 8, and 10 are similar.

We now turn our attention to the tests for divisibility for 3 and 9. The two explanations are similar, so we will only address 3. Also, we will just look at a three digit number to keep things simple. Suppose that ABC is a three digit number. This means that A, B, and C are digits and our number is

$$\begin{aligned} ABC &= A \times 100 + B \times 10 + C \\ &= A \times (99 + 1) + B \times (9 + 1) + C \\ &= A \times (3 \times 33 + 1) + B \times (3 \times 3 + 1) + C \\ &= 3 \times 33 \times A + A + 3 \times 3 \times B + B + C \\ &= 3 \times (33 \times A + 3 \times B) + A + B + C \end{aligned}$$

Now suppose that we have ABC beans and that we want to place them into piles of 3 (to see if ABC is divisible by 3). We can first split the beans into two piles, one with the size of the red number here and one with the size of the blue. The beans in the red numbered pile can be placed into $(33 \times A + 3 \times B)$ piles of size 3. To place all of the beans into piles of 3, we have to be able to place the beans in the blue pile into piles of three. This means that the number of beans in the blue pile must be divisible by 3. This number is exactly the sum of our digits.

Since $6 = 2 \times 3$, every multiple of 6 is also a multiple of 2 and of 3. The fact that every number which is a multiple of both 2 and 3 is also a multiple of 6 follows from the fact that 2 and 3 are prime – a notion we will investigate in the next section. The explanation of the divisibility test by 7 is beyond the scope of what we want to investigate here.

Prime Numbers

Prime numbers have been the primary object of study in number theory for hundreds of years. They are the source of many ongoing endeavors in mathematics, and they are essential to modern tools used for internet security.

A counting number other than 1 is **prime** if its only factors are itself and 1. A counting number other than 1 which is not prime is **composite**. It follows from this definition that composite numbers can be factored in *interesting* ways while prime numbers cannot. The primary reason we care about prime numbers is the Fundamental Theorem of Arithmetic:

Fundamental Theorem of Arithmetic: Every counting number other than 1 can be factored into primes in a unique way.

The object of number theory is to study arithmetic properties of counting numbers. The Fundamental Theorem of Arithmetic declares that prime numbers are the basic building blocks of counting numbers. If we understand prime numbers, and if we understand multiplication, we can understand the counting numbers.

The word "unique" in the Fundamental Theorem of Arithmetic means that there is only one way that any number can be factored into prime numbers. It happens to be that 2, 3, and 5 are prime and that $60 = 2 \times 2 \times 3 \times 5$. If we factor 60 into primes, we will always get two 2s, a 3, and a 5. The only thing that may vary is the order and notation.

Omitting One

We disregard the number 1 when we define prime and composite. It is tempting to include 1 as a prime number since it has no interesting factors. In fact, hundreds of years ago, some mathematicians did. However, modern mathematicians do not declare 1 to prime or composite. The number 1 should clearly not be composite because it has no interesting factors. The reason that we do not declare 1 to be prime is that this would violate the uniqueness guaranteed by the Fundamental Theorem of Arithmetic. If 1 were prime, then $60 = 2 \times 2 \times 3 \times 5$ and $60 = 1 \times 2 \times 2 \times 3 \times 5$ would be two different prime factorizations of 60. To avoid that confusion, we avoid 1.

Finding Primes

Eratosthenes was a Greek mathematician who lived during the third century before Christ. He was active in mathematics, astronomy, geography, poetry, and music. One of the feats that Eratosthenes is most famous for is his use of geometry to estimate the circumference of the Earth (yes, scholars thought the Earth was round in the third century BC). Eratosthenes gave us a simple method for finding primes that is still taught today. This is the **Sieve of Eratosthenes**. We will demonstrate how to use the Sieve to find all of the primes less than 50. First, we list the numbers through 50 and cross out 1.

■	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Now we will repeat these steps:

1. Circle or highlight the first number in the list that is not crossed out.
2. Cross out every higher multiple of the number just circled.

We repeat these steps until every number in our list is either highlighted or crossed out. At this point, the first number not crossed out is 2, so we highlight 2 and cross out higher multiples of 2.

■	2	3	■	5	■	7	■	9	■
11	■	13	■	15	■	17	■	19	■
21	■	23	■	25	■	27	■	29	■
31	■	33	■	35	■	37	■	39	■
41	■	43	■	45	■	47	■	49	■

Now, we highlight 3 (the next unmarked number) and cross out the multiples of 3.

■	2	3	■	5	■	7	■	■	■
11	■	13	■	■	■	17	■	19	■
■	■	23	■	25	■	■	■	29	■
31	■	■	■	35	■	37	■	■	■
41	■	43	■	■	■	47	■	49	■

The next unmarked number is 5, so we highlight 5 and cross out the other multiples of 5.

■	2	3	■	5	■	7	■	■	■
11	■	13	■	■	■	17	■	19	■
■	■	23	■	■	■	■	■	29	■
31	■	■	■	■	■	37	■	■	■
41	■	43	■	■	■	47	■	49	■

The next unmarked number is 7, so we highlight 7 and cross out the remaining multiples – which is just 49.

■	2	3	■	5	■	7	■	■	■
11	■	13	■	■	■	17	■	19	■
■	■	23	■	■	■	■	■	29	■
31	■	■	■	■	■	37	■	■	■
41	■	43	■	■	■	47	■	■	■

At this point, all of the multiples of the remaining unmarked numbers are already crossed out, so if we continue this process, we will end up simply highlight the remaining numbers.

■	2	3	■	5	■	7	■	■	■
11	■	13	■	■	■	17	■	19	■
■	■	23	■	■	■	■	■	29	■
31	■	■	■	■	■	37	■	■	■
41	■	43	■	■	■	47	■	■	■

Thus, the primes less than 50 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47.

Looking at the Sieve, we can immediately see some questions that mathematicians have pondered for thousands of years. The primes 3 and 5 are two consecutive odd numbers, as are the primes 5 and 7, 11 and 13, 17 and 19, and 41 and 43. These are called **twin primes**. One open question in number theory is, how many twin primes are there? The primes 3, 13, and 23 are three vertically adjacent primes in the Sieve. How often does this happen? Can you have four vertically adjacent primes? What is the highest number of vertically adjacent primes you can have?

Primality Testing

Given a counting number, we may want to know whether or not it is prime. If the number is not prime, then the Fundamental Theorem of Arithmetic declares that it should be divisible by a prime, so to check for primality, we will simply try to divide our number by the primes less than it. If we ever get a whole number quotient, the number is not prime. Otherwise, it is prime. This method is sometimes called **trial division**. It is another brute force technique.

Problem: Determine if 389 is prime or not.

We use trial division and attempt to divide 389 by primes.

$$\begin{aligned} 389 \div 2 &= 195.5 \\ 389 \div 3 &= 129.\overline{6} \\ 389 \div 5 &= 77.8 \\ 389 \div 7 &= 55.\overline{571428} \\ 389 \div 11 &= 35.\overline{36} \\ 389 \div 13 &= 29.\overline{923076} \\ 389 \div 17 &= 22.88235 \ldots \\ 389 \div 19 &= 20.47368 \ldots \\ 389 \div 23 &= 16.913043 \ldots \end{aligned}$$

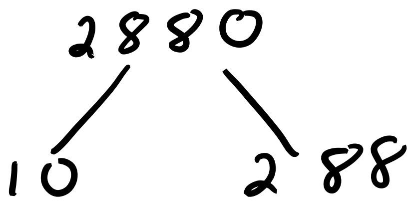
The question now becomes where to stop. We could try all of the primes up to 389, but this requires knowing those primes. It is also tedious. If we consider our work so far, we see that the primes we are dividing by are increasing while the quotients are decreasing. This is a similar situation to when we were looking for all factors of a number before. When the primes we are dividing by outgrow the quotients, we can stop. The number 389 is prime.

Factor Trees

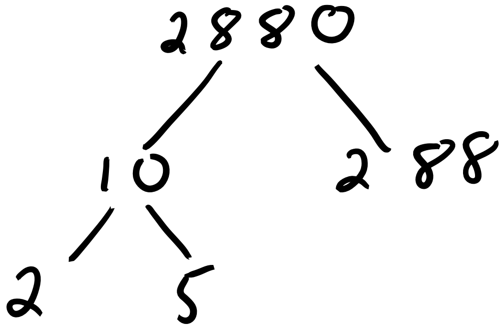
The Fundamental Theorem of Arithmetic declares that we can factor any counting number into primes. We demonstrate how to do that here using a factor tree. The method is to begin with a number and factor it into a product of two numbers. We then repeat the process and factor each of those numbers, and we repeat until we cannot factor any of our numbers. We organize our work in a diagram that is called a tree. It grows by branching, but it branches down rather than up.

Problem: Find the prime factorization of 2880.

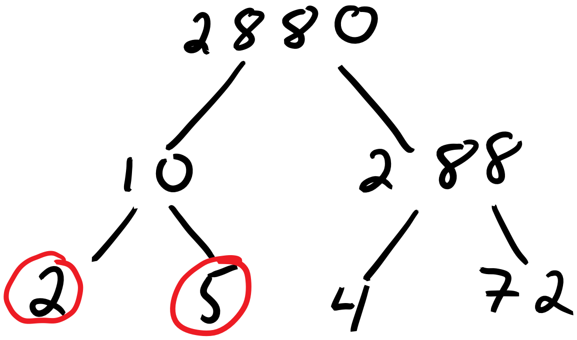
We first factor 2880 any way we can. Since the number ends in 0, we know that 10 is a factor, so we begin with $2880 = 10 \times 288$. We do not have to start with 10, any other legitimate factor would give us the same final answer. Graphically, we demonstrate that $2880 = 10 \times 288$ by drawing branches down from 2880 to 10 and 288.



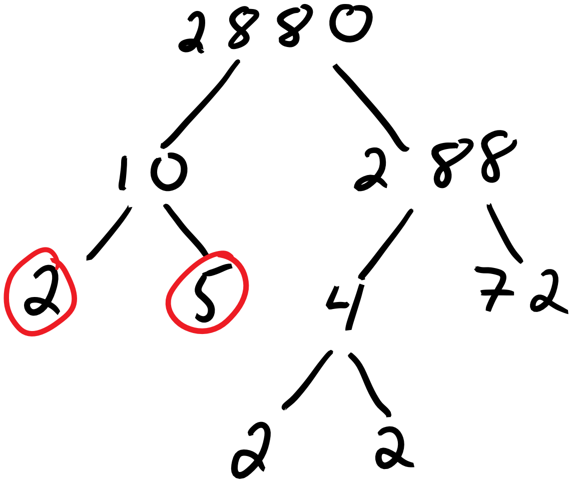
Next, we factor 10.



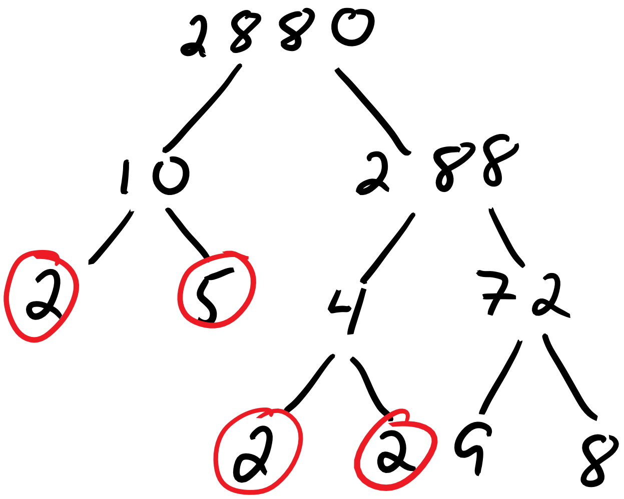
Since 2 and 5 are prime, we circle them and factor 288.



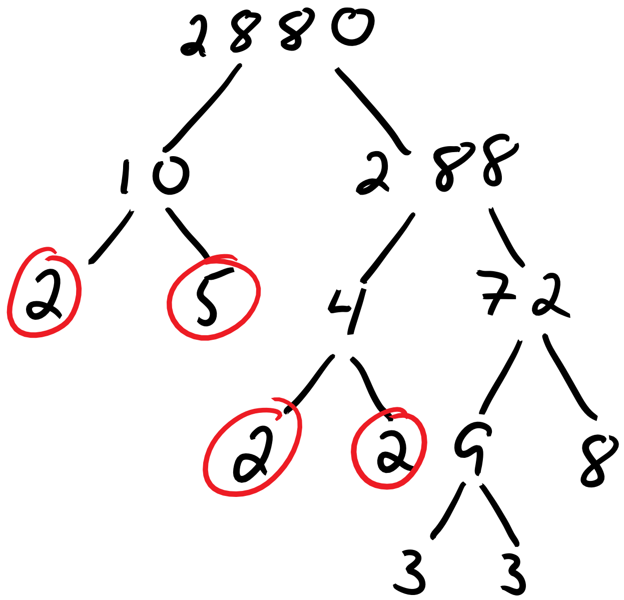
Next, we factor 4.



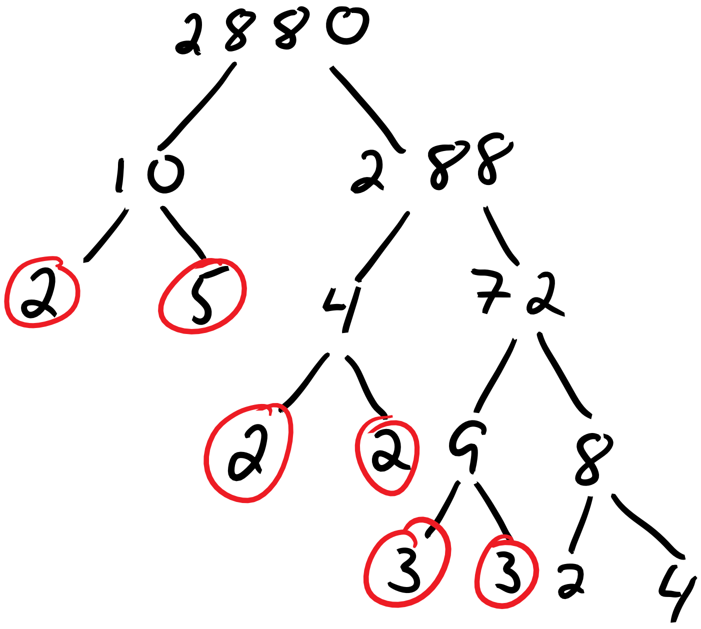
Since 2 is prime, we circle the 2s and factor the 72.



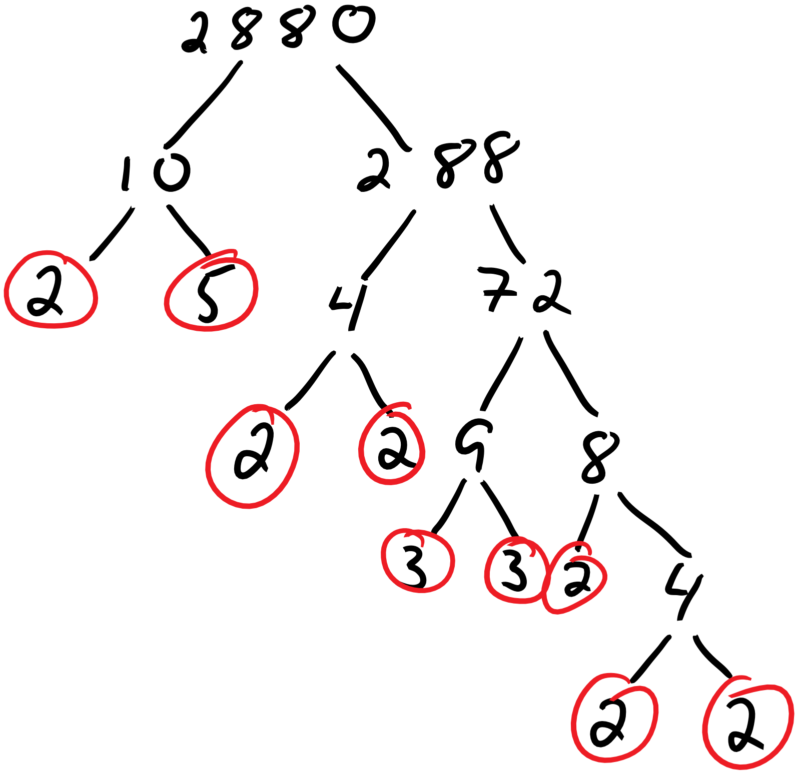
We factor 9.



Since 3 is prime, we circle the 3s and factor 8.



Now we circle the prime 2, factor 4, and circle the last two 2s.



This is our factor tree. We see that 2880 has six factors of 2, two factors of 3, and a factor of 5. We can write that using exponents: $2880 = 2^6 \times 3^2 \times 5$.

Common Factors and Multiples

Here are all of the (counting number) factors of 12 and 18:

Factors of 12: 1, 2, 3, 4, 6, 12
Factors of 18: 1, 2, 3, 6, 9, 18

Notice that some numbers (1, 2, 3, and 6) show up in both lists. These are called common factors of 12 and 18. Among these common factors, there is a largest, 6. This is the greatest common factor or GCF. If A and B are counting numbers, then any counting number which is a factor of A and a factor of B is a **common factor** of A and B. The largest number which is a common factor of A and B is the **greatest common factor** or **GCF** of A and B. Sometimes, we might express the greatest common factor of A and B as $GCF(A,B)$.

Here are all of the small multiples of 12 and 18:

Multiples of 12: 12, 24, 36, 48, 60, 72, 84, 96, 108…
Multiples of 18: 18, 36, 54, 72, 90, 108, 126…

We cannot list all of the multiples because there are infinitely many of them. Notice that there are some numbers that are in both lists (36, 72, 108…). These are common multiples. Since there are infinitely many common multiples, it makes no sense to talk about the greatest common multiple. However, there is a least number in both lists, 36. This is the least common multiple. If A and B are counting numbers, then any counting number which is a multiple of A and a factor of B is a **common multiple** of A and B. The least number which is a common multiple of A and B is the **least common multiple** or **LCM** of A and B. Sometimes, we might express the least common multiple of A and B as $LCM(A,B)$. Our first order of business in this section is addressing how to find GCFs and LCMs. We will have three methods – brute force, prime factorizations, and the slide method.

Finding GCFs and LCMs with Brute Force

One way to find the GCF of two numbers is to list all factors of both number, identify the common factors, and select the greatest among these. Similarly, one way to find the LCM of two numbers is to list all small multiples of the numbers, identify common factors, and select the least of these (which, incidentally, will be the first common factor encountered). This method is another brute force method.

Problem: Find the GCF and LCM of 54 and 72 by listing factors and multiples.

The factors of 54 and 72 are:

Factors of 54: **1**, **2**, **3**, **6**, **9**, **18**, 27, 54
Factors of 72: **1**, **2**, **3**, 4, **6**, 8, **9**, 12, **18**, 24, 36, 72

The common factors are shown in bold, and the greatest of these is 18. Thus $GCF(54,72) = 18$. Now, the small multiples of 54 and 72 are"

Multiples of 54: 54, 108, 162, **216**, 270, 324, 378, **432**, 486, 540, 594, **648**…
Multiples of 72: 72, 144, **216**, 288, 360, **432**, 504, 576, **648**, 720…

The common multiples are shown in bold, and the least of these is 216. Thus $LCM(54,72) = 216$.

Problem: Consider the common multiples of 12 and 18 above and the common multiples of 54 and 72 above. Can you see a relationship between these and the least common multiple?

The least common multiple of 12 and 18 is 36, and the common multiples are 36, 72, 108… The least common multiple of 54 and 72 is 216, and the common multiples are 216, 432, 648… It appears that the common multiples are exactly the multiples of the least common multiple. This happens to be the case always.

Using Prime Factorization to Find GCFs and LCMs

Consider the numbers 4200 and 4500. Finding the GCF and LCM of these numbers using brute force may be tedious. However, we can find the GCF and LCM using prime factorizations. The prime factorizations of 4200 and 4500 are

$$4200 = 2^3 \times 3 \times 5^2 \times 7 \; and \; 4500 = 2^2 \times 3^2 \times 5^3$$

(we could have used a factor tree to find these). These factorizations give us information about the prime factorizations of factors of these two numbers:

Any factor of	$2^3 \times 3 \times 5^2 \times 7$	$2^2 \times 3^2 \times 5^3$
May contain	Up to 3 factors of 2	Up to 2 factors of 2
	Up to 1 factor of 3	Up to 2 factors of 3
	Up to 2 factors of 5	Up to 3 factors of 5
	Up to 1 factor of 7	

The greatest common factor should contain the largest permissible number of each factor. For 2, the largest number which is "Up to 3" and "Up to 2" is 2, so the GCF should have a factor of 2^2. Similarly, for 3, the largest number which is "Up to 1" and "Up to 2" is 1, so the GCF should have a factor of 3. For 5, the largest number which is "Up to 2" and "Up to 3" is 3, so the GCF should have a factor of 5^2. Finally, the number on the right has no factor of 7, so the GCF cannot have a factor of 7. Thus, the GCF of 4200 and 4500 is $2^2 \times 3 \times 5^2 = 300$. Let us recap that process:

To find the GCF of two numbers using prime factorizations: Find the prime factorizations of both numbers. List those primes which show up in both factorizations. On each of these primes, place the lower of the exponents which appear in the two factorizations.

Let us now turn to least common multiples. Since

$$4200 = 2^3 \times 3 \times 5^2 \times 7 \; and \; 4500 = 2^2 \times 3^2 \times 5^3$$

we can derive this information about the prime factorizations of multiples of these two numbers:

Any multiple of	$2^3 \times 3 \times 5^2 \times 7$	$2^2 \times 3^2 \times 5^3$
Must contain	At least 3 factors of 2	At least 2 factors of 2
	At least 1 factor of 3	At least 2 factors of 3
	At least 2 factors of 5	At least 3 factors of 5
	At least 1 factor of 7	

The smallest number which satisfies all of these conditions is

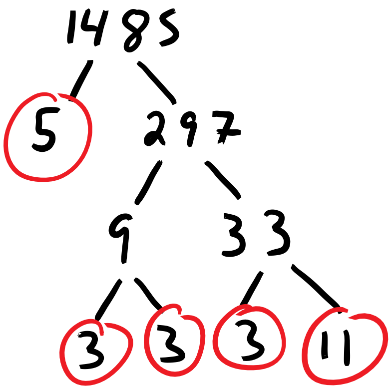
$$2^3 \times 3^2 \times 5^3 \times 7 = 63000$$

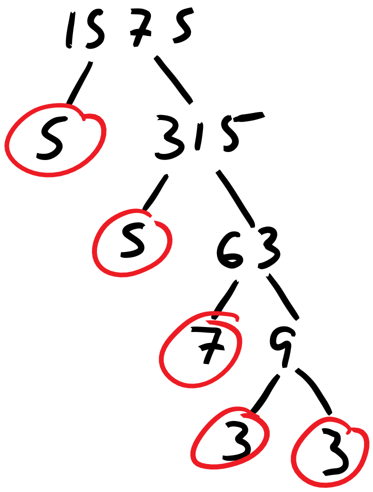
so the least common multiple of 4200 and 4500 is 63000. Let us recap that process:

To find the LCM of two number using prime factorizations: Find the prime factorizations of both numbers. List those primes that show up in one factorization or the other or both. On each of these primes, place the higher of the exponents that appear in the two factorizations.

Problem: Find the GCF and LCM of 1485 and 1575 using prime factorization.

First, we draw factor trees to find the prime factorizations of 1485 and 1575.





Thus, $1485 = 3^3 \times 5 \times 11$ and $1575 = 3^2 \times 5^2 \times 7$. For the GCF, we list the primes which the two numbers have in common: $GCF = 3^? \times 5^?$. For exponents, we select the lower of the exponents from the factorizations we found, so $GCF = 3^2 \times 5^1 = 45$. For the LCM, we list all of the primes which appear in one factorization or the other or both: $LCM = 3^? \times 5^? \times 7^? \times 11^?$. Then, for each prime, we select the higher of the exponents that appear in the factorizations, so $LCM = 3^3 \times 5^2 \times 7 \times 11 = 51975$.

Using the Slide Method to find GCFs and LCMs

The slide method is usually most students' favorite method for finding GCFs and LCMs. We illustrate by finding the GCF and LCM of 54 and 72 (which we already know from above). First, we write down the two numbers we are considering.

54 72

Next, we think of any common factor of the two numbers and write it to the left of the numbers. We choose 2, but any common factor will work.

2 54 72

This process starts to build three columns. Now, we divide the two numbers in the right-hand columns by the common factor in the left hand column to get $52 \div 2 = 27$ and $72 \div 2 = 36$. We place the quotients 27 and 36 beneath the numbers 54 and 72.

2	54	72
	27	36

We now repeat the process. Find a common factor of the two numbers at the bottoms of the two right hand columns, write it in the left hand column, and divide by it. We choose to use the common factor 3. Any common factor will work. We place $9 = 27 \div 3$ and $12 = 37 \div 3$ beneath 27 and 36.

2	54	72
3	27	36
	9	12

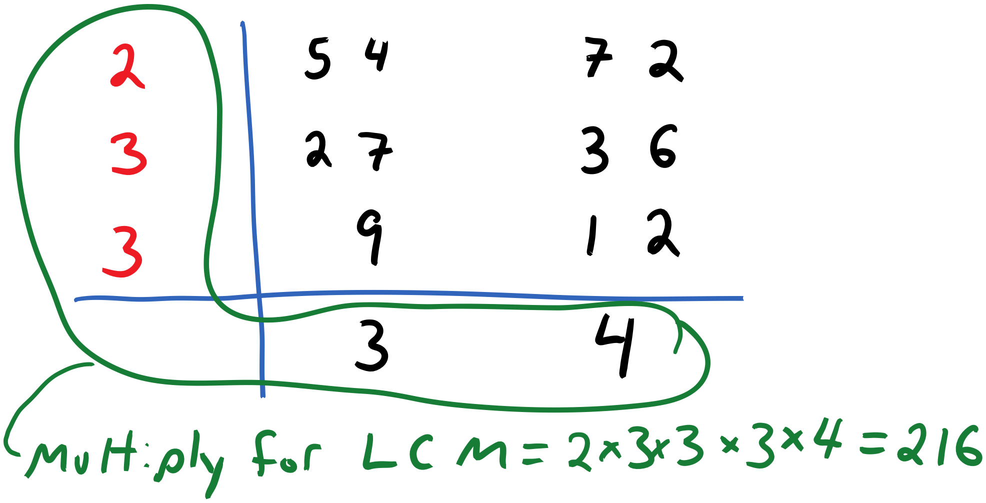
Again, we repeat the process. Find a common factor of the two numbers at the bottoms of the two right hand columns, write it in the left hand column, and divide by it. We choose to use the common factor 3. Any common factor will work.

2	54	72
3	27	36
3	9	12
	3	4

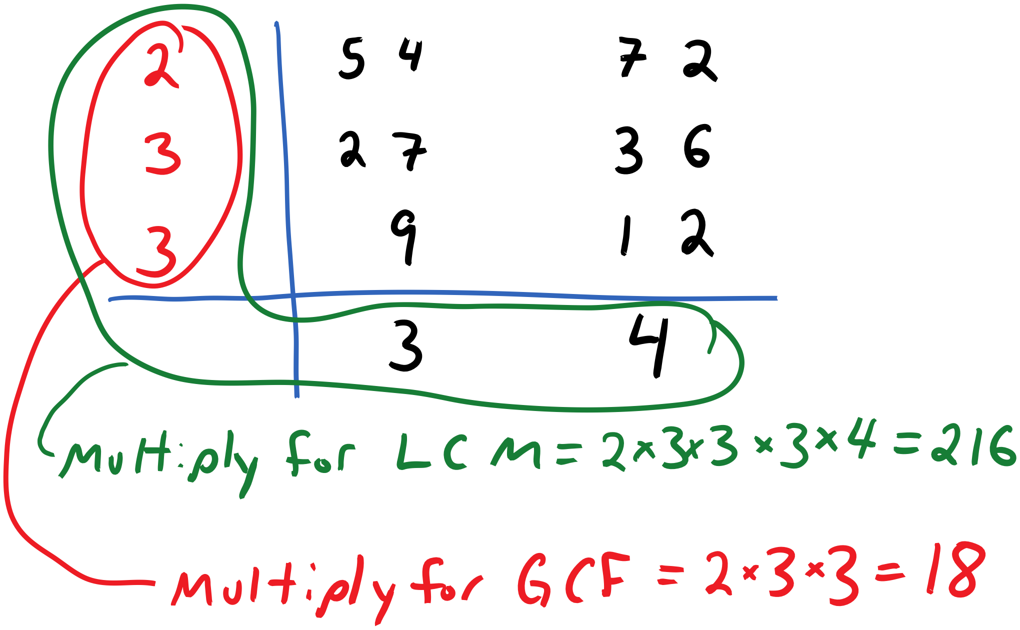
At this point, the two bottom numbers have no common factor other than 1, so we cannot continue the process. We draw a vertical line to separate the left hand column of common factors and a horizontal line to separate the bottom row.

2	54	72
3	27	36
3	9	12
	3	4

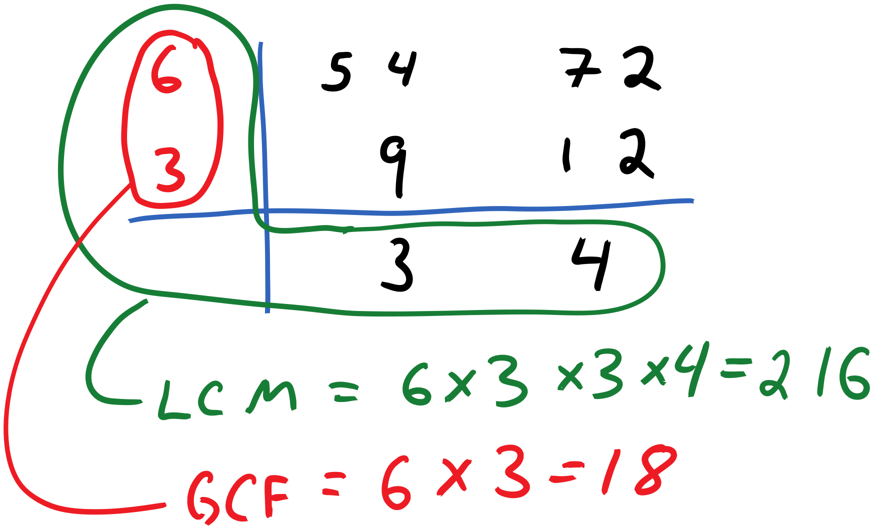
The idea now is that we have divided out everything the two numbers have in common, and these common values appear in the left hand column. What is left over, the bottom row, represents everything that is unique to the two numbers. To find the least common multiple, we multiply everything that the numbers have in common (the left column) along with everything that is unique to the numbers (the bottom row).



To find the greatest common factor, we multiple what the numbers have in common (the left column).

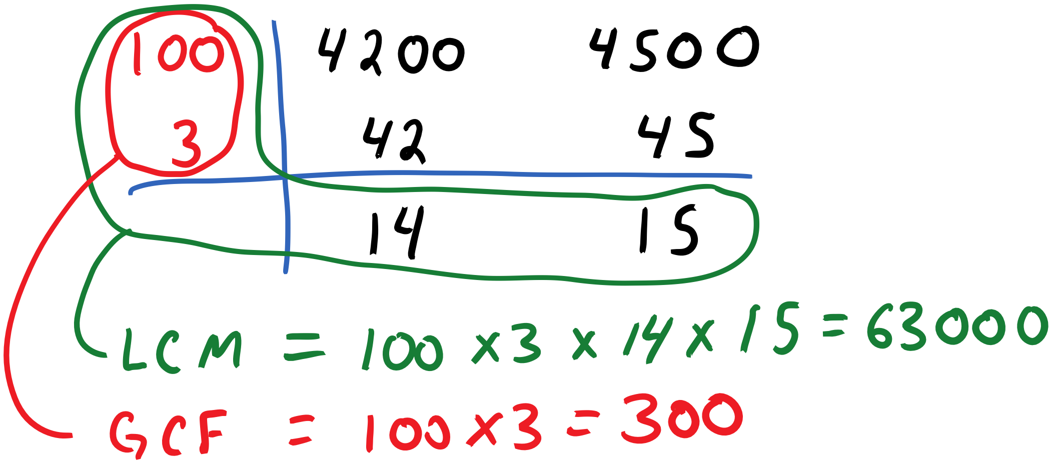


The choices we made for common factors in this process are almost irrelevant (as long as they are actually common factors). The larger the factors that are chosen, the faster the process will go. The smaller the factors are, the easier the dividing will be. Here is the process for the same numbers with a different choice of common factors.



Problem: Use the slide method to find the GCF and LCM of 4200 and 4500.

Notice that since 4200 and 4500 are obviously by divisible by 100, and since 100 is a somewhat large factor, we start by dividing by 100.



Luckily, this agrees with our work above.

Problem: Pencils come in packages of 24, and erasers come in packages of 16. Ariel wants to buy the same number of pencils and erasers. What is the fewest number of pencils and erasers she can buy? How many packages of pencils is this? How many packages of erasers is this?

The number of pencils that Ariel buys is a multiple of 24. The number of erasers she buys is a multiple of 16. Therefore, we are looking for a common multiple of 24 and 16. The smallest such number is the least common multiple of 24 and 16. We choose to find this by brute force. We list the small multiples of 24 and 16:

Multiples of 16: 16, 32, 48, 64, 80…
Multiples of 24: 24, 48, 72, 96…

The smallest number in each list is 48, so Ariel is going to buy 48 pencils and 48 erasers. Since pencils come in packs of 24, she will need $48 \div 25 = 2$ packages of pencils. Since erasers come in packs of 16, she will need $48 \div 16 = 3$ package of erasers.

Problem: A class is clapping and snapping to a steady beat. Half of the class is following this pattern:

snap, snap, snap, clap, snap, snap, snap, clap, snap, snap, snap, clap…

The other half of the class is following this pattern

snap, snap, clap, snap, snap, clap, snap, snap, clap…

On which beats will both groups clap together?

The first group is clapping on every fourth beat, so the beats on which the class clap together are multiples of 4. The second group is clapping on every third beat, so the beats on which the class clap together are multiples of 3. Therefore, they will clap together on those beats which are common multiples of 3 and 4. These are beats 12, 24, 36, 48… which are all multiples of 12.

Problem: Jasmine is making a quilt which is going to be three feet by five feet. The quilt is going to be made of squares, and she wants to use no fractional squares. She wants the dimensions of each square to be a whole number of inches to simplify measuring. What is the largest square she could use?

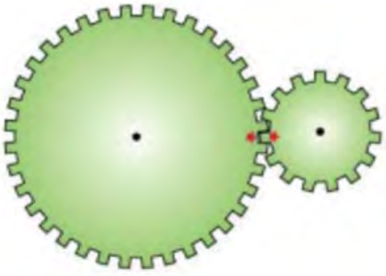
The quilt will measure 36 inches by 60 inches. If Jasmine is not going to have any fractional squares, then the dimensions of each square must be a factor of 36 and a factor of 60. The largest dimension would be the greatest common factor of 36 and 60, which we find with the slide method:

$$\begin{array}{c|cc} 6 & 36 & 60 \\ 2 & 6 & 10 \\ \hline & 3 & 5 \end{array}$$

$GCF = 6 \times 2 = 12$

The largest square she can use is 12 inches by 12 inches.

Problem: Two gears are meshed like in the picture below. The small gear has 15 teeth, and the big gear has 36 teeth. How many revolutions must the small gear make before the stars are again aligned?



The secret here is to count the number of teeth that pass the current location of the star as the gears rotate. For each revolution of the small gear, 15 teeth pass this point, so the number of teeth that pass this point before the stars align must be a multiple of 15. Similarly, for each revolution of the large gear, 36 teeth pass the location of the star, so the number of teeth that pass this point before the stars align is also a multiple of 36. The number of teeth that must pass this point before the stars align again is the least number which is a multiple of 15 and 36. This is the LCM of 15 and 36. To find this number, we use prime factorizations. The factorizations of 15 and 36 are $15 = 3 \times 5$ and $36 = 2^2 \times 3^2$. Therefore, the LCM is $LCM(15,36) = 2^2 \times 3^2 \times 5 = 180$. Thus 180 teeth must pass the center point before the stars align again. Since each revolution of the small gear accounts for 15 teeth, this is $180 \div 15 = 12$ revolutions of the small gear.

Problem: Fill in the table below and look for patterns.

A	B	$GCF(A,B)$	$LCM(A,B)$	$A \times B$	$GCF(A,B) \times LCM(A,B)$
8	10				
12	15				
8	15				
9	20				
18	24				
18	25				

Here is the table filled in:

A	B	$GCF(A,B)$	$LCM(A,B)$	$A \times B$	$GCF(A,B) \times LCM(A,B)$
8	10	2	40	80	80
12	15	3	60	180	180
8	15	1	120	120	120
9	20	1	180	180	180
18	24	6	72	432	432
18	25	1	450	450	450

We should immediately notice that the last two columns are identical. It appears as if

$$A \times B = GCF(A,B) \times LCM(A,B)$$

and this is, indeed, always true. Something else we may notice is that $LCM(A,B)$ is sometimes equal to $A \times B$, but not always. Using the equation $A \times B = GCF(A,B) \times LCM(A,B)$, we might conclude that $A \times B = LCM(A,B)$ can only happen when $GCF(A,B) = 1$, and the table supports this idea. When $GCF(A,B) = 1$, we say that A and B are **relatively prime**. In this case, A and B have no *interesting* common factors.

Least Common Multiples and Fraction Arithmetic

When simplifying fractions such as $\frac{76}{96}$, we divide the top and bottom of the fraction by common factors such as $\frac{76}{96} = \frac{76 \div 2}{96 \div 2} = \frac{38}{48}$. If the top and bottom still have a common factor, we can repeat the process: $\frac{76}{96} = \frac{38}{48} = \frac{38 \div 2}{48 \div 2} = \frac{19}{24}$. To make this process go as quickly as possible, we can divide by the GCF of the top and bottom at the beginning. It happens to be that $GCF(76,96) = 4$ and $\frac{76}{96} = \frac{76 \div 4}{96 \div 4} = \frac{19}{24}$.

When adding and subtracting fractions, we first have to get a common denominator. The common denominator should be a common multiple of the denominators of the fractions being added. Any common multiple will work. When we first started adding fractions, we simply multiplied the denominators. This might look something like this:

$$\frac{5}{12} + \frac{3}{8} = \frac{5 \times 8}{12 \times 8} + \frac{3 \times 12}{8 \times 12} = \frac{40}{96} + \frac{36}{96} = \frac{76}{96} = \frac{19}{24}.$$

Since any common factor of the denominators is adequate, we can keep our numbers small by using the least common multiple of the denominators as our common denominator. In fact, the least common multiple of the denominators of a pair of fractions is usually called the **least common denominator** or **LCD**. In this case, $LCM(12,8) = 24$ so

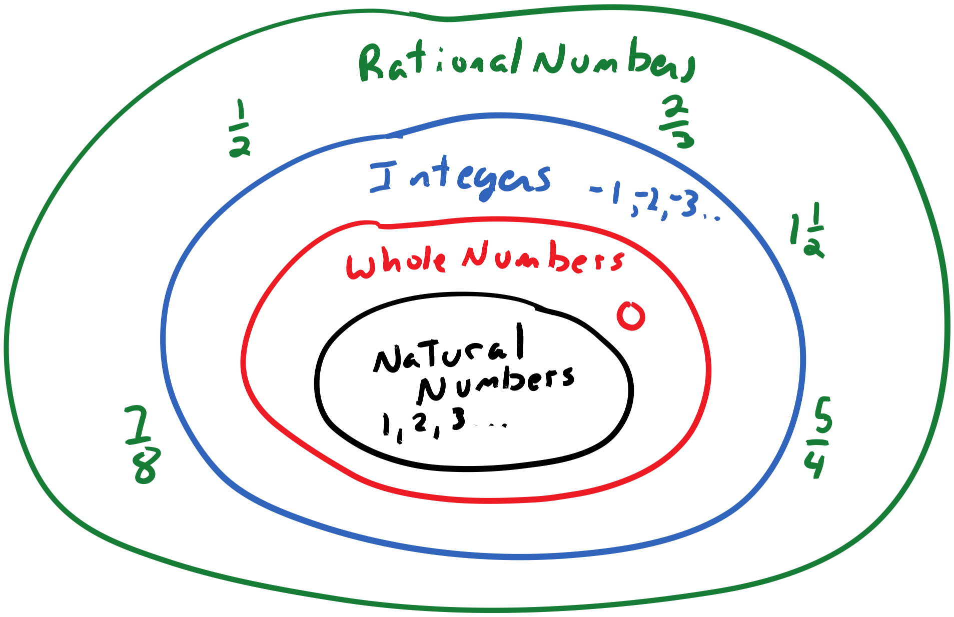
$$\frac{5}{12} + \frac{3}{8} = \frac{5 \times 2}{12 \times 2} + \frac{3 \times 3}{8 \times 3} = \frac{10}{24} + \frac{9}{24} = \frac{19}{24}.$$

Rational and Irrational Numbers

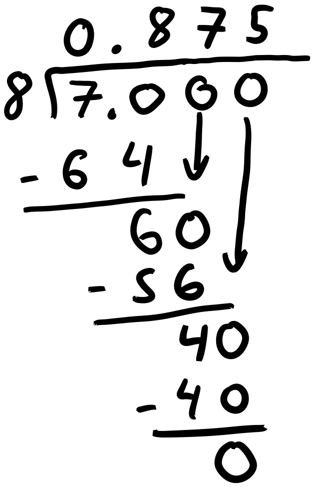
We earlier defined these number systems:

- The **counting numbers**: These are the names of the sizes of nonempty sets of objects. We denote them as 1, 2, 3, 4… The counting numbers were historically called the **natural numbers** by mathematicians.
- The **whole numbers**: These are the counting numbers along with zero. They include 0, 1, 2, 3, 4, 5… Most of our arithmetic algorithms were developed first at the level of the whole numbers.
- The **integers**: These numbers consist of the counting numbers, zero, and negatives of the counting numbers. We can list them as …-4, -3, -2, -1, 0, 1, 2, 3, 4, 5… We could also list them as 0, 1, -1, 2, -3, 3, -3, 4, -4…
- The **rational numbers**: These are all numbers which can be expressed as fractions of integers. They include all integers along with numbers like $\frac{5}{7}$, $-\frac{4}{5}$, $2\frac{1}{3}$, and $\frac{7}{3}$.

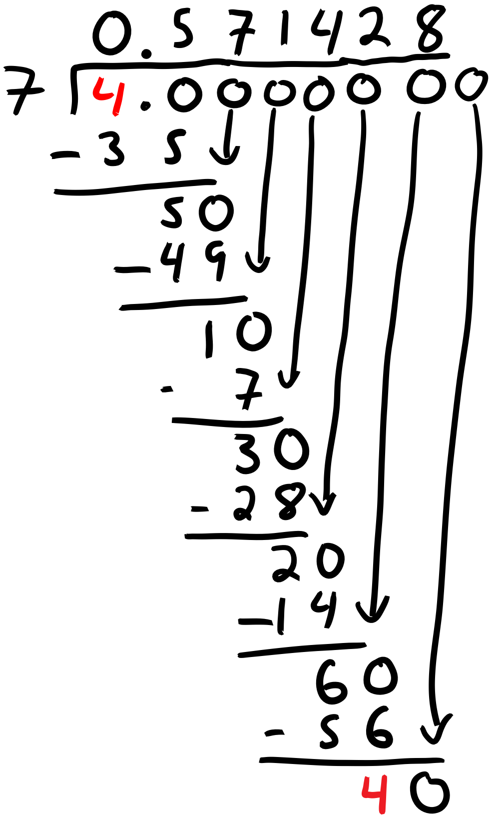
And we drew a Venn diagram illustrating the relationships between these sets of numbers:



All of the numbers in this diagram are rational and, thus, can be expressed as fractions of integers. We can also use division to express each of these fractions as decimals. For example, if we divide 7 by 8, we can get a decimal representation of $\frac{7}{8}$:



Therefore, $\frac{7}{8} = 0.875$. Since $\frac{7}{8} = 0.875$, then $4\frac{7}{8} = 4.875$. Converting fractions to decimals does not always work out quite so smoothly. Consider for example $\frac{4}{7}$. We start the division process here:



We pause for a moment here. The next question we would ask ourselves if we were to continue would be, what is $40 \div 7$. However, that is exactly the same question we started with at the first step. Therefore, the division process we have gone through would simple repeat itself. If we had the stamina, we would see that

$$4 \div 7 = 0.571428\ 571428\ 571428\ 571428\ 571428\ 571428\ 571428\ 571428\ 571428\ 571428\ \ldots$$

The digits “571428” repeat, in that order, over and over again. We have special notation for this:

$$4 \div 7 = 0.\overline{571428}$$

The “bar” over the digits indicate that this pattern repeats over and over again.

Consider a fraction of the form $\frac{A}{3}$. If we perform division to convert this fraction to a decimal, one of three things must happen:

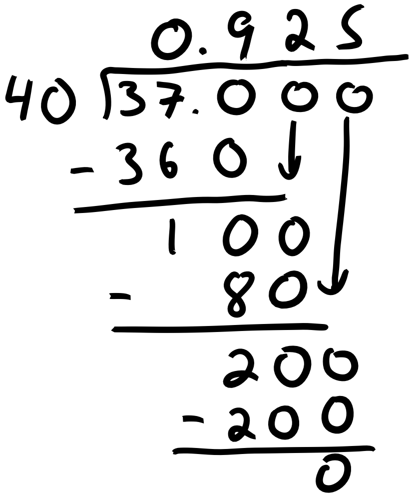
- We might encounter a remainder of 0 in the division process. In this case, we say that the process (and hence the decimal) terminates, and we get a decimal expression with finitely many digits to the right of the decimal.
- We might encounter a remainder of 1. In this case, we will end up with repeating 3s in our decimal expansion. In this case, the division process (and hence the decimal) is said to repeat.
- We might encounter a remainder of 2. In this case, we will end up with repeating 6s in our decimal expansion.

Something similar happens whenever we convert any rational number to a decimal. Either the process (decimal) terminates with a remainder of 0, or the process (decimal) repeats.

Nifty Fact: The decimal expansion of any rational number either terminates or repeats.

Problem: Convert $\frac{37}{40}$ to a decimal.

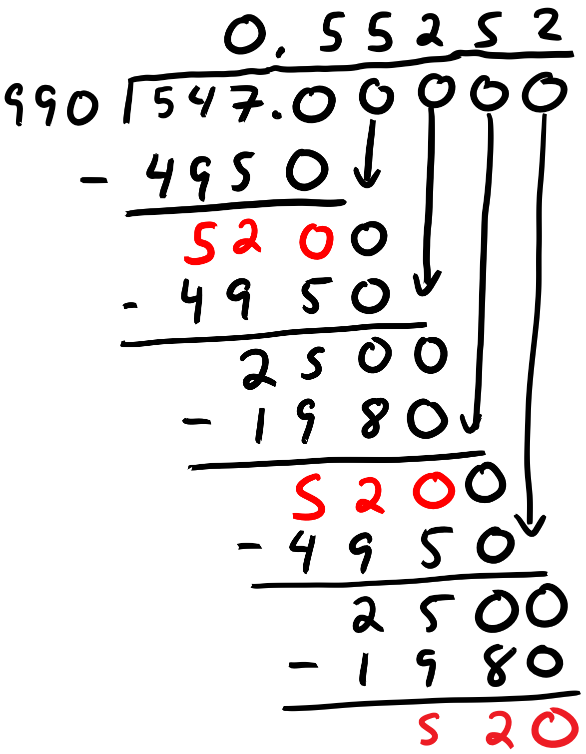
We simply divide:



We encounter a remainder of 0, so we have $\frac{37}{40} = 0.925$.

Problem: Convert $\frac{547}{990}$ to a decimal.

Again, we simply divide:



It appears that the digits “52” are repeating, so we can say that $\frac{547}{990} = 0.5\overline{52}$. We can write this a number of ways:

$$\frac{547}{990} = 0.5\overline{52} = 0.55\overline{25} = 0.5\overline{5252} = 0.552\overline{52}.$$

All are correct. Usually, we opt for the shortest expression (the first one), but sometimes we like to emphasize the pattern. If we look at the division process, we may notice that we divided further than we needed to in order to see the repetition. Once the second 520 appeared, we should have guessed there would be repetition. It is generally good, however, to continue dividing to make sure we are interpreting the repetition correctly.

Converting Decimals to Fractions

We have seen that every rational number can be expressed as a repeating or a terminating decimal. The converse is also true:

Another Nifty Fact: Every decimal that repeats or terminates is equal to a rational number.

We demonstrate how to convert decimals to fraction through examples.

Problem: Convert 1.234 to a fraction.

We will do this twice. The first approach will give an improper fraction. The second approach will give a mixed number. The simply way to convert terminating decimals to fractions is to simply place the digits of the terminating decimal over a power of 10 – that is, over a 1 with some 0s after it. The number of 0s is the same as the number of digits to the right of the decimal. Since 1.234 has three digits to the right of the decimal, we have three 0s on bottom: $1.234 = \frac{1234}{1000}$. That is, 1.234 is 1234 thousandths. Option two is to first ignore the digits to the left of the decimal, convert the portion of the number to the right of the decimal to a fraction, and then affix the digits that were ignored. That is, $0.234 = \frac{234}{1000}$ so $1.234 = 1\frac{234}{1000}$. This fraction could be reduced. However, for the purposes here, we will not reduce it.

Problem: Convert 89.98 to a fraction.

Since $0.98 = \frac{98}{100}$, then $89.98 = 89\frac{98}{100}$. This fraction could be reduced. However, for the purposes here, we will not reduce it.

Converting repeating decimals to fractions requires a bit more work (and some algebra).

Problem: Convert $7.8\overline{91}$ to a fraction.

To begin with, we ignore the 7 and convert $0.8\overline{91}$ to a fraction. The, we will affix the 7 to make a mixed number. Call the number we are converting x, so $x = 0.891\overline{91}$. It will be useful to write our number with one full extra copy of the repeating digits. Count how many digits repeat in x. In this case, we have 2 digits repeating, so we are going to multiply x by $10^2 = 100$ (this is a 1 followed by the same number of 0s as there are digits repeating). That is $100x = 89.1\overline{91}$. Now we have:

$$\begin{aligned} 100x &= 89.1\overline{91} \\ x &= 0.8\overline{91} \end{aligned}$$

Notice how the repeating parts of the decimals in $100x$ and x line up. This was the reason for multiplying by 100. Now we subtract these two expressions.

$$\begin{array}{rr} & 100x = 89.1\overline{91} \\ - & x = 0.8\overline{91} \\ \hline & 99x = 88.3 \end{array}$$

On the left hand side of the equal sign, $100x - x = 99x$. On the right hand side, the repeating portions of the decimal cancel each other out. Again, this was the reason we multiplied by 100. We can now solve for x by dividing.

$$\begin{aligned} 100x &= 89.1\overline{91} \\ - \quad x &= 0.8\overline{91} \\ \hline 99x &= 88.3 \\ x &= \frac{88.3}{99} \end{aligned}$$

We now know x. However, we generally do not mix fractions and decimals, so we multiply the top and bottom of this expression by 10.

$$\begin{aligned} 100x &= 89.1\overline{91} \\ - \quad x &= 0.8\overline{91} \\ \hline 99x &= 88.3 \\ x &= \frac{88.3}{99} \\ x &= \frac{883}{990} \end{aligned}$$

We now have that $0.8\overline{91} = \frac{883}{990}$. It follows that $7.8\overline{91} = 7\frac{883}{990}$. When we follow this process, we might end up with fractions that can be simplified. Since that is not the focus of our work here, we will not simplify them.

Problem: Convert $0.1\overline{012}$ to a fraction.

We use the same process as above, but we do not have to worry about the whole part to the left of the decimal. We let $x = 0.1012\overline{012}$, showing an extra copy of the repeating portion of the number. Then we multiply by 1000 (with three 0s since three digits repeat) to get $1000x = 101.2\overline{012}$. And then we dive into our algebra:

$$\begin{array}{rrcl} & 1000x & = & 101.2\overline{012} \\ - & x & = & 0.1\overline{012} \\ \hline & 999x & = & 101.1 \end{array}$$

$$x = \frac{101.1}{999}$$

$$x = \frac{1011}{9990}$$

Therefore, $0.1\overline{012} = \frac{1011}{9990}$.

Decimals That Do Not Repeat or Terminate

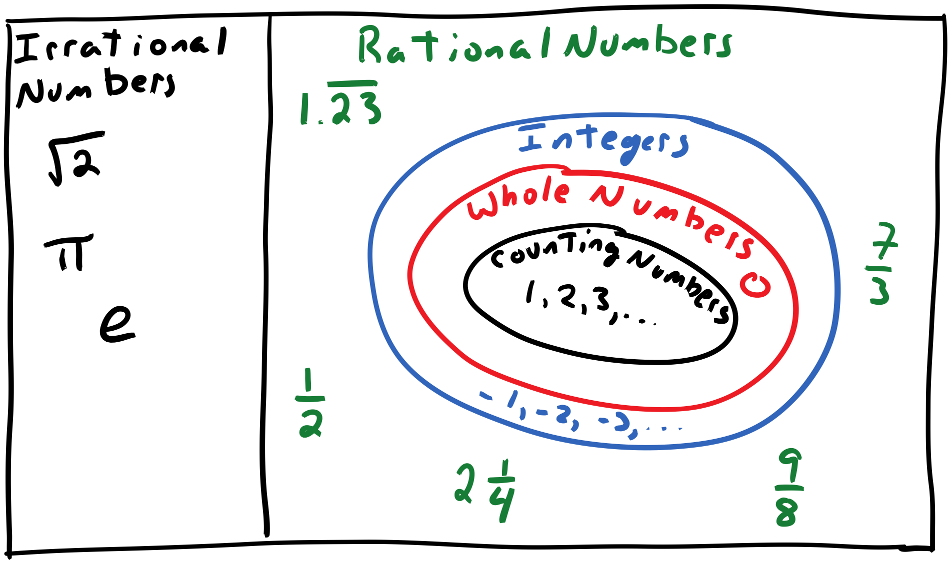
We saw above that the rational numbers are exactly those decimals that repeat or terminate. It is easy to make up decimals that do not repeat and do not terminate. For example:

$$0.101001000100001000001000000100000001000000001000000000100000000001 \ldots$$

The pattern here is intended to continue – the number of 0s between the 1s increases by 1 each step along the way. This decimal does not repeat and does not terminate, but it does represent a number (the fact that any decimal we make up represents a number is a deep fact that uses calculus to explain). This number cannot be represented as a fraction of integers. It is not rational. Numbers that are not rational we call **irrational**. Some numbers that are not rational are $\sqrt{2}$ (which we will demonstrate is not rational below) along with $\pi \approx 3.14159$ and $e \approx 2.71828$. We can now adjust or add to our list of number systems.

- The **counting numbers**: These are the names of the sizes of nonempty sets of objects. We denote them as 1, 2, 3, 4… The counting numbers were historically called the **natural numbers** by mathematicians.
- The **whole numbers**: These are the counting numbers along with zero. They include 0, 1, 2, 3, 4, 5… Most of our arithmetic algorithms were developed first at the level of the whole numbers.
- The **integers**: These numbers consist of the counting numbers, zero, and negatives of the counting numbers. We can list them as …-4, -3, -2, -1, 0, 1, 2, 3, 4, 5… We could also list them as 0, 1, -1, 2, -3, 3, -3, 4, -4…
- The **real numbers**: These are all numbers that can be expressed with decimals.
- The **rational numbers**: These are all numbers which can be expressed as fractions of integers. They include all integers along with numbers like $\frac{5}{7}$, $-\frac{4}{5}$, $2\frac{1}{3}$, and $\frac{7}{3}$.
- The **irrational numbers**: These are all real numbers that are not rational, such as $\sqrt{2}, \pi$, and e.

Here is an updated Venn diagram including all of these number systems. The entire rectangle represents the real numbers. The left side represents irrational numbers, and the right side represents rational. A rather surprising fact is that there are vastly more irrational numbers than rational.

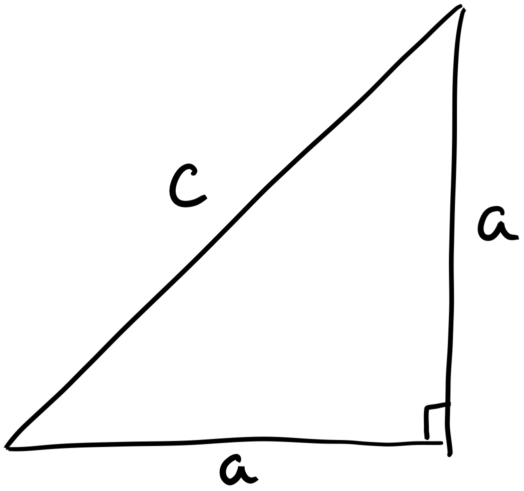


The Square Root of Two

That the square root of two is not a rational number is a fact that was known to the followers of Pythagoras around 500 BC. The Pythagoreans had developed a philosophy and religion based on relationships between the counting numbers. They used ratios of counting numbers (rational numbers in our terminology) to explain everything in the universe. The discovery of a number which is not rational challenged their entire religion. The discovery that the square root of two is not rational is attributed to a Pythagorean by the name of Hippasus. Some legends say that Hippasus was so distraught over the discovery that he threw himself from a ship. Others claim that the Pythagoreans threw Hippasus from a ship to hide his discovery. Either way, it apparently did not turn out well for Hippasus.

We give here an explanation why the square root of two cannot be rational. The explanation will use some geometry we have not encountered yet. Be patient. The alternative is to use algebra, but at least this way we get to draw a neat picture. The approach we will take is something called proof by contradiction. We will assume that the square root of 2 is rational, and we will use this assumption to arrive at a contradiction – a statement that cannot be true. At that point, we will be forced to reject the idea that the square root of two is rational.

Assume, then, that the square root of two is rational. This means that there are counting numbers c and a so that $2 = \frac{c^2}{a^2}$. To avoid fractions, we can rewrite this as $2a^2 = c^2$. This we can rewrite as $a^2 + a^2 = c^2$. A Pythagorean might notice this equation as being a special case of the Pythagorean Theorem for a right triangle with legs of length a and hypotenuse of length c.

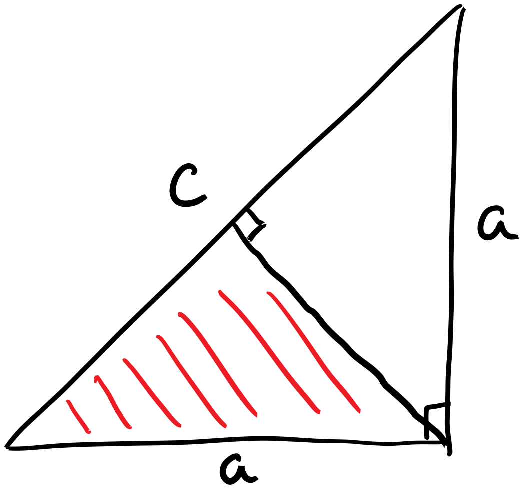


Since the two legs of this right triangle have the same length, the two non-right angles must both be 45 degrees. This is what we might have called a 45-45-90 triangle in grade school. In particular, it is a 45-45-90 triangle whose sides are all counting numbers. Something special happens in this case. Since

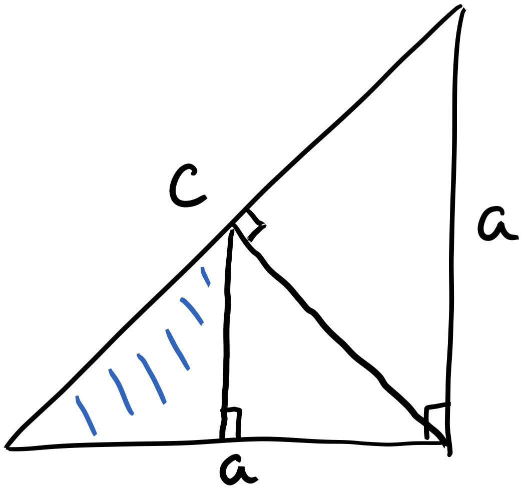
$$c^2 = a^2 + a^2 = 2a^2$$

it has to be that c^2 is even. Then only way this can happen is if c is also even. At this point, we have a 45-45-90 triangle whose sides are all counting numbers, and we know that in any such triangle the length of the hypotenuse is even. We will use this repeatedly.

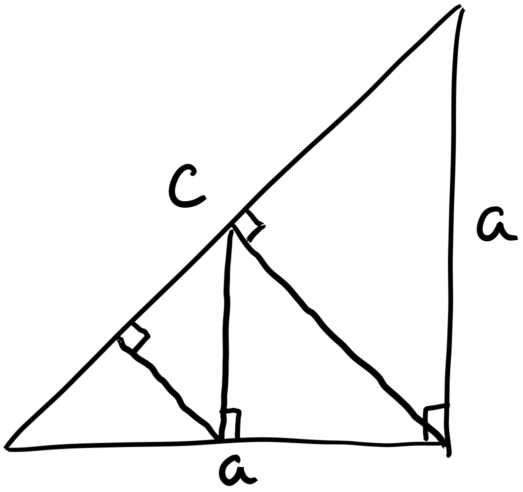
Now, we will draw a perpendicular line from the vertex with the right angle to the hypotenuse of our triangle.



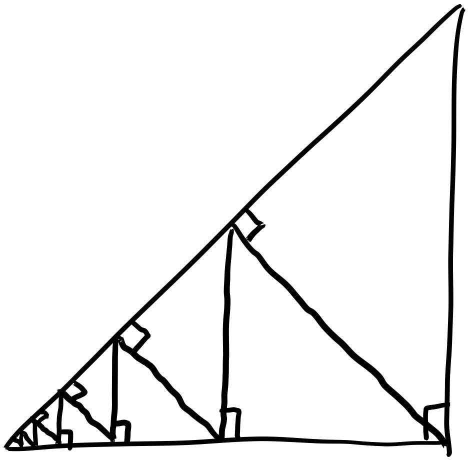
This forms a new right triangle (here shaded red). Since the right triangle includes an angle from the original triangle, we know that it is a 45-45-90 triangle. This implies that the two legs have equal length. Since c was even, and since the perpendicular will bisect the hypotenuse, each of these legs has length equal to half of c, which is a counting number. Also, the hypotenuse of the new right triangle has length a, which is also a counting number. Thus, the red-shaded triangle is a 45-45-90 right triangle whose sides are counting numbers. This forces the hypotenuse a to be even, so we can repeat the process. Drop a perpendicular to the hypotenuse of the red triangle.



This forms a new triangle (here shaded blue) with all of the nice properties of our original triangle. It is a 45-45-90 triangle whose sides are counting numbers and whose hypotenuse is even. We can repeat the process on this triangle.



And we can repeat the process again, and again, as many times as we like.



Consider now the segments along the bottom of this diagram, along the way, these segments get cut repeatedly in half. The lengths go something like

$$a, \frac{a}{2}, \frac{a}{4}, \frac{a}{8}, \frac{a}{16}, \frac{a}{32}, \frac{a}{64}, \frac{a}{128}, \frac{a}{256}, \frac{a}{512}, \frac{a}{1024} \cdots$$

Eventually, these numbers have to be less than one. HOWEVER, they are all supposed to be counting numbers. This is our contradiction. If the square root of two were rational, then we could construct counting numbers which would be strictly less than one. Since this cannot happen, then the square root of two cannot be rational.

An Algebraic Approach

In case you did not like the geometric approach to showing that the square root of two is irrational, here is an algebraic approach. Suppose that the square root of two is rational. This means that there are counting numbers c and a so that $2 = \frac{c^2}{a^2}$. By reducing the fraction, we can assume that $\frac{c}{a}$ is in lowest terms. That is, c and a have no common factors greater than 1. Now, since $2 = \frac{c^2}{a^2}$, then $2a^2 = c^2$. This implies that c^2 is even, so c has to be even. Therefore, there is a counting number k so that $c = 2k$. If we substitute $c = 2k$ into $2a^2 = c^2$ we get $2a^2 = (2k)^2 = 4k^2$. Dividing by 2 gives $a^2 = 2k^2$. This implies that a^2 is even, so a has to be even. However, that means that a and c are both even and have a common factor of 2. This is a contradiction since we assumed that a and c had no common factors greater than 1.

Geometry

The word geometry is derived from two Greek words, geo+metria, meaning “Earth measure.” The field of geometry originated for practical reasons. People had to know how to measure in order to survey land and build buildings. However, after contributions by Thales and later Euclid, geometry began to be studied for more academic reasons. Since 300 BC, geometry has been studied by scholars in every field for one primary reason:

We study geometry to learn how to think.

Thales

Thales of Miletus was a Greek scholar who traveled to Egypt around 600 BC. In Egypt, Thales studied mathematics and astronomy. Thales was a question-asker. He saw the pyramids and asked, “How old are they?” The response was, “We do not know.” He received the same response to the question, “How tall are they?” Egyptian scholars taught Thales that every triangle inscribed in a semicircle is a right triangle. He asked why, and he received the same answer again. However, Thales was able to use assumptions about triangles to *prove* that every such triangle is a right triangle. At that moment, geometry (and mathematics) ceased to be a practical science based on observation and became a discipline based on pure reasoning. Thales introduced a new way of doing mathematics and science. Thales taught that mathematics should be constructed by proof from a handful of basic assumptions. His ideas paved the way for the *axiomatic* approach to mathematics that would be developed more fully by Euclid. Thales’ approach to science was similar. He observed the natural world and stated hypotheses to explain how the world works. His school of thought would only accept those hypotheses which were supported by observation and which had the power to explain occurrences in nature. This approach was the precursor to the modern scientific method. In the realm of geometry, Thales proved theorems about similar triangles. He was able to use his results about similar triangles to measure the height of the pyramids. He also developed a technique to use similar triangles to measure the distance to a ship seen in the distance.

Euclid

Thales’ approach to mathematics and science was adopted by Greek mathematicians and grew until about 300 BC when Euclid set out to write the *Elements*, a text that was intended to systematically develop and record all mathematical knowledge of the time. Euclid himself made few contributions to mathematics in the form of new discoveries. His main contribution was in how he developed and presented the mathematics of his day. His *Elements* are the first know use of the *axiomatic method* for developing mathematics. Euclid begins with *Definitions*. Here are some examples:

1. A *point* is that which has no part.
2. A *line* is breadthless length.
3. The ends of a line are points.
4. A *straight line* is a line which lies evenly with the points on itself.
5. A *surface* is that which has length and breadth only.

Euclid also includes a list of five *Common Notions*. The Common Notions are statements that are assumed to be true that are (more or less) independent of geometry. They are:

1. Things which equal the same thing also equal one another.

2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things which coincide with one another equal one another.
5. The whole is greater than the part.

The first three Common Notions simply allow us to manipulate equations to do algebra. The last two common notions allow us to compare sizes of geometric objects. In addition to the Common Notions, Euclid includes five *Postulates* or *Axioms*. These are statements about geometry that are simply assumed to be true. Euclid set out to derive all of geometry from these five statements. Euclid's five postulates are:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and radius.
4. That all right angles equal one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The Greeks were largely concerned with geometry related to constructing shapes using a compass (a tool for drawing circles) and a straight edge. The first three postulates relate to constructions. The fifth postulate, commonly called the Parallel Postulate, is the most complex to state and the least intuitive of the postulates. Mathematicians spent 2000 years trying either to simplify the Parallel Postulate or to prove that it is unnecessary. We will see a couple of alternative statements of the Parallel Postulate after we have more geometry vocabulary below. Using the Common Notions and Postulates, Euclid set out to prove *Propositions* or *Theorems*. These are statements whose truth follows from known truths (previously proven propositions) or assumptions (postulates). Here are a few of Euclid's Propositions:

1. To construct an equilateral triangle on a given finite straight line.
2. To place a straight line equal to a given straight line with one end at a given point.
3. To cut off from the greater of two given unequal straight lines a straight line equal to the less.
4. If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.
5. In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

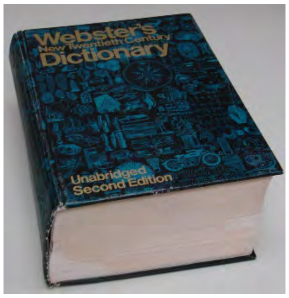
In all, Euclid's *Elements* contain 131 definitions, 5 common notions, 5 postulates, and 465 propositions. The revolutionary characteristic of the *Elements* is that Euclid tries to develop all of geometry from 5 simple assumptions. This is the second step toward modern mathematics (the first having been taken by Thales).

While Euclid's approach to mathematics was revolutions, by modern standards his efforts fell short. Consider, for example, his definition of point, "that which has no part." What is a part, and what does it mean to have one? This definition actually has no content without answers to those questions. The definition of line is similarly plagued by the lack of definitions for breadth and length. Euclid's proofs are even more problematic than his definitions. Starting with his very first proof, he makes use of assumptions that do not appear in his postulates or common notions. While the *Elements* are lacking in this way, it is true that Euclid's approach was revolutionary. The impact on how humans reason of his attempt to use axiomatic reasoning simply cannot be overestimated.

The Modern Axiomatic Method

Modern mathematics has developed the axiomatic method to reconcile some of the deficiencies of Euclid's approach. The modern ingredients of an axiomatic system are:

- **Primitives**: A *primitive* is a word that we do not define. Here is a rather large dictionary (that was given to me by my mother-in-law):

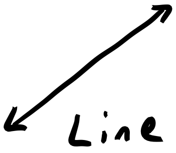


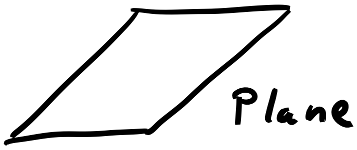
  If we look up the word "small" in this dictionary, we will find the definition, "little in size." If we look up the word "little" in this dictionary, we will find, "small in size." This issue is known as a circular definition. If we not know what small and little mean, then this dictionary is useless. To avoid circular definitions, mathematicians avoid trying to define every word and begin with primitives. The use of primitives also avoids empty definitions such as those in Eculid.
- **Definitions**: Using primitives and natural language, mathematicians define words that will be used in the process of doing mathematics.
- **Axioms**: An axiom is a statements that is assumed to be true. Sometimes axioms are based on observations. Other times, mathematicians play the game of "What if?" And address what other statements must be true if we make certain assumptions.
- **Proofs**: A proof is merely a list of statements so that every statement in the list is either an axiom, a known truth, or *follows logically* from earlier statements in the list. Mathematician use a rather rigorous definition of what "follows logically" may mean.
- **Theorems**: A theorem is a statement which is proven to be true.

The Axiomatic Method in Geometry (for us)

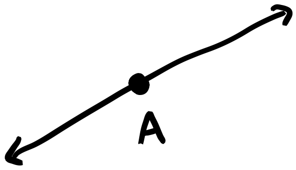
The actual primitives and axioms used in geometry may vary somewhat between textbooks and environments. However, we give here some examples of what may constitute some primitives, definitions, axioms, and theorems in geometry.

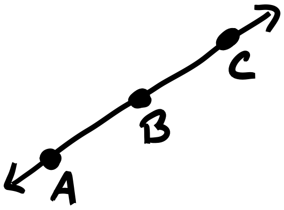
- **Primitives**: Some common primitives in geometry could be: point, line, plane, on, and between. We will never say what these words mean, but we may draw pictures that represent the concepts.

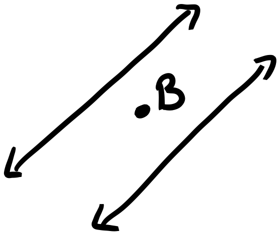




Here, the line is intended to “extend forever” in the directions of the arrows. The plane is intended to “extend forever” in all directions.





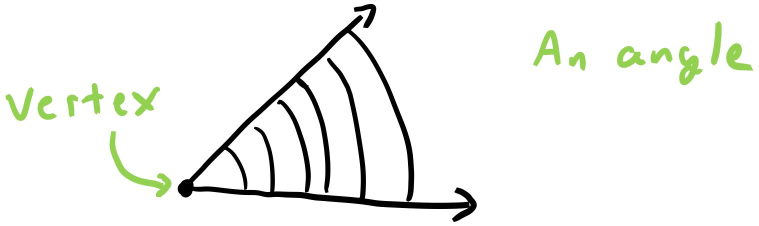


In the left picture here, the point labeled A is on the line. Some might also say that the line is on the point. In the middle picture, the point B is between the points A and C. In the right picture, the point is between the two lines.

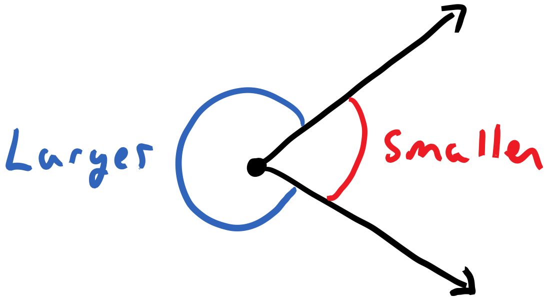
- **Definitions**: Here are some examples of definitions.
  - Two points along with all of the points between them is a **line segment**. The two points are called the **endpoints** of the line segment.
  - If A, B, and C are three points on a line, then B and C are **on the same side of** A if either B is between A and C or C is between A and B.
  - If A and B are points on a line, then A along with all of the points on the line on the same side of A as B are a **ray**. The point A is the **endpoint** of the ray.
  - Two lines **intersect** at a point if that point is on both of the lines.
  - Two lines are **parallel** if they do not intersect at any point.
- **Axioms**: The axioms used in geometry vary from book to book, but we note that all of the sets of axioms are in some way equivalent (sort of). Some of the axioms will address existence (There are at least two points. Any two points are on a line.) Some of the axioms will address measurement of distance and angles. Some of the axioms may address congruence of triangles. Some axioms may address area. All systems which prove the same theorems as Euclid also include the Parallel Postulate or a replacement for it.

Angles

Informally, an angle represents a certain amount of rotation around a point. More formally, if two rays have a common endpoint, then the two rays along with the region between them form an **angle**. The common endpoint of the two rays is the **vertex** of the angle.



Notice that every pair of rays actually determines two angles, usually a smaller one and a larger one.



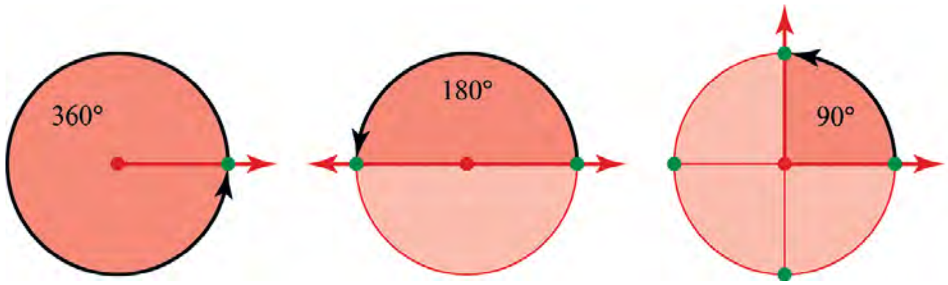
(Actually, many texts only call the smaller one an angle.)

Angle Measure

We assume (in something called the Protractor Postulate) that any angle has a measure in a unit called degrees. We assume that:

- If the two rays forming an angle are actually the same ray, then the measure of the smaller angle (representing no rotation) is 0 degrees or 0°. The measure of the larger angle (representing a full rotation) is 360°.
- If the two rays forming an angle are opposite sides of a line, then the measure of the angle is 180°. In this case, the angle is called a *straight angle*.

If an angle represents one quarter of a turn, then its measure is 90°.

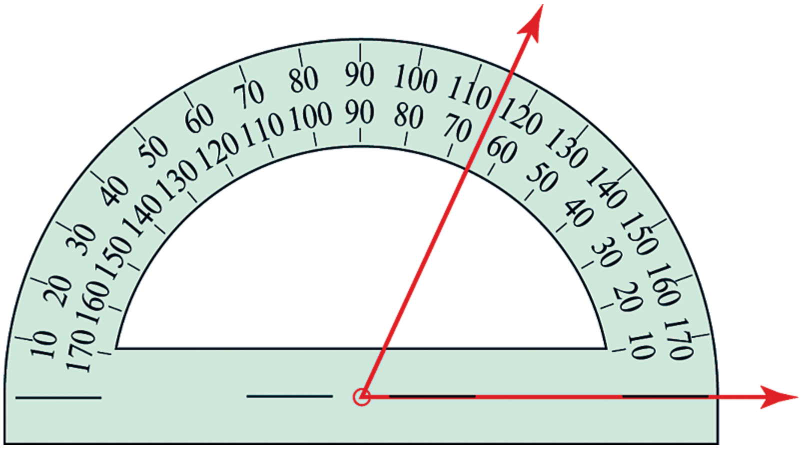


The choice of 360 degrees for the measurement of complete rotation is arbitrary. If we search history books for a reason for using 360, we will find a variety of tales. Ancient Babylonians and Indians divided the circle into 360 parts. One reason for this might be that they both know that a year is about 360 days. It could also be because the Babylonians used a base sixty system (however, this use itself might be based on the number of days in a year). A mathematical reason to use 360 rather than 365 (the number of days in a year) or 100 (a nice base ten number) is that 360 has many divisors. The truth behind why the Babylonians and Indians both used 360 divisions of a circle is lost to history.

If two angles have the same measure, then we say that those angles are **congruent**. We may also informally say that two angles with the same measure are equal; however, in this case it is the measures of the angles that are technically equal, not the angles themselves.

Protractor

The tool used to measure angles is a **protractor**, pictured here:

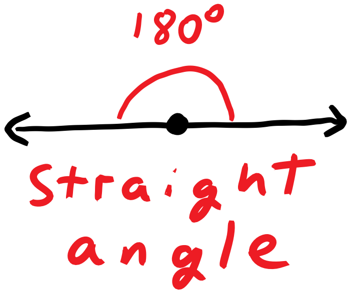


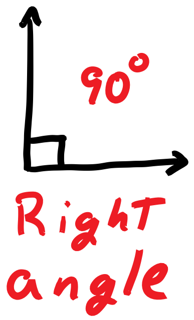
To use a protractor, we place the vertex of our angle in the small circle at the bottom center of the protractor, and we make sure one ray of the angle is along the bottom line on the protractor. That bottom angle can point to the left or to the right. The other (top) ray then should cross the numbers on the protractor. On this picture, if the bottom ray is pointing right, we use the inside ring of numbers. If the bottom ray is pointing left, we use the outside ring of numbers. The top ray of this angle crosses the inside ring of numbers between 60 and 70. The measure seems to be 65°.

Types of Angles

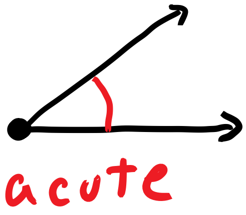
Angles are classified by their measure.

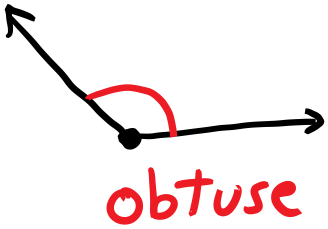
- A **straight angle** is an angle whose measure is 180°.
- A **right angle** is an angle whose measure is 90°. Since we cannot draw perfect pictures, we will usually draw a little square in an angle to indicate that it is a right angle.





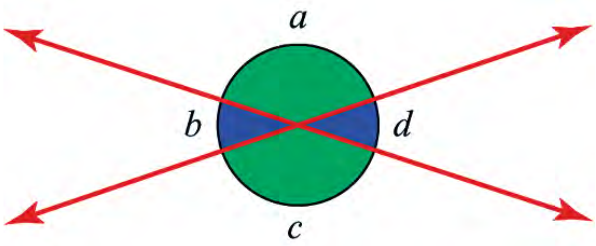
- An **acute angle** is an angle whose measure is less than 90°.
- An **obtuse angle** is an angle whose measure is more than 90° but less than 180°.





Intersecting Lines

Two lines **intersect** at a point if that point is on both of the lines. When two distinct lines intersect at a point, then the rays with that point as an endpoint form four different angles as in this diagram.



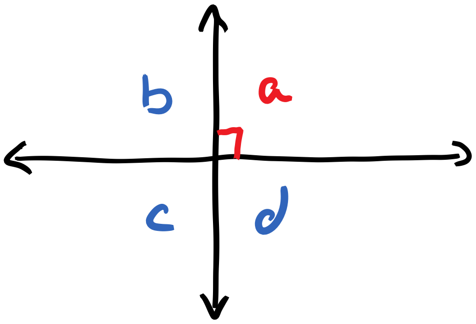
In this case, the angles a and c are called **vertical angles**, and the angles b and d are also called vertical angles. In the discussion that follows, we are going to abuse notation a little, and we will use a, b, c, and d to represent both the angles and the measures of those angles. We mentioned earlier that Thales changed mathematics by introducing the idea that all results in mathematics should be able to be proven from a handful of results. We also mentioned that Euclid extended this view to introduce the idea of the axiomatic method. The first theorem (supposedly) proven by Thales was about vertical angles.

Consider the four angles formed by the two intersecting lines above. We know that the angles a and d combine to form a straight angles. Therefore, it has to be that $a + d = 180°$. Similarly, the angles c and d combine to form a straight angle. Therefore, $c + d = 180°$. If we solve the first equation for a and the second equation for c, then we have $a = 180° - d$ and $c = 180° - d$. Therefore, $a = c$. Thus, we have proven this theorem (due to Thales, supposedly):

Vertical Angles Theorem: The vertical angles formed when two lines intersect are congruent.

Problem: Suppose that two lines intersect as in the diagram below and that angle a is a right angle.

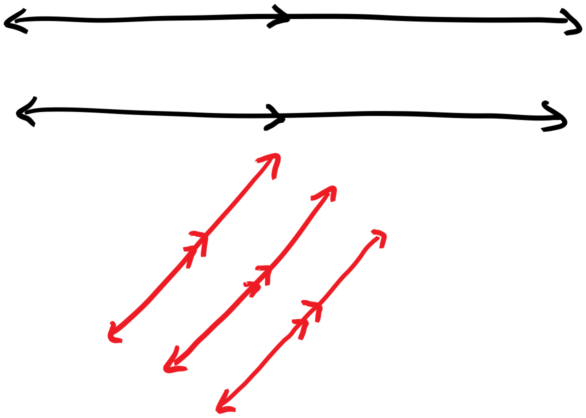
1. What is the measure of angle b? Why?
2. What is the measure of angle c? Why?
3. What is the measure of angle d? Why?



First, since a is a right angle, we know that $a = 90°$. Since a and b combine to form a straight angle, it has to be that $a + b = 180°$. Since $A = 90°$, then $b = 90°$ also. Now a and c are vertical angles, so $c = a = 90°$. Similarly, b and d are vertical angles, so $d = b = 90°$. Thus, we have proven that if two line intersect, and if one of the angles they form is a right angle, then all of the angles are right angles. In this case, we say that the lines are **perpendicular**.

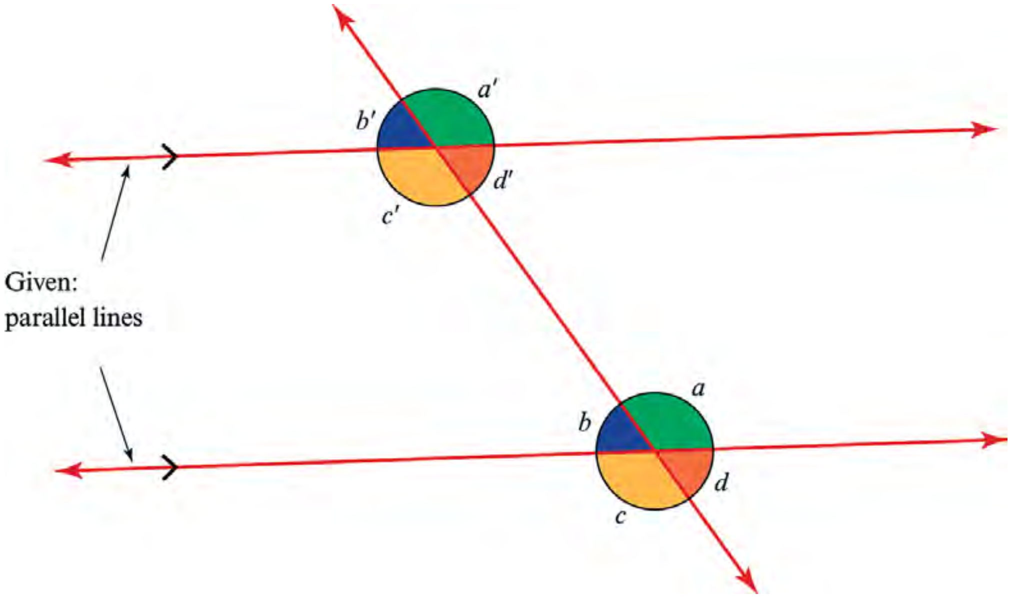
Parallel Lines

Two lines which do not intersect at any point are said to be **parallel**. To indicate that two lines in a diagram are parallel, we will place small arrows on them. The number of arrows indicate which lines are parallel. For example:



Here, the single arrows in the centers of the black lines indicate they are parallel to each other. The double arrows in the centers of the red lines indicate they are all parallel to each other (but not necessarily to the black lines).

In our discussion about Euclid above, we mentioned that his Fifth Postulate (called the Parallel Postulate) had many translations related to parallel lines. We state one of those here. Suppose that we have two parallel lines which each intersect a third line as in this diagram:



The third line forms two sets of four angles with the parallel lines. Those angles which are colored the same here are called **corresponding angles**. The angles c' and b are called same **side interior angles**. The word interior indicates the angles are between the parallel lines. The words same side indicate that the angles are on the same side of the non-parallel line. Similarly, the angles c' and a are **alternate interior angles**. The word alternate indicates that the angles are on opposites sides of the non-parallel lines. The Parallel Postulate (or our version of it) relates to the corresponding angles formed when two parallel lines are intersected by a third line.

Parallel Postulate: The corresponding angles formed when two parallel lines both intersect a third line are congruent.

In the diagram above, this means that $a = a'$, $b = b'$, $c = c'$, and $d = d'$. We will use the Parallel Postulate to prove a couple of related theorems.

Problem: Suppose that two parallel lines are intersected by a third as in the diagram above.

1. How is c' related to a'? Why?
2. How is a' related to a? Why?
3. How is c' related to a? Why?

Since c' and a' are vertical angles, we know that $c' = a'$ by the Vertical Angles Theorem. Since a' and a are corresponding angles, we know that $a' = a$ by the Parallel Postulate. Since $c' = a'$ and $a' = a$, we know that $c' = a$. Thus, we have proven:

Alternate Interior Angles Theorem: The alternate interior angles formed when two parallel lines intersect a third line are congruent.

Problem: Again consider the diagram above the Parallel Postulate.

1. How are c' and a related? Why?

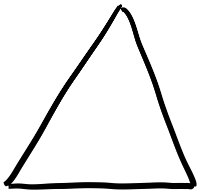
2. What is $b + a$? Why?
3. What is $b + c'$? Why?

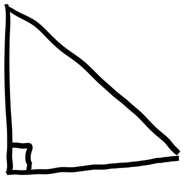
The angles c' and a are alternate interior angles, so they are congruent by the alternate interior angles theorem. Since the angles b and a combine to form a straight angle, we know that $b + a = 180°$. Since $c' = a$, it follows that $b + c' = 180°$ also. When two angles add to $180°$, we call them **supplementary**. We have proven:

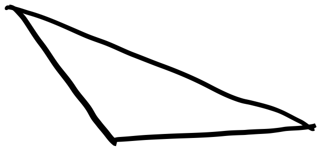
Same Side Interior Angles Theorem: The same side interior angles formed when two parallel lines intersect a third line are supplementary.

The Angles in a Triangle

The Parallel Postulate allows us to determine the sum of the angles in a triangle. First, we have to define triangle. A **triangle** is a closed shape in the plane made by three line segments. We will return to triangles and their properties at a later time, but here we want to discuss the angles in a triangle. Here are some triangles:



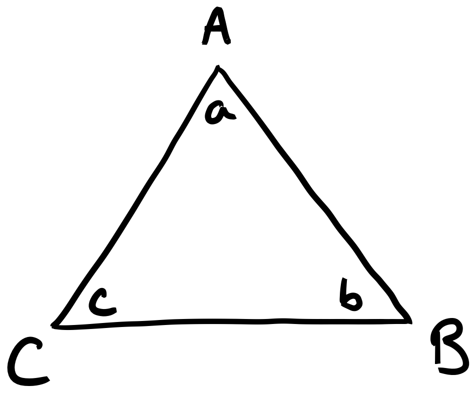




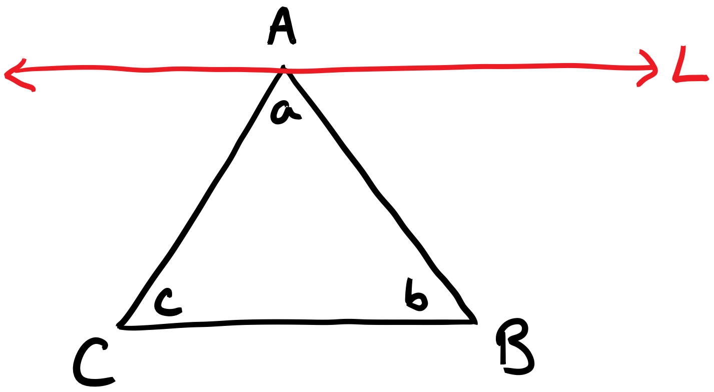
If we draw a triangle on paper, cut it out, rip off the corners, and fit them together, it will appear as if the three angles together form a straight angle (the reader should try this process before continuing). This actually is the case, and we will prove it using the Parallel Postulate. To do so, we will follow these steps:

1. Draw a triangle.
2. Label the vertices A, B, and C
3. Label the angle at A as a, the angle at B as b, and the angle at C as c.
4. Draw a line L through A parallel to $\overline{BC}$
5. The side $\overline{AC}$ forms an angle with L. Label this angle c'.
6. The side $\overline{AB}$ forms an angle with L. Label this angle b'.
7. How do c and c' compare? What about b and b'?
8. What is $a + b' + c'$?
9. What is $a + b + c$?

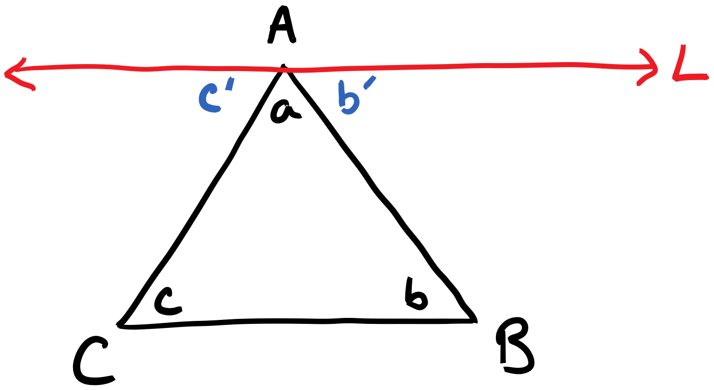
The notation $\overline{AB}$ here means the line segment with endpoints A and B. First, we follow steps 1, 2, and 3 to draw and label out triangle:



For step 4, we draw a line L through A parallel to $\overline{BC}$.



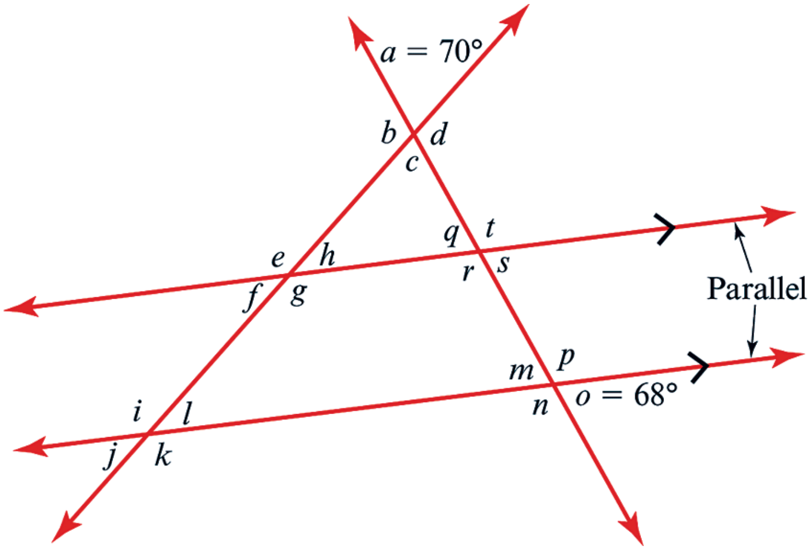
Now we label b' and c' for steps 5 and 6.



We can now address the questions in steps 7, 8, and 9. Since c and c' are alternate interior angles, we know that $c = c'$. Since b and b' are alternate interior angles, we know that $b = b'$. Since c', a, and b' combine to form a straight angle, we know that $a + b' + c' = 180°$. Now, since $b = b'$ and $c = c'$, this tells us that $a + b + c = 180°$. Thus we have proven:

Angles in a Triangle Theorem: The angles in a triangle add to $180°$.

Problem: Use results from this section to find all of the angles in the diagram below.



We start in the lower right hand corner. Since o and p combine to make a straight angle, we know that $0 + p = 180^\circ$, so $p = 180^\circ - o = 112^\circ$. Also, since o and m are vertical angles, and since p and n are vertical angles, we know that $m = 0 = 68^\circ$ and $n = p = 112^\circ$.

The Parallel Postulate tells us that the angles r, s, t, and q are equal to the angles n, o, p, and m, respectively, so

$$r = n = 112^\circ$$
$$s = o = 68^\circ$$
$$t = p = 112^\circ$$
$$q = m = 68^\circ.$$

Now we move to the top point of intersection. Since a and b combine to make a straight angle, we know that $a + b = 180^\circ$. Then

$$b = 180^\circ - a = 110^\circ.$$

Since a and c are vertical angles, and since b and d are vertical angles, we know that

$$c = a = 70^\circ\ and\ d = b = 110^\circ.$$

Consider now the triangle with angles $h, c, and\ q$. We know that $h + c + q = 180^\circ$, and we know that $c = 70^\circ$ and $q = 68^\circ$. Therefore $h = 180^\circ - c - q = 42^\circ$.

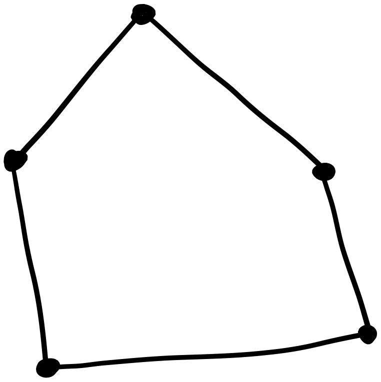
If we apply the same reasoning to the four angles e, f, g, and h that we did to n, o, p, and m and to a, b, c, and d, then we find

$$f = h = 42^\circ$$
$$e = g = 138^\circ.$$

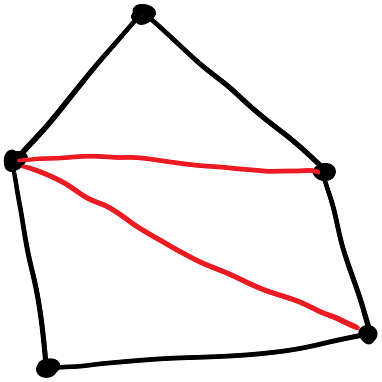
The Parallel Postulate now gives

$$j = l = 42^\circ$$
$$i = k = 138^\circ.$$

Problem: Find the sum of the angles in the shape below.



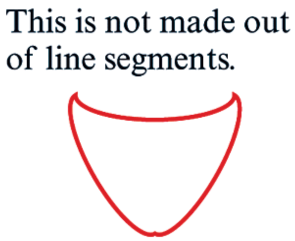
We know how to add up the angles in triangle (a shape made of three line segments), but this shape is made of five line segments (later, we will call this a pentagon). The secret is to select one corner and draw line segments to the other, non-adjacent corners, cutting the shape into three triangles.

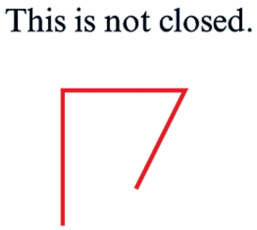


Now, all of our original angles are formed by combining the angles in the triangles, so all we have to do is add all of the angles from the three triangles. Since the angles in one triangle add to 180°, the angles in three triangles add to $3 \times 180^\circ = 540^\circ$.

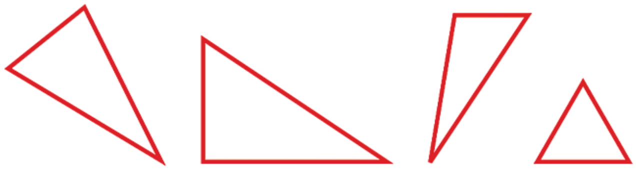
Triangles

A triangle is a **closed** shape in the plane made of three line segments. These two shapes are not triangles:





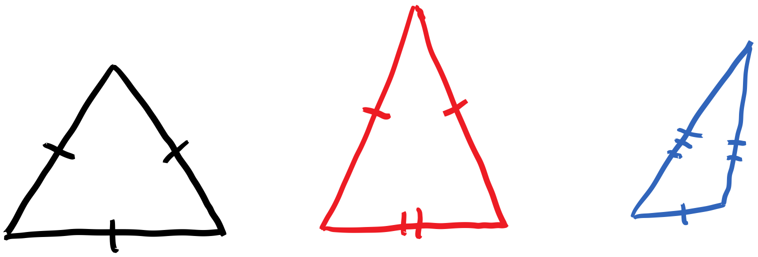
These shapes are triangles:



The line segments which make up a triangle are called **sides**. The point at which two sides of a triangle intersect is a **vertex**. The plural of vertex is **vertices**.

Classifying Triangles

Triangles can be classified by how their angles relate to each other. It happens to be that how the angles in a triangle relate to each other is intimately related to how the sides of the triangle relate to each other. When drawing a triangle, we will sometimes place hash marks on sides to indicate that they are the same length. Any two sides with the same number of hash marks are intended to have the same length.

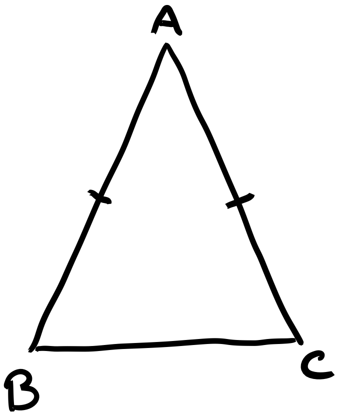


The black triangle here is intended to have three equal length sides (which we will call equilateral below). All three sides are marked with a single hash mark, indicating they are the same length. The red triangle has two equal length sides (we will call this isosceles below). The two sides with single hash marks are the same length. The third side with two hash marks is a different length. The blue triangle three sides with different lengths (we will call this scalene below). Each side has a different number of hash marks, indicating different lengths.

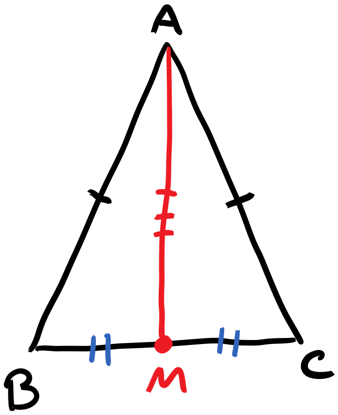
That the relationship between the angles in a triangle is related to the relationship between the angles follows from this theorem, which is due to Thales:

Equal Side Equal Angle Theorem: If a triangle has two sides of equal length, then the angles opposite those sides are congruent.

Suppose that we have this triangle with two sides of equal length:



We have labeled the vertices of the triangle A, B, and C to make it easier to reference sides and angles. The diagram indicates that the length of $\overline{AB}$ is the same as the length of $\overline{BC}$. We will argue that the angle at vertex B is congruent to the angle at vertex C. To do so really requires the notion of congruence of triangles (or symmetry) which we will not encounter until later in our notes, but we can get the gist of the argument here. First, we select point M to be the midpoint of $\overline{BC}$. That is, the length of $\overline{BM}$ is the same as the length of $\overline{MC}$. Now draw the line segment $\overline{AM}$. This gives us these line segments with the corresponding lengths.



This diagram now includes two small triangles. On uses the points A, M, and B, and the other uses points A, M, and C. Notice that the two triangles have exactly the same side lengths. They are simply two copies of the same triangle (in technical terms, they are congruent). The angle at B is the angle between the side with one hash and the side with two hashes. The angle at C is also the angle between the side with one hash and the side with two hashes. Therefore, these two angles are two copies of the same angle – they are congruent.

Now we can return to classifying triangles. It could be that all of the angles in a triangle are congruent to each other, that two are congruent but the third is different, or that all three angles are different measures. This exactly corresponds to the situations where all three sides are equal in length, where two are equal in length, or where all three sides have different lengths.

- An **equilateral triangle** has three sides of equal length.
- An **isosceles triangle** has at least two sides of equal length.
- A **scalene triangle** has three sides with three different lengths.

Notice in our definition of isosceles that we require at least two sides to have the same length. Since three is at least two, *every equilateral triangle is also isosceles*.

Problem: What is the measure of each angle in an equilateral triangle?

As mentioned above, every equilateral triangle is also equi-angular. That is, all three angles have the same measure. We know that the angles in a triangle must add to $180°$, so each angle must be one third of $180°$, or $60°$.

There is one more type of triangle that we will name:

- A **right triangle** is a triangle which contains a right angle.

Problem: How many right angles can a triangle have?

We know that the angles in a triangle must add to $180°$. If a triangle had two right angles, then those two angles would already add to $180°$, and there would be nothing left for the third angle. Therefore, no triangle can have two right angles. A triangle can have at most one right angle.

Problem: Can a right triangle be equilateral?

All of the angles in an equilateral triangle are $60°$, so none of them is a right angle. Therefore, no right triangle can be an equilateral triangle.

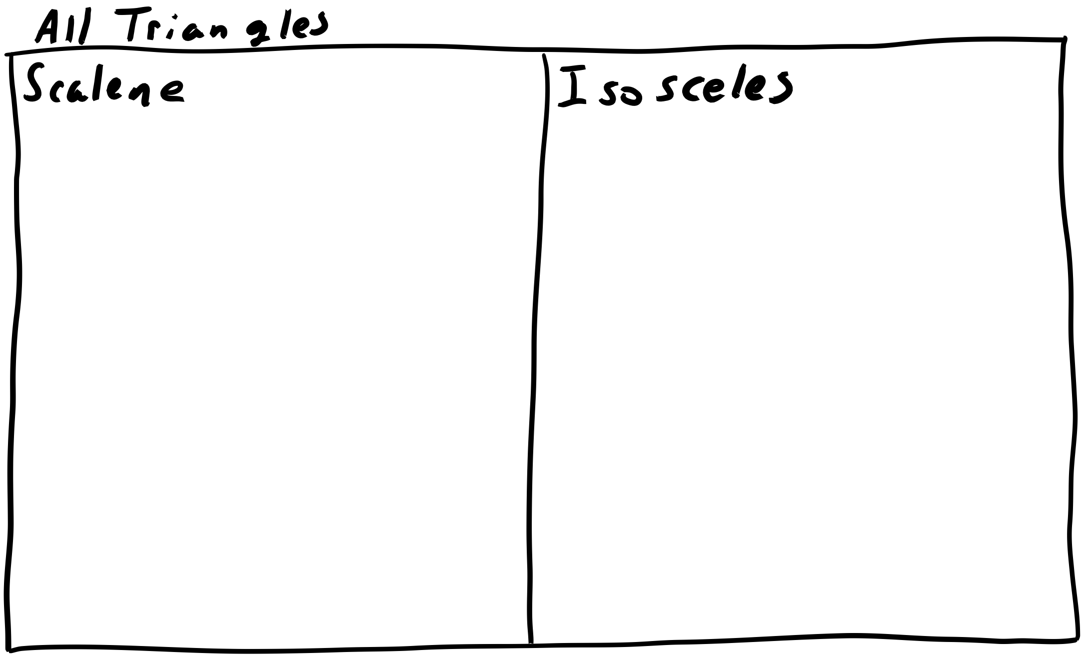
Problem: What are the angles in an isosceles right triangle?

Exactly one of the angles in an isosceles right triangle must be $90°$. By the Equal Side Equal Angle Theorem, the other two angles must be equal in measure. Since all of the angles must add to $180°$, this means that the other two angles must be equal and add to $90°$. This tells us the other two angles are each $45°$.

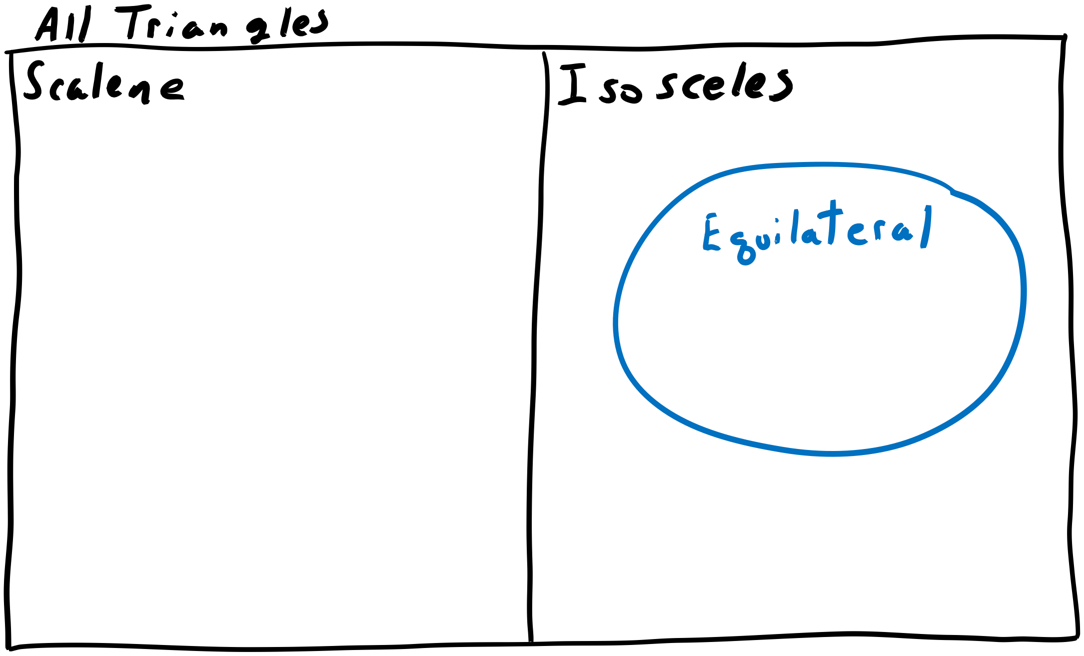
Venn Diagram for Triangles

Problem: Draw a Venn diagram showing the relationships between all triangles, equilateral, isosceles, scalene, and right triangles. Draw an example triangle in each region of the Venn diagram.

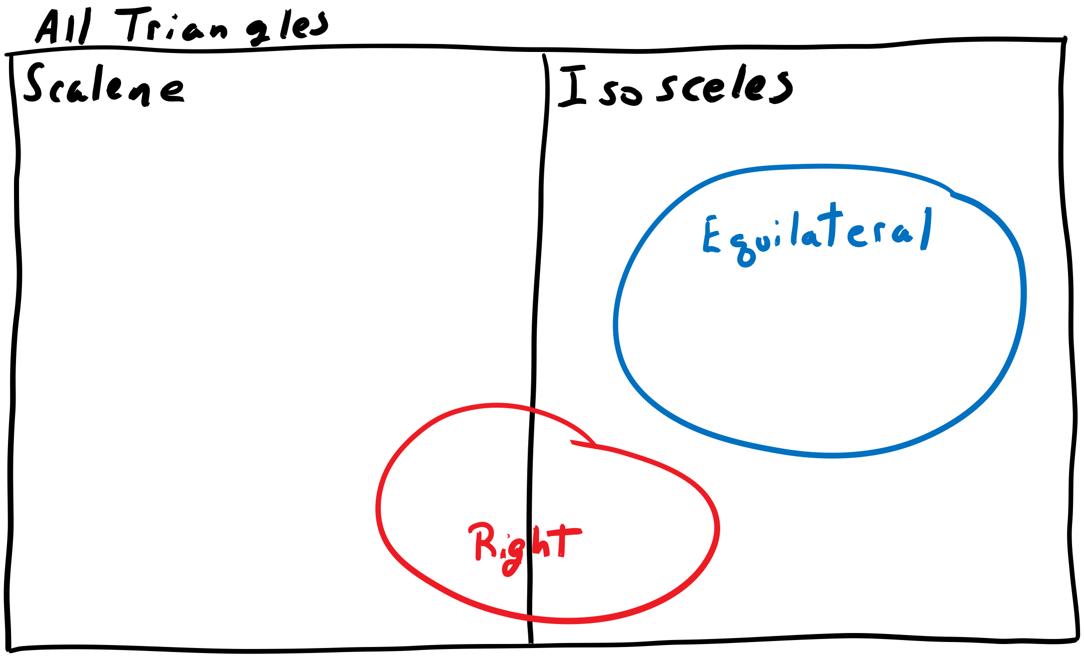
First of all, every triangle either has three sides of different lengths, or it has at least two that have equal length. This means that every triangle is either scalene or isosceles. Therefore, our region for all triangles should be divided into two parts – scalene and isosceles:



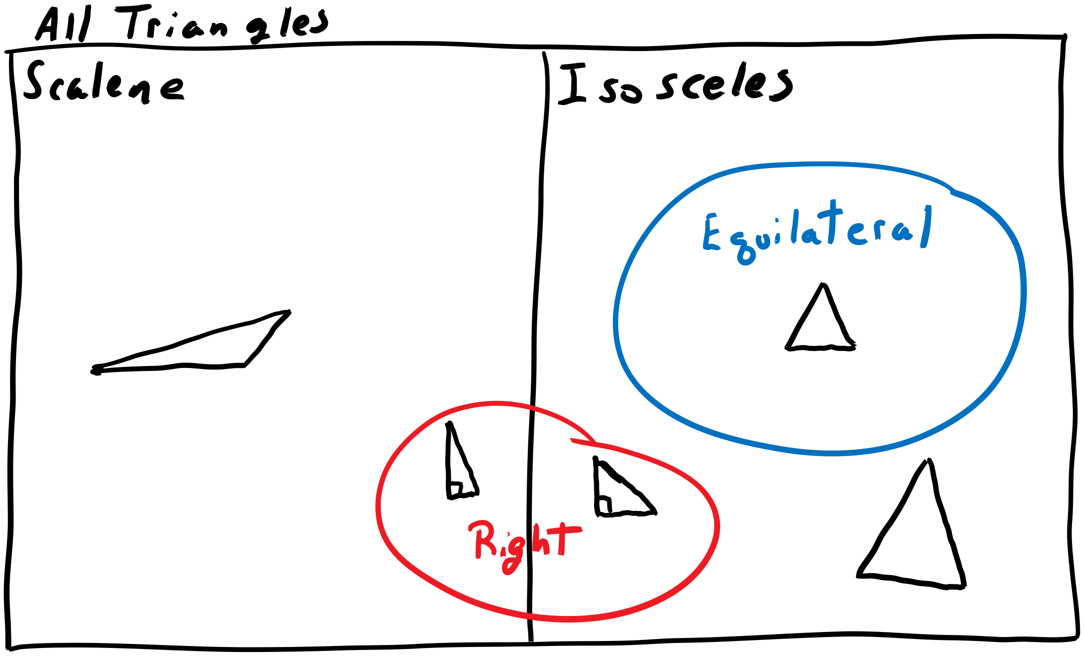
This should be reminiscent of the way we divided all real numbers into irrational and rational numbers. Since every equilateral triangle must be isosceles, we place an oval for equilateral triangles inside the region for isosceles.



Now, no right triangle can be equilateral, so the region for right triangles cannot overlap the region for equilateral. However, a right triangle may be scalene or isosceles, so we draw an oval for right triangles overlapping these regions.



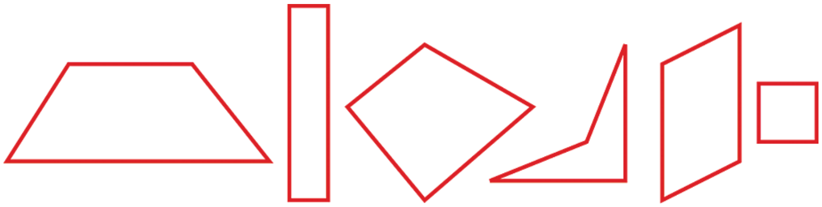
Now we draw an example triangle in each region: scalene but not right, scalene and right, isosceles and neither right nor equilateral, isosceles and right, and equilateral.



Quadrilaterals and Other Polygons

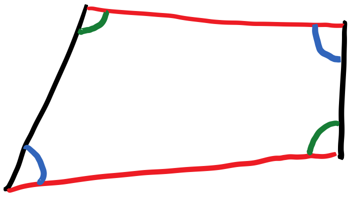
We defined a triangle as a closed shape in the plane made of three line segments. We will call the corresponding four sided shape a quadrilateral, but we have to be slightly more careful in our definition. These are three shapes that we do not want to be considered quadrilaterals for different reasons:

The shape on the left is not made of line segments. The shape in the middle is not closed. The shape on the right seems to have too many points of intersection. Our definition of quadrilateral avoids these possible problems: A **quadrilateral** is a closed shape in the plane made of four line segments that do not cross each other except at endpoints. The line segments are the **sides** of the quadrilateral, and the point of intersection of two sides is a **vertex**. These shapes are quadrilaterals:



Classifying Quadrilaterals

In the quadrilateral below, the two red sides are considered **opposite sides** as are the two black sides. The two green angles are considered **opposite angles** as are the two blue angles.



Two angles in a quadrilateral with one side between them are **adjacent angles**. There are a variety of properties we might look at in quadrilaterals in order to classify them. Some are:

- Properties Related to Sides
  - All sides equal
  - Opposite sides equal
  - Opposite sides parallel
- Properties Related to Angles
  - All angles equal
  - Opposite angles equal
  - Adjacent angles sum to 180°

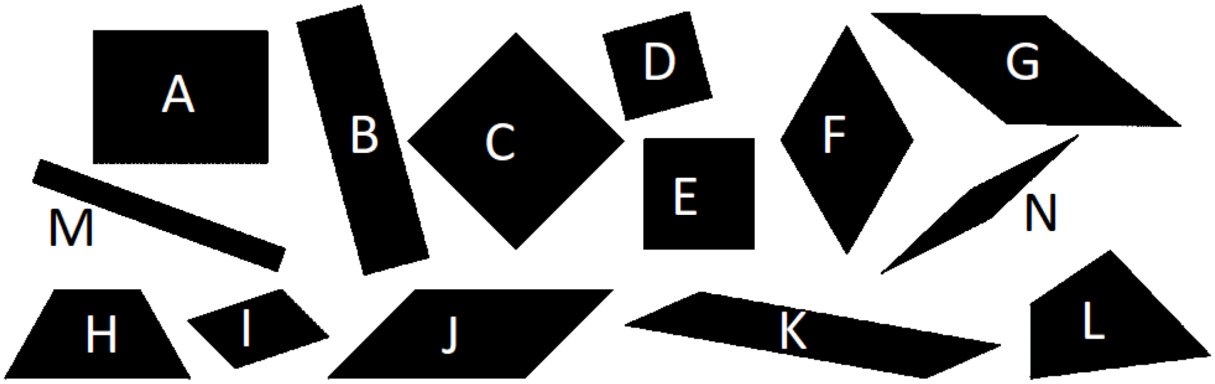
Considering sides and angles, we make these definitions:

- A **square** is a quadrilateral with four right angles and four sides of equal length.
- A **rectangle** is a quadrilateral with four right angles.
- A **rhombus** is a quadrilateral with four sides of equal length.

- A **parallelogram** is a quadrilateral in which opposite sides are parallel.
- A **trapezoid** is a quadrilateral in which *at least one* pair of opposite sides are parallel.

Note that in our definition of trapezoid we require at least one pair of opposite side to be parallel. Since two pair is at least one pair, with this definition every parallelogram is also a trapezoid. Some books define trapezoids to have exactly one pair of parallel sides. In those texts, parallelograms would not be trapezoids. Also note that a quadrilateral must meet two requirements to be a square. One of those requirements (four right angles) is the condition to be a rectangle, while the other (four equal sides) is the condition to be a rhombus. This implies that every square is both a rhombus and a rectangle.

Problem: Give each label (square, rectangle, rhombus, parallelogram, trapezoid) that applies to each shape below. Include all labels that apply.

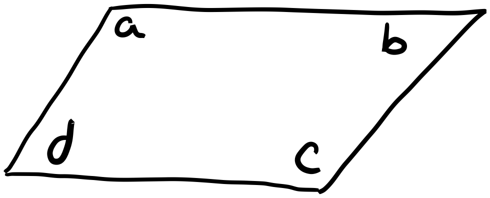


We first look for squares. These shapes must have four right angles and four equal length sides. The squares include C, D, and E. Next, we look for rectangles. These must have four right angles. The rectangles include A, B, C, D, E, and M. Next, we look for rhombuses. These must have four equal sides. The rhombuses include C, D, E, F, G, and N. Parallelograms must include two pair of parallel sides. The parallelograms are A, B, C, D, E, F, G, I, J, K, M, and N. Finally, we consider trapezoids. To be a trapezoid, a quadrilateral only needs on pair of parallel sides. The trapezoids include every shape here except L. Here is a summary in a chart:

	Square	Rectangle	Rhombus	Parallelogram	Trapezoid
A	No	Yes	No	Yes	Yes
B	No	Yes	No	Yes	Yes
C	Yes	Yes	Yes	Yes	Yes
D	Yes	Yes	Yes	Yes	Yes
E	Yes	Yes	Yes	Yes	Yes
F	No	No	Yes	Yes	Yes
G	No	No	Yes	Yes	Yes
H	No	No	No	No	Yes
I	No	No	No	Yes	Yes
J	No	No	No	Yes	Yes
K	No	No	No	Yes	Yes
L	No	No	No	No	No
M	No	Yes	No	Yes	Yes
N	No	No	Yes	Yes	Yes

Problem: Show that opposite angles are congruent in every quadrilateral in which adjacent angles add to $180°$.

Consider this quadrilateral. The actual shape of the quadrilateral is irrelevant, we only care about the names of the angles so that we know which ones are opposite and which are adjacent.



Assume that adjacent angles in this quadrilateral add to $180°$. We will argue that $a = c$. Since a and b are adjacent, we know that $a + b = 180°$ and that $a = 180° - b$. Since b and c are adjacent, we know that $b + c = 180°$ and that $c = 180° - b$. Thus, $a = 180° - b = c$ as desired.

Problem: Determine which of the properties below each type of quadrilateral seems to have.

Property		Square	Rectangle	Rhombus	Parallelogram	Trapezoid
Sides	1. All sides equal					
	2. Opposite sides equal					
	3. Opposite sides parallel					
Angles	4. All angles equal					
	5. Opposite angles equal					
	6. Adjacent angles sum to $180°$					

Squares: A square must satisfy properties 1, 2, 4, 5, and 6 by definition. Every square in the problem above happens to be a parallelogram, so it appears that squares satisfy property 3 as well.

Rectangles: A rectangle must satisfy properties 4, 5, and 6 by definition. However, rectangle B above fails property 1. In the problem above, every rectangle seems to satisfy property 2. In the problem above, every rectangle is a parallelogram, so it appears that rectangles also satisfy property 3.

Rhombuses: A rhombus must satisfy properties 1 and 2 by definition. Rhombus F above fails property 4. All of the rhombuses above are parallelograms, so it appears as if rhombuses satisfy property 3.

Properties 5 and 6 are a little more difficult to see. Property 6 holds by the Same Side Interior Angle Theorem since the opposite sides of a rhombus are parallel. Property 5 actually follows from property 6 as the problem above shows.

Parallelograms: Property 3 holds by definition. Property 6 and 5 hold just like they did for rhombuses. Properties 1 and 4 fail for parallelogram K above. Property 2 seems to hold for the parallelogram in the problem above, so we will go out on a limb and guess yes.

Trapezoids: Trapezoid H above fails all 6 properties!

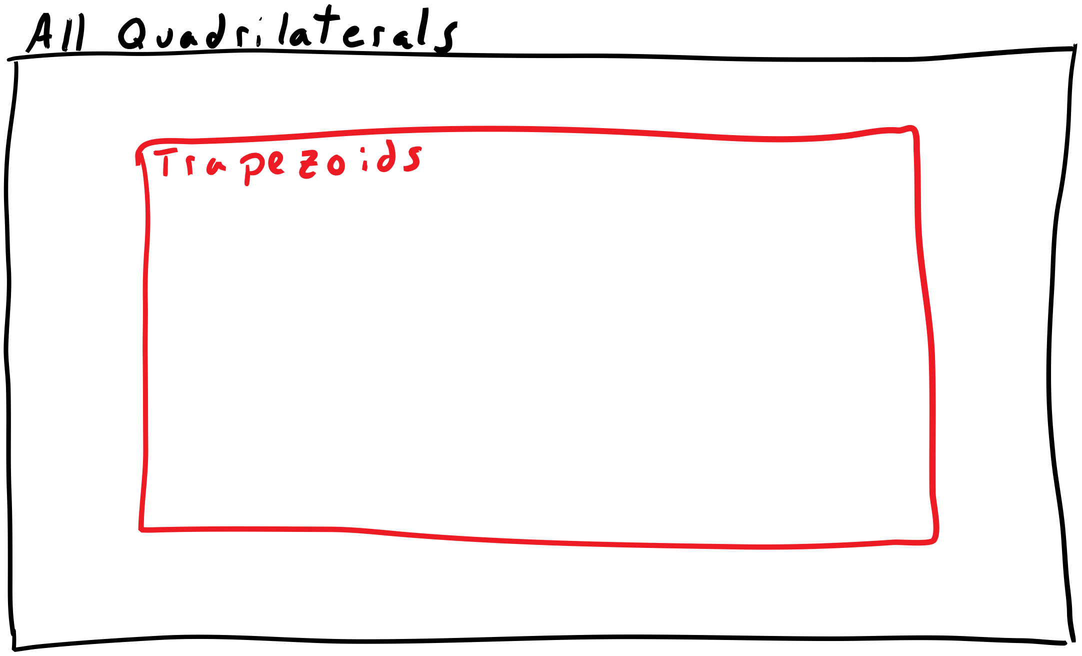
Here is a summary of our guesses:

Property		Square	Rectangle	Rhombus	Parallelogram	Trapezoid
Sides	1. All sides equal	Yes	No	Yes	No	No
	2. Opposite sides equal	Yes	Yes	Yes	Yes	No
	3. Opposite sides parallel	Yes	Yes	Yes	Yes	No
Angles	4. All angles equal	Yes	Yes	No	No	No
	5. Opposite angles equal	Yes	Yes	Yes	Yes	No
	6. Adjacent angles sum to 180°	Yes	Yes	Yes	Yes	No

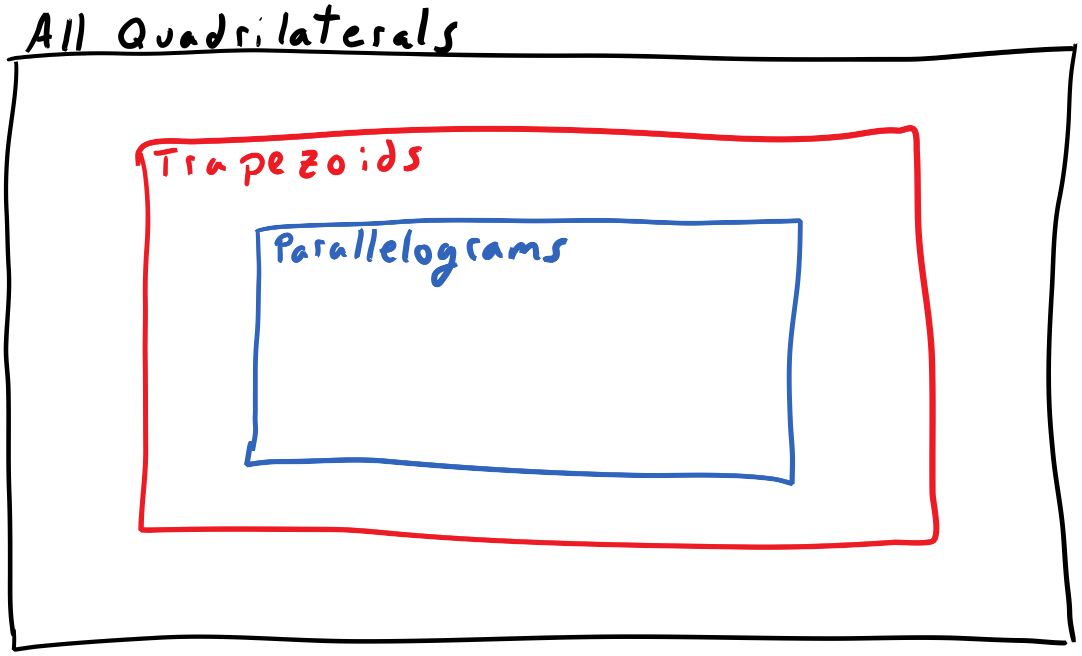
Venn Diagram for Quadrilaterals

Problem: Draw a Venn diagram depicting the relationships between all quadrilaterals, squares, rectangles, rhombuses, parallelograms, and trapezoids.

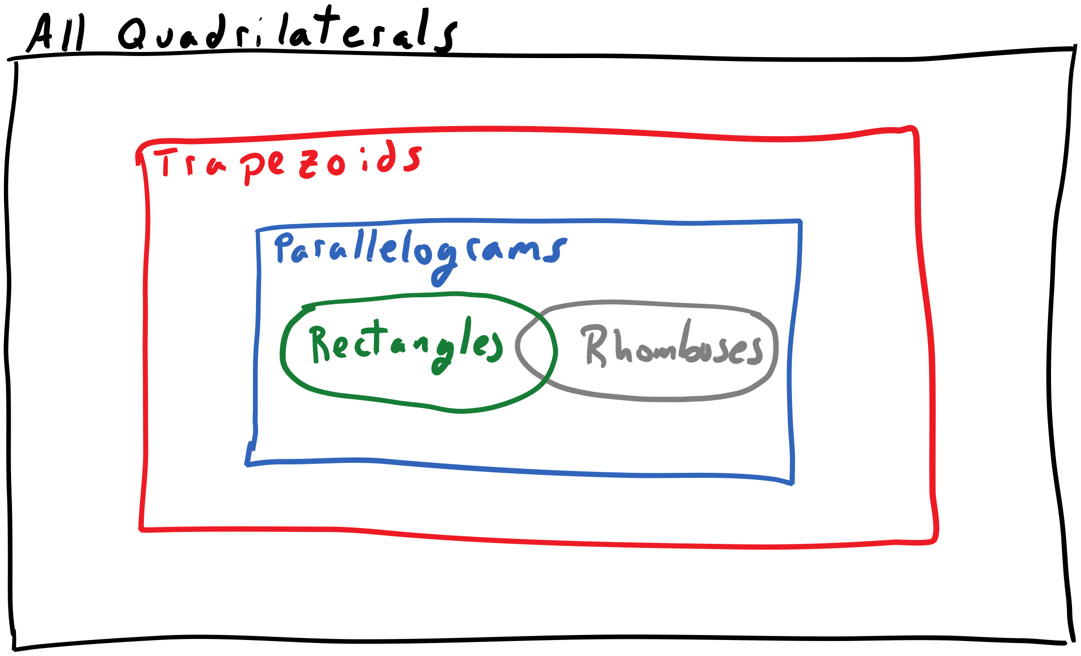
First, every type of quadrilateral we have discussed has at least one pair of parallel sides, so every type of quadrilateral we have will be inside a region for trapezoids. We draw that region.



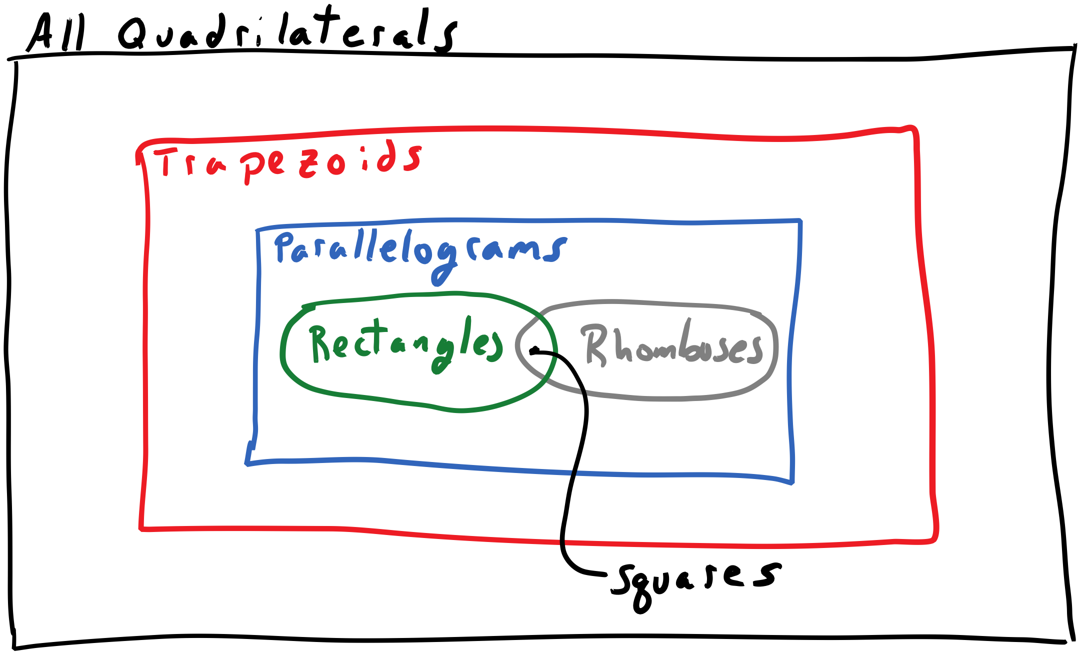
Next, rectangles, rhombuses, and squares are all parallelograms, so we add parallelograms.



Rhombuses and rectangles are parallelograms, and they can overlap:

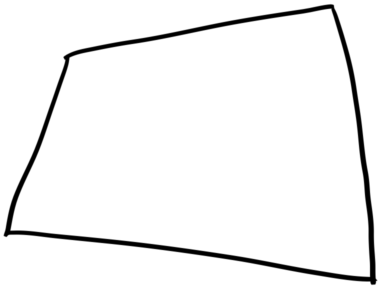


Finally, squares are exactly where rectangles and rhombuses overlap.

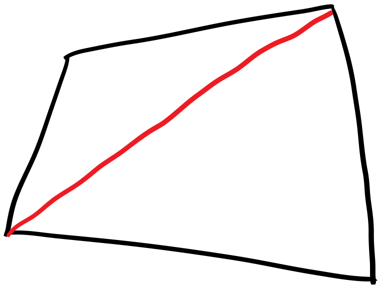


Sum of Angles

Problem: Find the sum of the angles in the quadrilateral pictured below.



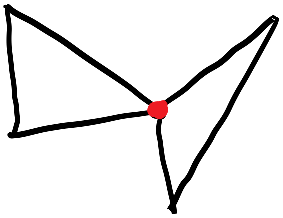
We are going to exploit the fact that we know the sum of the angles in a triangle is $180°$. First we draw a diagonal in the quadrilateral connecting opposite vertices.



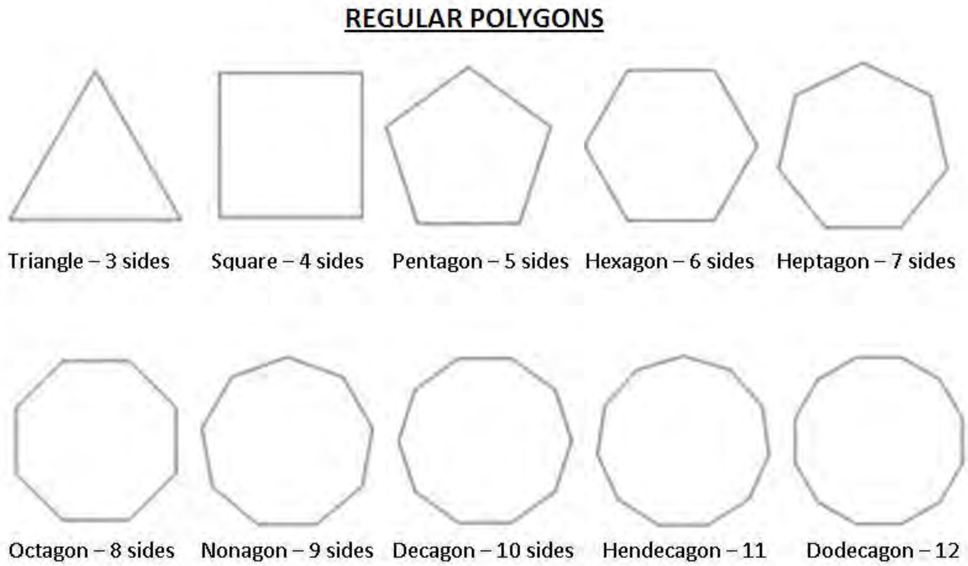
Now, the angles in the quadrilateral are made up of the angles from two triangles. The sum of the angles from the two triangles is $2 \times 180° = 360°$, so the sum of the angles in the quadrilateral is $360°$.

Polygons

A triangle is a closed shape in the plane made of three line segments. A quadrilateral is a closed shape in the plane made of four line segments that do not cross except at endpoints. We would like to extend this idea to an arbitrary number of line segments. To do so, we need to add one more condition to avoid issues like this "bow tie."

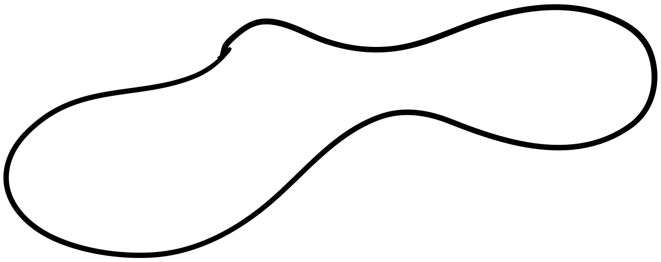


The problem is at the center red dot. We have too many line segments intersecting here. A **polygon** is a closed shape in the plane made of finitely many line segments that do not intersect each other except at endpoints so that no more than two line segments intersect at any endpoint. The line segments are called the **sides** of the polygon, and the point of intersection of two sides is a **vertex**. The word polygon is derived from Greek poly+gon or many+angles. Polygons are classified by their number of sides (which is equal to their number of angles). A polygon in which all sides are equal and all angles are equal is called a **regular polygon**. The diagram below shows several small regular polygons along with their names.



For large polygons, we adopt the same naming convention, but we dispense with the Greek. A polygon with 48 sides is a 48-gon. A polygon with 1024 sides is a 1024-gon. We could also call a square a 4-gon, but that is rarely done. A polygon with n sides is called an n-gon.

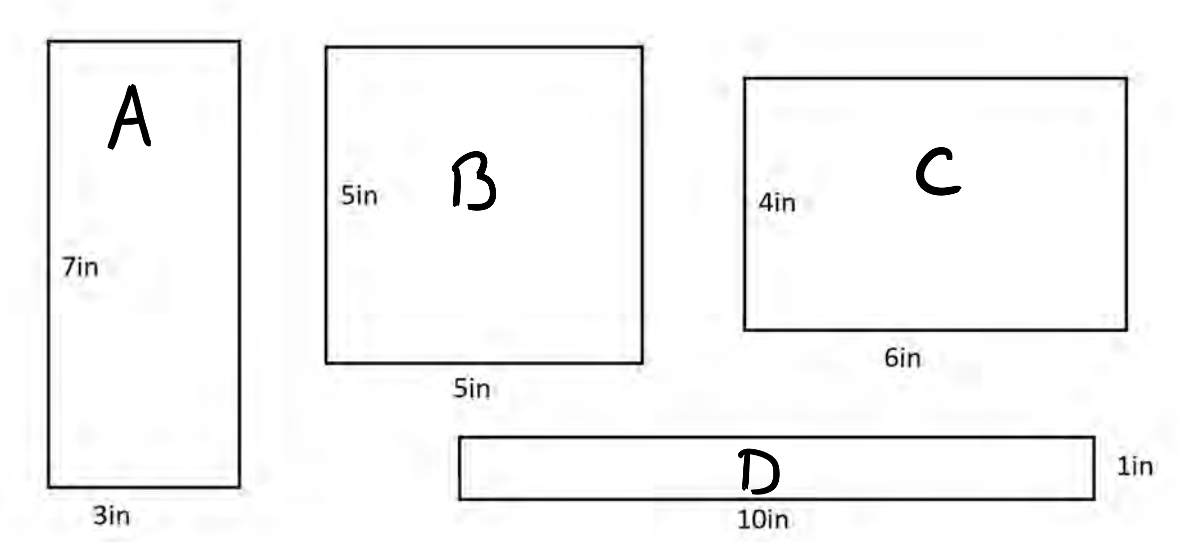
Problem: Why is the shape below not a polygon?



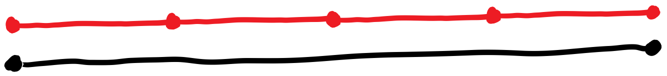
This shape is not a polygon because it is not made of line segments.

Measurement

Which shape below is the largest?



This exercise may seem somewhat academic (and it is), but the same sort of question might be asked when comparing two buildings or two trees or two people. We might say that A is largest because it is tallest. We might say that D is largest because it is widest or because it has the largest perimeter (distance around). We might also say that B is largest because it takes up the most area. The point is that if we want to declare one of these shapes to be the largest, then we first have to identify a feature of the shapes to compare. Also notice that we seem to be comparing the width and height of each of these shapes to something called an "in" (which is short hand for an inch). The width of A seems to be three times as long as an inch. The height of A is seven times as long as an inch. When we compare a quantity to a fixed amount or unit (such as an inch), that is called measuring. To **measure** a quantity is to compare that quantity to a fixed unit.



In the picture above, if we treat the red segments as our unit (they are all identical), then the black segment seems to be equivalent to four units.

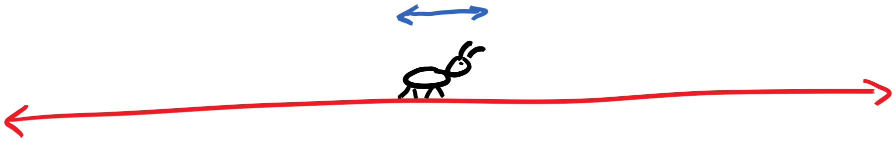
Structure of Measurement

There are a couple of assumptions or expectations we have about measurements. First, the purpose of measuring is to compare objects. Therefore, we assume that measurement imposes some sort of **order** on objects that allows us to compare them. If we measure peoples' heights, then we can order the people by height. If we measure what a student knows using an exam of some sort, then we can use that measurement to compare what the student knows now to what the student knew at some point in the past (to see if the student is learning).

Most measurement is also **additive**. If we combine two objects that weight 3 pounds and 5 pounds, the combined objects should weigh 8 pounds. If we place a box that is 3 feet tall on top of a box that is 5 feet tall, then the combined structure should be 8 feet tall. Not all measurement is additive though. If we pour water that is 75° into water that is 25°, the mixture is not going to be 100°.

Dimension

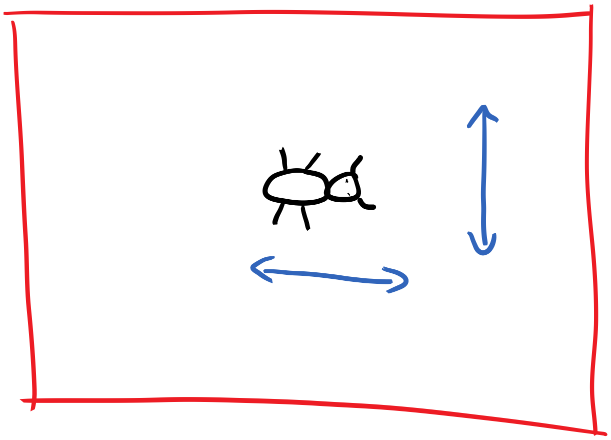
Most of the measuring we will do will be of one, two, or three dimensional features. An object is **one-dimensional** if at any point on the object motion is possible forward or backwards along one direction. A measurement of a one-dimensional object is called **length**. The canonical example of a one dimensional object is a line. If a bug is standing on a line (which is difficult because the line has no width) then that bug can move forward or backward along the line, and that is the only way the bug can move.



Problem: Name some one-dimensional features of a book.

Some one-dimensional features of a book may include the edge of a page, the line where two pages meet at the center of the book, or a line drawn on the page. We have to take care with identifying one-dimensional features. A line on a page actually has some width, so it is technically not one dimensional. However, it does represent a one-dimensional feature. Pages have thickness, so the edge of a page has the same issue.

An object is **two-dimensional** if at any point on the object motion is possible forward or backwards in two directions. A measurement of a two-dimensional object is called **area**. The canonical example of a two-dimensional object is the plane. If a bug is standing on the plane, then the bug can move forward and backwards along lines moving left and right or up and down.



Problem: Identify some two-dimensional features of a book.

Some two-dimensional features of a book are the front of the book or one side of a page in the book.

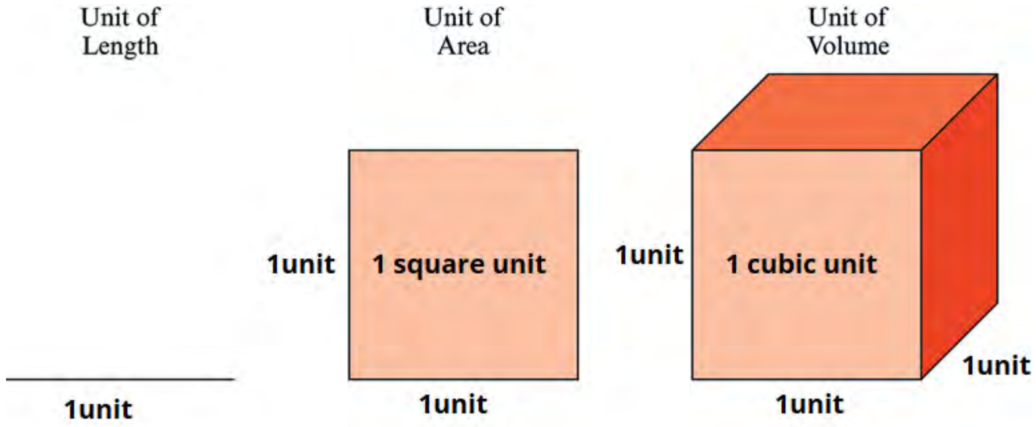
An object is **three-dimensional** if at any point on the object motion if possible forward or backwards in three directions. A measurement of a three-dimensional object is called **volume**. The standard example of a three dimensional object is space. If a bug is in space, then it can move forwards and backwards, left and right, or up and down.

Problem: Identify a three-dimensional feature of a book.

A book itself is a three-dimensional object.

Ideal, Unified System of Measurement

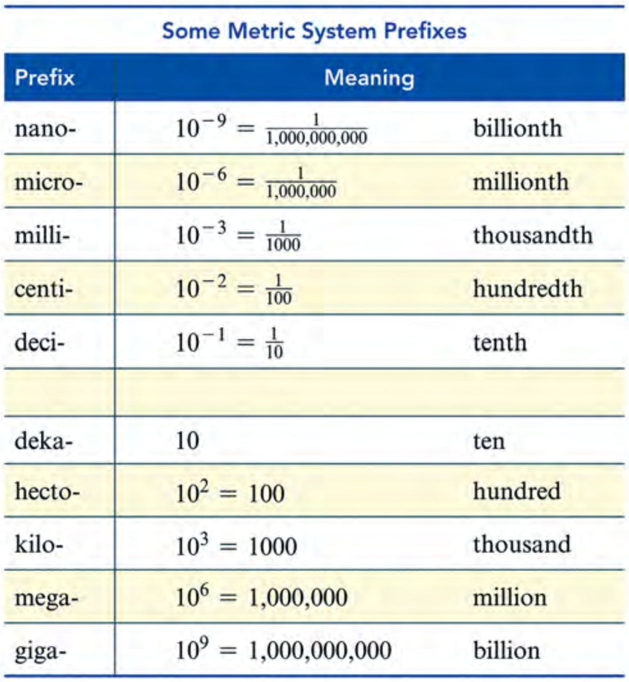
An ideal, unified system of measurement would first identify a unit with which to measure length. Call this $1\ unit$ or $1u$. Then the standard unit for measuring area would be a square whose edges have length $1u$. This would be called $1\ square\ unit$ or $1\ unit$ squared or $1u^2$. The standard unit for volume would be a cube whose edge lengths are each $1u$. This would be called 1 cubic unit or 1 unit cubed or $1u^3$.



There are currently two systems of measurement used in the United States (where this is being written), the U.S. Customary system of measurements and the metric system. The U.S. Customary system is based on the English system of measurements used in Britain up until 1824. The metric system or International System of Units (SI) was introduce in France in 1790 and has been adopted by every country in the world except for the United States, Liberia, and Myanmar. The U.S. Customary system does not have an apparent connection between some units of length, area, and volume. The metric system was intended to unify one, two, and three dimensional units and to make it easier to convert between large and small units. According to the Metric Conversion Act of 1975, the metric system is the "preferred" system of measurement in the United States for trade and commerce. The country has been slow in making a complete conversion to the metric system. If we had made this conversion, all of the exercises in the next section would be much simpler.

Basic Idea of the Metric System

The basic idea of the metric system was to introduce a unit for length (the meter), volume (the liter), and mass (the gram) and to use standard prefixes to make larger and smaller units. Here is a table of prefixes:



Some Metric System Prefixes

Prefix	Meaning	
nano-	$10^{-9} = \frac{1}{1,000,000,000}$	billionth
micro-	$10^{-6} = \frac{1}{1,000,000}$	millionth
milli-	$10^{-3} = \frac{1}{1000}$	thousandth
centi-	$10^{-2} = \frac{1}{100}$	hundredth
deci-	$10^{-1} = \frac{1}{10}$	tenth
deka-	10	ten
hecto-	$10^2 = 100$	hundred
kilo-	$10^3 = 1000$	thousand
mega-	$10^6 = 1,000,000$	million
giga-	$10^9 = 1,000,000,000$	billion

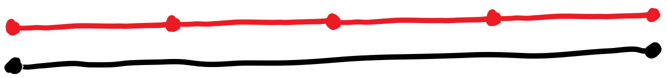
If the standard measure of length is a meter, then one tenth of a meter is a decimeter. A hundredth of a meter is a centimeter. A million meters is a megameter. Similarly, if the standard measure of mass is a gram, then a million grams is a megagram. Converting between large and small units amounts to multiplying or dividing by 10, 100, 1000, and so on. By comparison, a small unit of length in the U.S. Customary system is an inch. Twelve inches makes a foot. Three feet makes a yard. One mile is 1760 yards. There is no apparent rhyme or reason to the factors 12, 3, and 1760.

The metric system is further unified by a standard relationship between measures of length, volume, and weight (really mass): **One cubic centimeter of water has a volume of one milliliter and weighs one gram.**

Units of Length

Some standard U.S. Customary units of length are: an inch (about the width of a quarter, abbreviated in), a foot (twelve inches, ft), a yard (three feet, yd), and a mile (5280 feet, mi).

The standard metric unit of length is the meter (which is a little longer than a yard). Some commonly used metric units for length aside from the meter are millimeter (0.001 meters, about the thickness of a dime, abbreviated mm), centimeter (0.01 meters, about the width of a small fingernail, cm), and kilometer (1000 meters, km).



To say that an object has a length of 4 feet means that the object can be covered end to end with 4 one-foot segments without gaps and without overlaps.

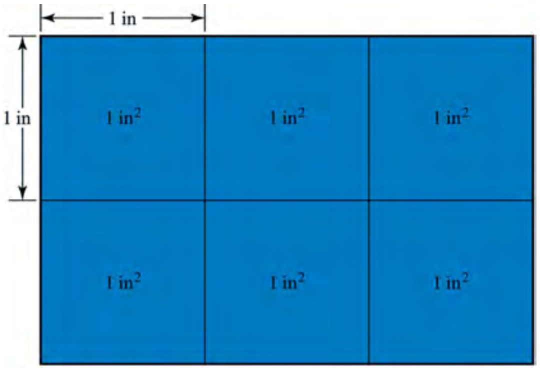
Problem: Which U.S. Customary and which metric units would you use to measure: the width of a book, the height of a person, the distance between two buildings on a college campus, and the distance between two cities?

The width of a book is small, so we would measure it in inches or centimeters. We would not use millimeters because these would be for tiny objects. The height of a person is usually measured in feet or meters. We could measure the height of a person in inches or centimeters, but that would make for larger numbers. A person who is six feet tall would measure 72 inches, 1.83 meters, or 183 centimeters. We would probably measure the distance between two buildings in feet or meters. Yards might actually be more appropriate, but they are not actually used that much. Cities tend to be far apart, so we would use the larger measures mile and kilometer to measure the distance between cities.

Units of Area

Some U.S. Customary units of area are: square inch (abbreviated in^2), square foot (ft^2), square mile (mi^2), and acre (one square mile is 640 acres).

Some metric units of area are: square centimeter (cm^2), square meter (m^2), and square kilometer (km^2).



To say that an object has an area of 6 square inches means that the object can be covered with 6 one-inch squares without gaps, without overlaps, but maybe with some cutting.

Problem: Which U.S. customary and which metric units would you use to measure the area of a page in a book, the floor in a room, or a city?

A page in a book is pretty small, so we would likely use square inches or square centimeters. We would probably use square feet or square yards to measure the area of the floor in a room. Another option would be square yards, but that is not too common these days. Cities are large, so we would likely measure their area with square miles, acres, or square meters.

Units of Volume

Some U.S. Customary units of volume are gallon (gal), cup (c), quart (qt), cubic inch (in^3), and cubic foot (ft^3).

Some metric units of volume are liter (L), milliliter (mL), cubic centimeter (cm^3), and cubic meter (m^3).

To say that an object has a volume of 10 cubic feet means that the object (or the space it occupies) can be filled with 10 one-foot cubes without gaps, without overlaps, but maybe with some cutting. Sometimes, we see the word **capacity** rather than volume. Suppose that we have a gallon jug of milk. When discussing how much the jug – a container – holds, we may use the word capacity. The jug has a capacity of one gallon. This means that the amount of milk necessary to fill the jug is one gallon. If we are talking about the amount of milk, we use the word volume. The jug has a capacity to hold a volume of one gallon of milk. This difference is small and can be confusing. We will usually refer to capacity and volume as volume.

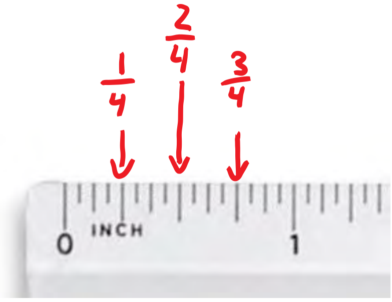
Problem: What units would we use to measure the volume of a drinking glass, a bathtub, a room, or a swimming pool?

A drinking glass is small. We would use ounces or cubic inches if using U.S. Customary units. For metric units, we would probably use cubic centimeters or a fractional liter. A bathtub is significantly larger than a glass. We could measure its volume with cubic feet, gallons, cubic meters, or liters. Rooms are even larger than bathtubs. We would measure the volume of a room in cubic feet or cubic meters. We would measure the volume of a swimming pool in cubic feet or cubic meters also. We might also use gallons (and have a large number of gallons) or cubic yards.

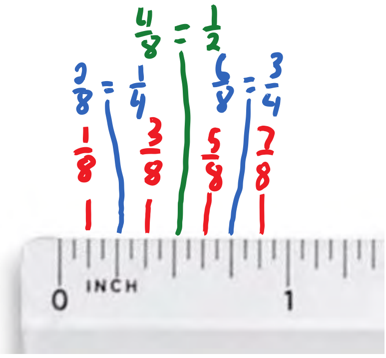
Using a Ruler

We said earlier that for an object to have a length of five inches means that the object can be covered from end to end with five one inch segments without gaps and overlaps. A tool which has one inch segments stacked end to end without gaps or overlaps is a ruler.

This ruler has one inch segments on one side and one centimeter segments on the other side. The one inch segments have hash marks indicating fractions of an inch. On this ruler, the long hash marks (not corresponding to whole inches) are quarters.



The medium hash marks are eighths.



Notice how some of the eighths line up with quarters. The smallest hash marks on this ruler measure sixteenths. Some of these will line up with quarters and eighths. To measure a line segment in inches with this ruler, we first line the end of the segment up with the 0 on the inches side.

We then identify the hash mark that is closest to the other end of the segment.

By counting hash marks, we determine this is the $\frac{3}{4}$ hash mark between 4 and 5, so this line segment is $4\frac{3}{4}$ inches long.

The centimeter side of the ruler is easier to use as far as fractions go. The small hash marks are each $1mm$ or $\frac{1}{10}cm$ or $0.1cm$ apart. When measuring with the centimeter side, our measurements will usually be in decimals. First line up the end of the segment with the 0 on the centimeter side of the ruler.

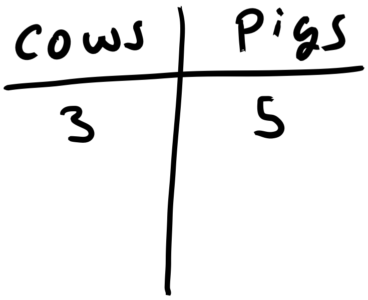
Then identify the hash mark closest to the other end. In this case, the other end is at the fourth hash mark between 11 and 12, so the line segment is $11.4cm$ long.

Unit Conversions

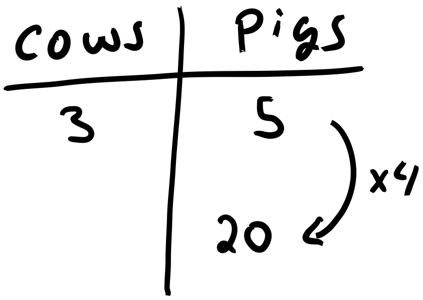
Sometimes we may measure an object with one unit – say feet – and then need to know the same measurement in another unit – say meters. Rather than measuring again, we can convert one unit into the other. The basis for unit conversions is the idea of a proportional relationship.

Problem: Jimmy has beads shaped like cows and pigs. When Jimmy lines up the cows next to the pigs, 3 cows have the same length as 5 pigs. Jimmy measures his notebook with pigs and discovers it is 20 pigs wide. How wide is the notebook in cows?

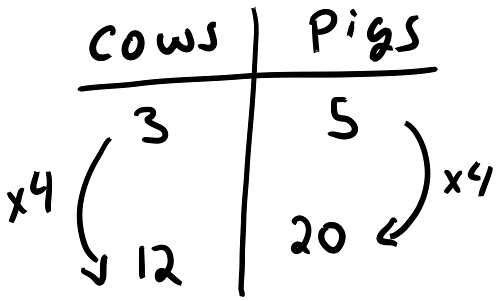
The secret here is that the numbers of cows and pigs are related proportionally, every 3 cows corresponds to 5 pigs. We could solve this problem with a ratio table. We draw a table with columns for cows and pigs and first fill in one row with the only known correspondence we have between cows and pigs.



We then manipulate the pigs column using multiplication until we have 20 pigs.



And then we perform the same operation to the cows column.



We now know that if 3 $cows = 5\ pigs$ then 20 $pigs = 12\ cows$.

If we think back to proportional relationships and ratio tables, we should recall that most proportion problems could be solved using *unit rates*. To convert $20\ pigs$ to $cows$, we simply multiply the number of $pigs$ by the *unit rate of cows per pig*. To find the unit rate of cows per pig, we simply take any equation we know relating cows and pig such as $3\ cows = 5\ pigs$ and divide both sides by the expression with pigs (here, $5\ pigs$). Since $3\ cows = 5\ pigs$, then

$$\frac{3\ cows}{5\ pigs} = \frac{5\ pigs}{5\ pigs} = 1.$$

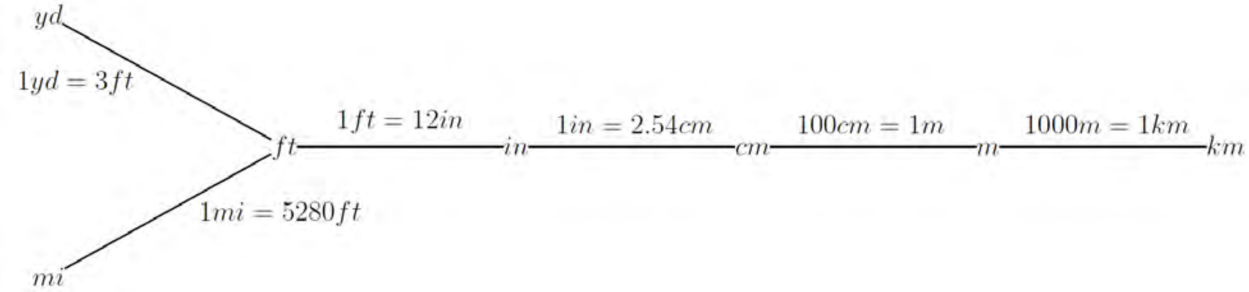
The unit rate of cows per pig is $\frac{3\ cows}{5\ pigs}$. To convert from pigs to cows, we simply multiply by this unit rate:

$$20\ pigs = \frac{20\ pigs}{1} \times \frac{3\ cows}{5\ pigs} = \frac{20\ \cancel{pigs}}{1} \times \frac{3\ cows}{5\ \cancel{pigs}} = \frac{60\ cows}{5} = 12\ cows.$$

Notice how we can visualize the $pigs$ cancelling. This helps to remind us which unit goes on bottom of our unit rate. If we have pigs and want cows, then cows should go on top and pigs on bottom.

Conversion Maps

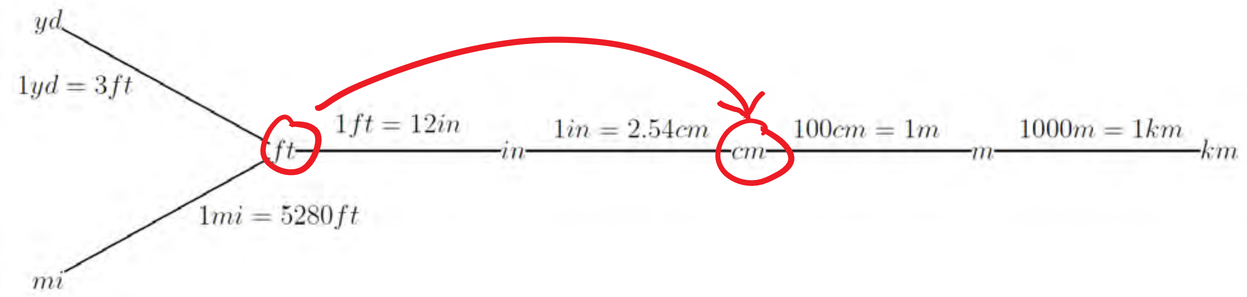
Of course, we do not usually measure lengths in cows and pigs. We have other units for that – inches, feet, yards, miles, meters, and centimeters to name a few. For any pair of units measure the same type of quantity there is a unit rate that can be used to convert from one to the other. We can convert from inches to feet or from inches to yards or from inches to meters, and so on. This would be quite the number of unit rates to memorize. Instead of keeping track of all of these unit rates, we use a graphic called a **conversion map** that keeps track of how units are related and minimizes the number of unit rates we need to know. Here is a conversion map for units of length:



The map allows us to find some unit rates quickly. For example, it is easy to get from yard to feet by using unit rate $\frac{3\ ft}{1\ yd}$ based on the equality $1\ yd = 3\ ft$. We could just as easily get from feet to inches or inches to centimeters. If we want to convert from yards to centimeters, we would do the arithmetic to convert from yards to feet and then to inches and then to centimeters by multiplying by three unit rates. Examples of this process are below.

Problem: Convert $0.75\ ft$ to centimeters.

First, we locate feet and centimeters on the conversion map for lengths, and we indicate a path from feet to centimeters.



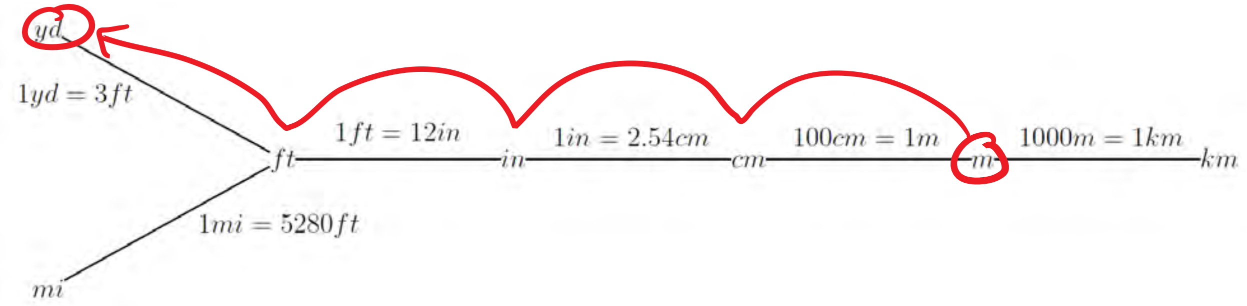
It is two steps from feet to centimeters on this map – one step from feet to inches and one from inches to centimeters. We will first multiply by a unit rate for the step from feet to inches and then multiply by a unit rate for the step from inches to centimeters. The unit rate for the first step is $\frac{12\ in}{1\ ft}$. Note how we put the feet on bottom because we have feet already and want to convert to something else. Multiplication by this unit rate will give us inches. The unit rate for inches to centimeters is $\frac{2.54\ cm}{1\ in}$. Notice how we put the inches on bottom and the centimeters on top because we have inches and want centimeters. Here is all of the multiplication together:

$$\begin{aligned} 0.75\ ft &= \frac{0.75\ ft}{1} \times \frac{12\ in}{1\ ft} \times \frac{2.54\ cm}{1\ in} \\ &= \frac{0.75\ \cancel{ft}}{1} \times \frac{12\ \cancel{in}}{1\ \cancel{ft}} \times \frac{2.54\ cm}{1\ \cancel{in}} \\ &= \frac{0.75 \times 12 \times 2.54\ cm}{1} \\ &= 22.86\ cm. \end{aligned}$$

Notice how the feet and the inches cancel to leave us with centimeters for units.

Problem: Convert $10m$ to yards.

We locate the appropriate units on the length conversion map:



This conversion will take four steps or four unit rates – meters to centimeters, centimeters to inches, inches to feet, and feet to yards. At each step of the way, the unit we have and are trying to get rid of goes on bottom of the unit rate. The unit we want goes on top. Since we are moving from right to left, this means that the left unit in each equation goes on top.

$$\begin{aligned} 10\ m &= \frac{10\ m}{1} \times \frac{100\ cm}{1\ m} \times \frac{1\ in}{2.54\ cm} \times \frac{1\ ft}{12\ in} \times \frac{1\ yd}{3\ ft} \\ &= \frac{10\ \cancel{m}}{1} \times \frac{100\ \cancel{cm}}{1\ \cancel{m}} \times \frac{1\ \cancel{in}}{2.54\ \cancel{cm}} \times \frac{1\ \cancel{ft}}{12\ \cancel{in}} \times \frac{1\ yd}{3\ \cancel{ft}} \end{aligned}$$

$$\begin{aligned} &= \frac{10 \times 100\ yd}{2.54 \times 12 \times 3} \\ &= \frac{1000}{91.44} yd \\ &= 10.94 yd. \end{aligned}$$

Note that we rounded the final answer to two decimal places. At our level, with no other directions, two decimal places is appropriate.

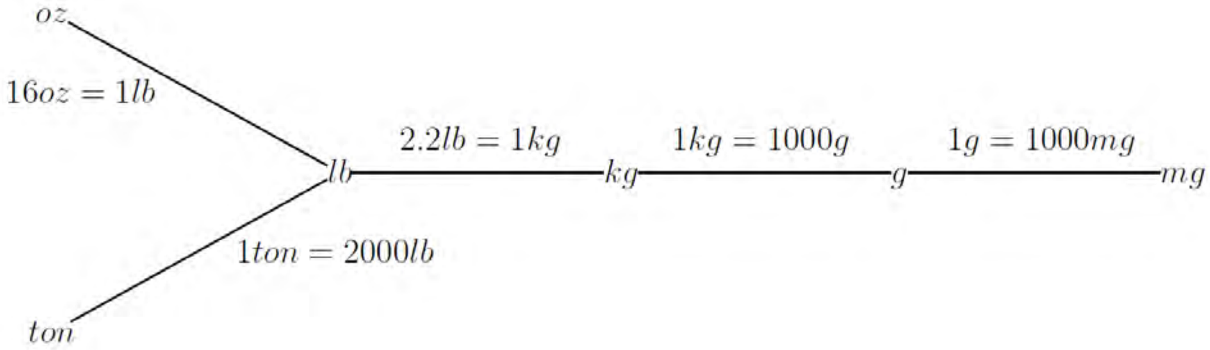
Problem: Convert 10 ft^2 to in^2.

What makes this problem different is that it is an area conversion problem as indicated by the exponents. One square foot is a square which measures one foot or 12 inches on each side. This square can be thought of as 12 rows of 12 one inch squares. Thus one square foot is 12 × 12 square inches. To find the unit rate for converting from square feet to square inches, we simply square the unit rate for converting from feet to inches. Our notation should help us remember that. To cancel ft^2 think of ft^2 as $ft \times ft$ and multiply by $\frac{12\ in}{1\ ft}$ twice. Thus

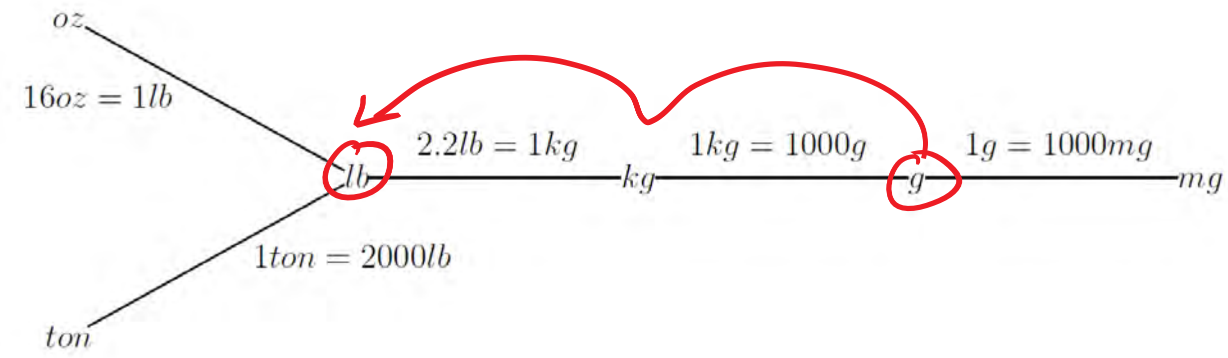
$$\begin{aligned} 10\ ft^2 &= \frac{10\ ft^2}{1} \times \frac{12\ in}{1\ ft} \times \frac{12\ in}{1\ ft} \\ &= 10 \times 12 \times 12 \times in \times in \\ &= 1440\ in^2. \end{aligned}$$

Problem: Convert $500g$ to pounds.

This is a weight problem, but that is fine because we have a weight conversion map:



As usual, we find the path from the unit we are given to the unit we want:

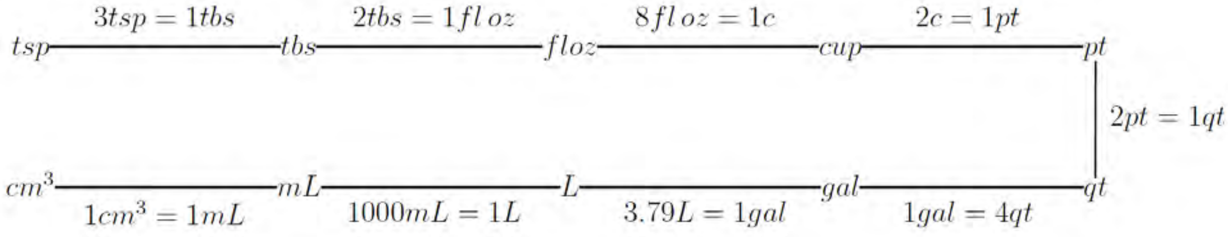


This is two steps, which requires two unit rates:

$$\begin{aligned}500\ g &= \frac{500\ g}{1} \times \frac{1\ kg}{1000\ g} \times \frac{2.2lb}{1\ kg} \\ &= \frac{500\ \cancel{g}}{1} \times \frac{1\ \cancel{kg}}{1000\ \cancel{g}} \times \frac{2.2lb}{1\ \cancel{kg}} \\ &= \frac{500 \times 2.2\ lb}{1000} \\ &= \frac{1100}{1000} lb \\ &= 1.1\ lb.\end{aligned}$$

Problem: Convert $1L$ to cubic inches.

This is a volume problem, but that is fine because we have a volume conversion map:



While we may be able to find liters on this conversion map, there are no cubic inches. However, there are cubic centimeters. What we will have to do is first convert liters to cubic centimeters and then convert cubic centimeters to cubic inches. Converting liters to cubic centimeters is two steps:

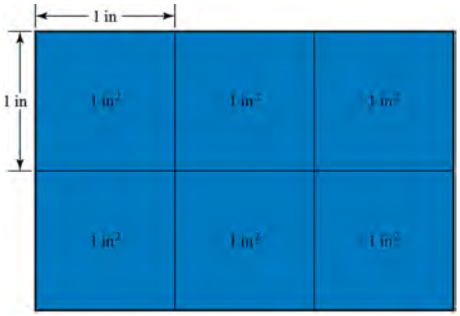
$$1\ L = \frac{1\ L}{1} \times \frac{1000\ mL}{1\ L} \times \frac{1\ cm^3}{1\ mL}.$$

To convert cm^3 to in^3 we have to follow a process similar to what we did for area above – we must multiply by our unit rate **three times** to cancel the exponent:

$$\begin{aligned}1\ L &= \frac{1\ L}{1} \times \frac{1000\ mL}{1\ L} \times \frac{1\ cm^3}{1\ mL} \times \frac{1\ in}{2.54\ cm} \times \frac{1\ in}{2.54\ cm} \times \frac{1\ in}{2.54\ cm} \\ &= \frac{1000\ in \times in \times in}{2.54 \times 2.54 \times 2.54} \\ &= \frac{1000}{16.387064} in^3 \\ &= 61.02\ in^3.\end{aligned}$$

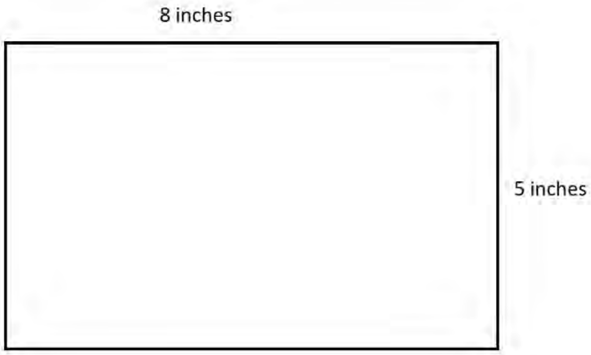
Area

Recall that when we say that the area of a two dimensional shape is 6 square inches we mean that the shape can be covered with exactly 6 one inch squares without gaps and without overlaps but possibly with some cutting.



In this section, we will find areas of a variety of shapes on the plane. we start with rectangles.

Problem: Without referring to any formulas, explain why the area of this rectangle is 40 square inches.



We can exactly cover this shape with 5 rows of one inch squares where each row contains 8 squares. Since 5 groups of 8 objects contains $5 \times 8 = 40$ objects, we exactly cover this shape with 40 one inch squares. That makes the area 40 square inches.

If we call the horizontal measure of a rectangle its width and the vertical measure its length (this assignment is arbitrary), then this approach tells us we can cover a rectangle with length l and width w with l rows of w one inch squares, so the area will be $l \times w$.

Area of a Rectangle: The area of a rectangle is $length \times width$.

Notice that to calculate area we are multiplying measure of length together. If we measure length in inches, then the area has units $inches \times inches$ or $inches^2$. This is the source of the square notation for area.

Fundamental Principles of Area

We use the formula for the area of a rectangle as the basis from which we will derive every other area formula. For these derivations, we need the following two principles (which are axioms of geometry).

Moving Principle: If a shape is moved rigidly without stretching it, then its area does not change.

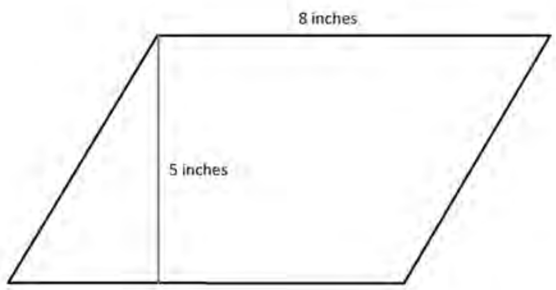
Additivity Principle: If two or more shapes are combined without overlaps, then the area of the new shape is the sum of the areas of the original shapes.

Our strategy to discover new area formulas is going to be to take a shape, cut it into pieces, and rearrange the pieces until we can apply the rectangle formula. The two principles declare that the cutting and rearranging do not change the total area.

Areas of Parallelograms

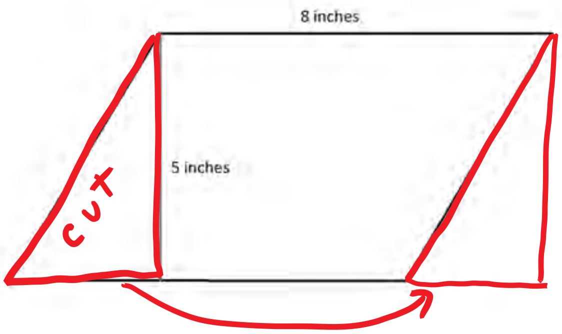
First, we use the area formula for a rectangle to find the area formula for a parallelogram.

Problem: Use the moving and additivity principles along with the area of a rectangle to explain why the area of this parallelogram is 40 square inches.

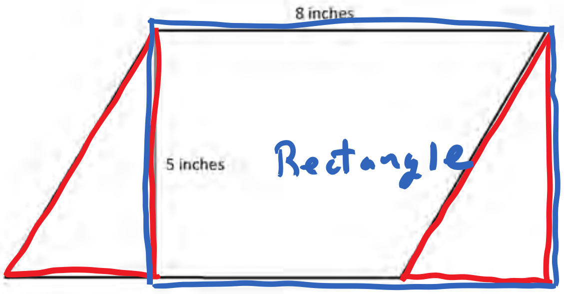


We call the sides with length 8 inches the **bases** of our parallelogram. The other distance here, 5 inches, is the length of a line segment which is perpendicular to lines containing the bases. The length of such a line segment is the **height** of our parallelogram. We note that the selection of which side we call a base is irrelevant. If we had selected the non-horizontal sides as a bases, then the height would be the length of a line segment perpendicular to lines containing those sides.

The height that we have drawn in this parallelogram creates a triangle on the left side of the parallelogram. We could cut that triangle off and move it to the right end of the parallelogram.



Doing so would not change the area of the figure. Moving the triangle would give us a rectangle with dimensions 8 $inches$ by 5 $inches$.



The rectangle area formula now tells us the area is $8 \times 5 = 40$ square inches.

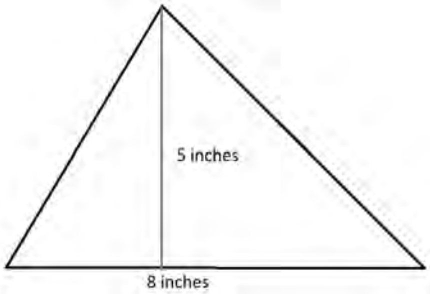
Notice here that we ended up with the area of the parallelogram being the length of the base times the height. Something similar happens for any parallelogram, although the cutting and moving might be more complicated for some parallelograms.

Area of a Parallelogram: The area of a parallelogram is $(length\ of\ base) \times height$.

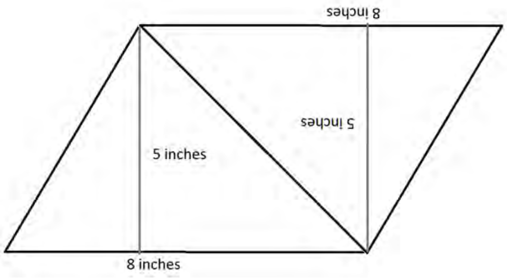
Areas of Triangles

We now turn to triangles.

Problem: Use the moving and additivity principles along with the formula for the area of a parallelogram to explain why the area of the triangle below is 20 square inches.



First, we have selected one of the sides of the triangle and draw it horizontally. We call this the **base** of the triangle. The selection of the base is arbitrary. Any side can be considered the base. Then, we have drawn a perpendicular line segment from the base to the vertex not on the base. The length of this line segment is the height of the triangle. We can copy the triangle, rotate it, and place the copy on top of the original like so:



When we do this, the result is a parallelogram with base length 8 inches and height 5 inches. The area of this parallelogram is

$$area\ of\ parallelogram = (length\ of\ base) \times height = 8 \times 5 = 40\ in^2.$$

Since the parallelogram is made of two copies of our triangle, the area of the triangle is half of this area:

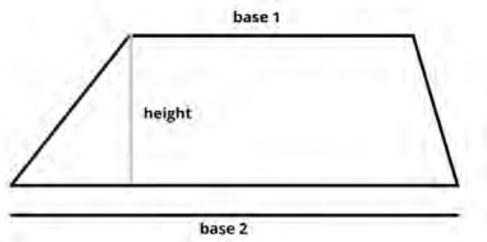
$$area\ of\ triangle = \frac{1}{2} \times (area\ of\ parallelogram) = \frac{1}{2} \times (length\ of\ base) \times height = 20\ in^2.$$

Notice that the area of the triangle ended up being half of the length of the base times the height. Something similar will happen with any triangle, although the diagrams may be more difficult to draw with some.

Area of a Triangle: The area of a triangle is $\frac{1}{2} \times (length\ of\ base) \times height.$

Area Formulas

We can also cut and rearrange trapezoids to find their areas in terms of rectangles. The two parallel sides of a rectangle are the bases of the trapezoid. The distance between these bases is the height of the trapezoid.



The area turns out to be:

Area of a Trapezoid: The area of a trapezoid is $\frac{1}{2} \times (sum\ of\ the\ lengths\ of\ the\ bases) \times height.$

In this formula, the expression $\frac{1}{2} \times (sum\ of\ the\ lengths\ of\ the\ bases)$ is the average of the lengths of the bases, so the area is the average base length times the height. This is consistent with parallelograms and rectangles.

A quick summary of what we have covered in relation to area formulas is this:

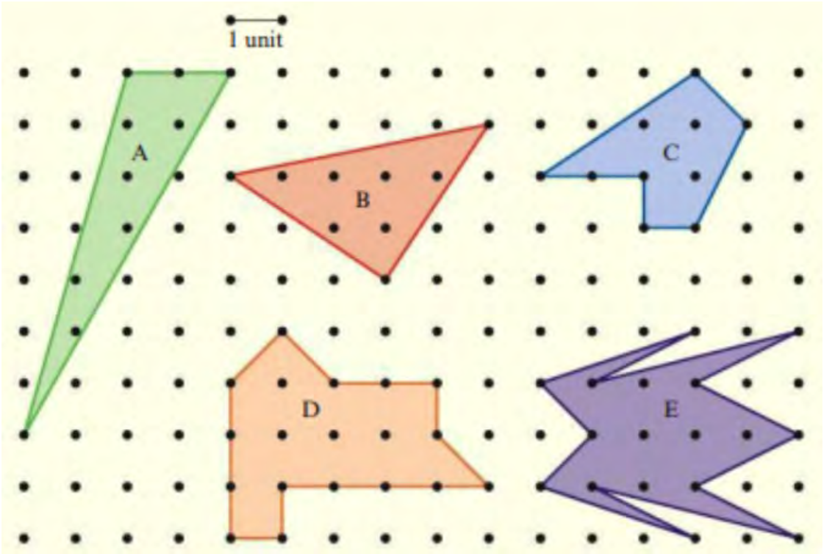
- Rectangles can be covered with rows of squares. The formula for area is simply the number of rows times the number of squares in each row.
- Parallelograms and trapezoids can be cut up and rearranged to fit into rectangles with the same base length and height. Then the rectangle formula can be used.
- A triangle is half of a parallelogram, so its area formula is half of the parallelogram area formula.

And here are all of our formulas in one place:

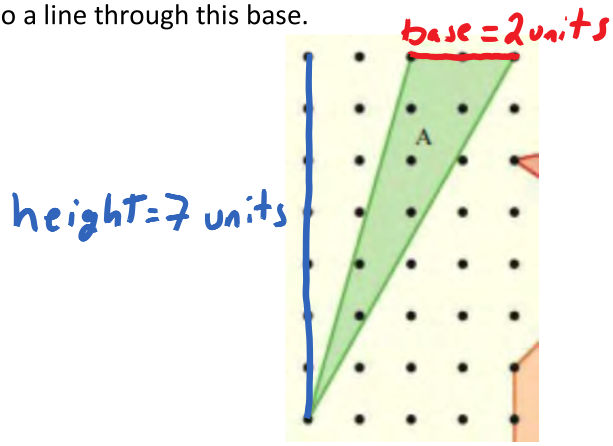
Area of a Rectangle: The area of a rectangle is $length \times width.$
Area of a Parallelogram: The area of a parallelogram is $(length\ of\ base) \times height.$
Area of a Triangle: The area of a triangle is $\frac{1}{2} \times (length\ of\ base) \times height.$
Area of a Trapezoid: The area of a trapezoid is $\frac{1}{2} \times (sum\ of\ the\ lengths\ of\ the\ bases) \times height.$

Area Examples

Problem: Find the area of each of the shapes below.

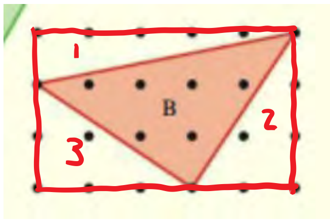


A. We start with shape A, which is a triangle. All we need to do is to identify the base and height of the triangle. The base needs to be selected so that its length is easy to identify. We choose the top, horizontal side. We then can measure the height by drawing a perpendicular from the other vertex to a line through this base.



The area is now $area = \frac{1}{2} \times (length\ of\ base) \times height = \frac{1}{2} \times 2 \times 7 = 7\ units^2$.

B. Shape B is also a triangle, but it is difficult to find the length of any of the sides because they are not horizontal or vertical. We draw a rectangle around the triangle and get creative:

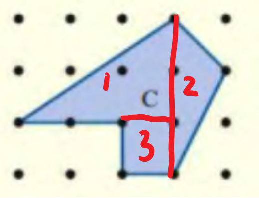


This rectangle is now made up of four triangles labeled 1, 2, 3, and B. If we find the area of the rectangle and subtract from that the area of triangles 1, 2, and 3, then we will be left with the area of B (this is the additive principle of area). The area of the rectangle is $3 \times 5 = 15\ units^2$. The area of the triangles are:

- Triangle 1: $\frac{1}{2} \times 5 \times 1 = \frac{5}{2} units^2$
- Triangle 2: $\frac{1}{2} \times 2 \times 3 = 3\ units^2$
- Triangle 3: $\frac{1}{2} \times 3 \times 2 = 3\ units^2$

Therefore, the area of Triangle B is $15 - \frac{5}{2} - 3 - 3 = \frac{13}{2} units^2$.

C. Shape C is not one of our basic shapes, but we can cut it into pieces that are.

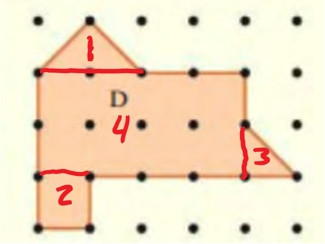


Shape 1 is now a triangle with base 3 units and height 2 units. Shape 2 is a triangle with a vertical base of length 3 units and a height of 1 unit. Shape 3 is a square with edge length 1 unit. The areas of the shapes are:

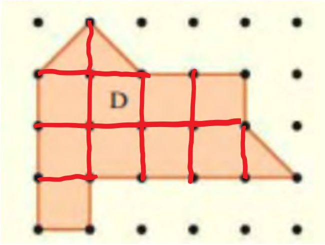
- Triangle 1: $\frac{1}{2} \times 3 \times 2 = 3\ units^2$
- Triangle 2: $\frac{1}{2} \times 3 \times 1 = \frac{3}{2}\ units^2$
- Square 2: $1 \times 1 = 1\ unit^2$

The total area for C is then $3 + \frac{3}{2} + 1 = \frac{11}{2} units^2$.

D. Shape D can also be broken down into simple shapes:

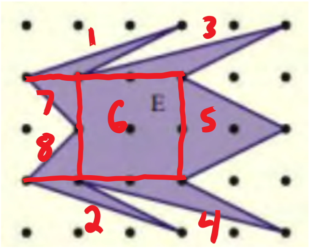


Shape 1 is a triangle with base $2\ units$, height $1\ unit$, and area $1\ unit^2$. Shape 2 is a square with edge $1\ unit$ and area $1\ unit^2$. Shape 3 is half of one of these squares, so its area is $\frac{1}{2} unit^2$. Shape 4 is a rectangle with base $4\ units$, height $2\ units$, and area $8\ units^2$. The total area is then $1 + 1 + \frac{1}{2} + 8 = \frac{21}{2} units^2$. Just for fun, here is another way to chop up shape D:



Each large piece is $1\ unit^2$. Each small piece is $\frac{1}{2} unit^2$. We could not find the area by counting.

E. We handle shape E similarly by chopping it up:

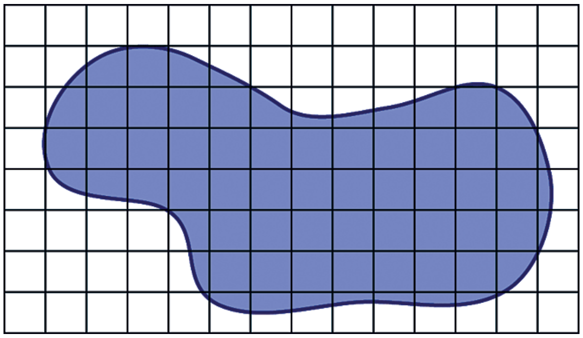


- Shapes 1, 2, 7, and 8 are all triangles with base and height $1\ unit$. Each has area $\frac{1}{2} unit^2$.
- Shapes 3 and 4 are both triangles with base $2\ units$ and height $1\ unit$. Each has area $1\ unit^2$.
- Shape 5 is a triangle with base (vertical) and height (horizontal) both equal to $2\ units$. Its area is $2\ units^2$.
- Shape 6 is a $2\ unit$ square with area $4\ units^2$.

The total area is $4 \times \frac{1}{2} + 2 \times 1 + 2 + 4 = 10\ units^2$.

Areas of Irregular Shapes

There are some shapes for which we may want the area but for which we do not have a formula. For example:

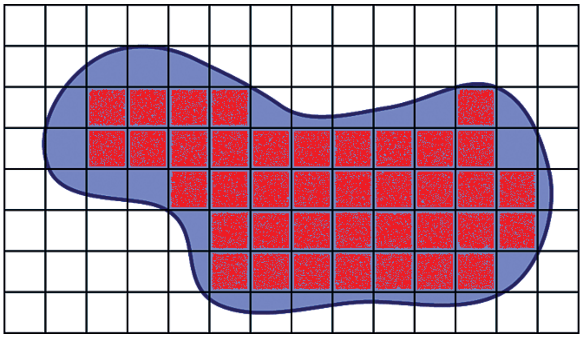


In these circumstances, our best hope is sometimes to approximate the area.

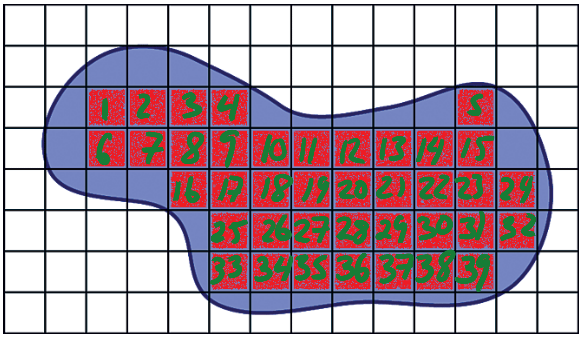
Problem: Assume that the length of an edge of each square in the image above is $1\ unit$. Approximate the area of the shape.

To approach this problem, we are going to find a rough estimate that is too small (which we will call and under-estimate), and we will find a rough estimate that is too large (called an over-estimate). Then we will average these two numbers. This will give a repeatable process that tries to minimize subjectivity.

First, we address the under-estimate. Each square here has area $1\ unit^2$. All we are going to do for the under-estimate is count all of the squares which are *entirely inside* of the shape. First, we mark those entirely inside to help with counting:

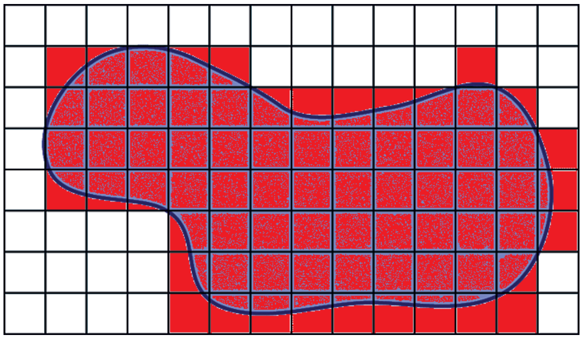


Then we count:

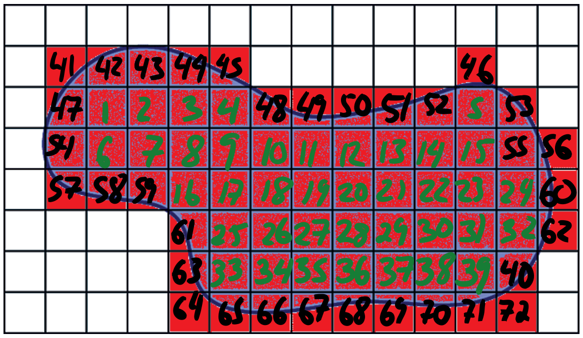


We could probably make counting go faster by grouping the squares, but numbering them works too. Our under-estimate is 39 $units^2$. However, this estimate is clearly too small.

For our over-estimate, we count every square that the shape even touches. First we mark them all:



Then, we count. When we count, we go ahead and make use of the numbering that we did with the under-estimate so that we do not have to recount those squares.



Our over-estimate is 72 $units^2$. This estimate is obviously too big. To get a better estimate, we average our under-estimate and over-estimate:

$$Area \approx \frac{Under + Over}{2} = \frac{39 + 72}{2} = 55.5\ units^2.$$

Problem: Assume now that each edge of each square in the image above is 3 miles long. Approximate the area of this shape in acres.

If each edge is 3 miles long, then each square is $3 \times 3 = 9\ mi^2$. The entire area is now approximately

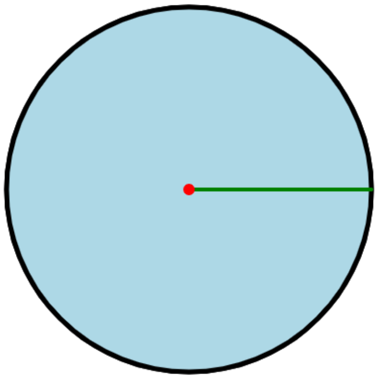
$$area \approx 55.5 \times 9 = 499.5\ mi^2.$$

Now, one square mile is 640 acres (we can check our conversion maps for that), so this blue ares is approximately

$$area \approx 499.5\ mi^2 \times 640 = 319680\ acres.$$

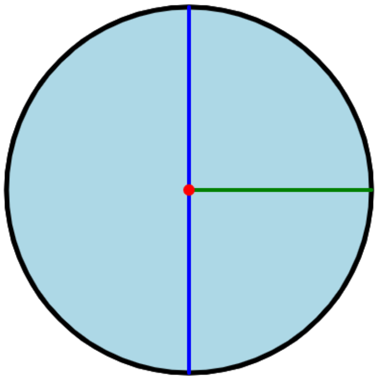
Circles

A **circle** is the set of all points on the plane which are a certain fixed distance from a certain fixed point. The fixed point is the center of the circle, and the fixed distance is the radius of the circle.



More precisely, if P is any point on the plane and r is any positive number, then the circle centered at P with radius r is the set of all points in the plane which are exactly r units from P. In the diagram above, the red point indicates the center of the circle. The green line segment has a length equal to the radius of the circle, and the circle is depicted in black. Note that the region that is light blue is not the circle, but is inside the circle. These are the points which are less than the radius from the center. Points outside the circle (which are white in the diagram) are more than the radius from the center.

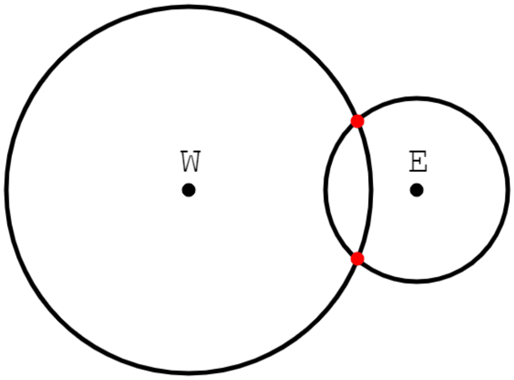
We use the word radius not only to refer to the distance from the center of a circle to the circle but also to refer to any line segment which extends from the center of the circle to the circle. With this convention, the green line segment above is a radius of the circle. The **diameter** of a circle is twice the radius. We also use the word diameter to mean a line segment which passes through the center of a circle and which has both endpoints on the circle. In the diagram below, the blue line segment is a diameter.



Notice that the blue diameter here is made of two radii (radii is the plural of radius).

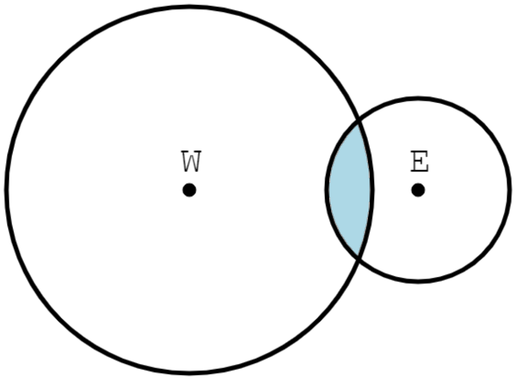
Problem: The town centers of Eastville and Westville are 25 miles apart. A new mall is to be built which is exactly 10 miles from the center of Eastville and exactly 20 miles from the center of Westville. Use circles to show where the mall might be located in relation to the towns.

We draw a rough map. The town centers of Eastville and Westville are represented as points 25 miles apart labeled by E and W. We draw a circle of radius 10 miles around E. The mall must lie on this circle. We also draw a circle of radius 20 miles around W. The mall must lie on this circle also. The only points which lie on both circles are colored red in the map. The mall must be at one of these points.



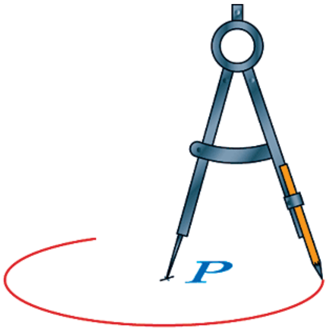
Problem: The town centers of Eastville and Westville are 25 miles apart. A new mall is to be built which is within 10 miles from the center of Eastville and within 20 miles from the center of Westville. Use circles to show where the mall might be located in relation to the towns.

Notice that this problem is almost identical to the last. We have simple replaced “exactly” with “within.” This means that we can draw the same map, except that the mall has to be inside of both circles. The region inside of both circles is light blue on this map:



Drawing Circles

We draw circles with a tool called a **compass**. A compass has two arms. At the end of one arm is a sharp point which is placed at the center of the circle. At the end of the other arm is a pencil. The two arms are pulled apart until the distance between the pencil and point is the desired radius. The point is placed at the center of the circle, and the pencil is dragged around the center to draw a circle. The compass keeps the distance from the center and the pencil fixed.



Drawing a circle with a compass

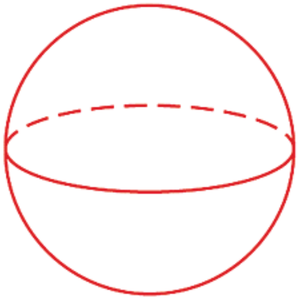
When a compass is not available, any object that can hold a point at the center of a circle and fix the distance from that point to a pencil can be used to draw a circle. Here is a circle being drawn with a paperclip:

Drawing a circle with a paper clip

Popsicle sticks with holes in them can also be used to draw circles.

Spheres

A **sphere** is the set of all points in space which are a fixed distance from a fixed point. The fixed point is the center of the sphere, and the fixed distance is the radius.



Notice how this definition is almost identical to the definition of circle with "plane" replaced by "space." Spheres can be used to locate points or regions in space just like circles can be used on the plane.

The Number π

The **circumference** of a circle is the distance around the circle. If we measure the circumference and diameter of any circle and make the fraction

$$\frac{circumference}{diameter}$$

we will always get a number a little over 3. For different circles, the fraction may seem to vary a little, but this variation is due to error in measurement. It happens to be that no matter what circle we use, the exact value of this fraction is always the same. (The reason for this, using terminology from later in the notes, is that all circles are *similar*.) We call this fraction π (the Greek letter pronounced "pie"):

The number π is the ratio of the circumference of any circle over its diameter.

The number π is an irrational number. It cannot be expressed as a fraction of integers. Its decimal expansion does not repeat and does not terminate. An approximation of π is

$$\pi \approx 3.14159265358979323846264338327950288419716939937510 5\ldots$$

For our calculations, we will always use $\pi \approx 3.14$.

Circumference

Suppose that a circle has circumference C and diameter D. By definition we have

$$\pi = \frac{C}{D}.$$

We can solve this for equation for C and get a formula for the circumference of a circle: $C = \pi D$. If we denote the radius of the circle by R, then $D = 2R$, so $C = 2\pi R$.

Circumference of a Circle: If a circle has radius R, diameter D, and circumference C, then

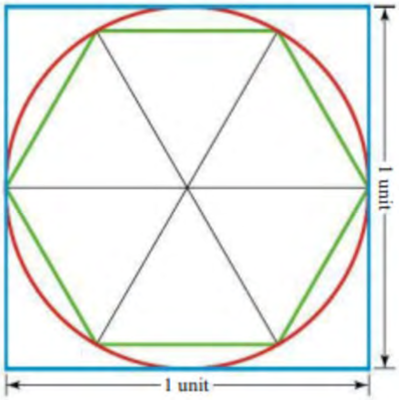
$$C = \pi D \text{ and } C = 2\pi R.$$

Problem: Suppose that a circle has a radius of 5 in. Find the circumference of the circle.

We will use the approximation $\pi = 3.14$. The circumference of a circle is $C = 2\,\pi R$, so here:

$$\begin{aligned} C &= 2\,\pi R \\ &= 2 \times 3.14 \times 5\ in \\ &= 31.4\ in. \end{aligned}$$

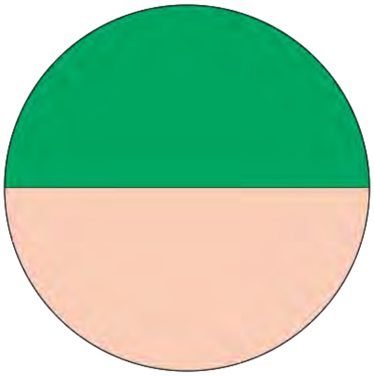
Problem: Below is a hexagon inscribed in a circle inscribed in a square. The circle has diameter 1 $unit$. Find the perimeter of the square, the circumference of the circle, and the perimeter of the hexagon. What does this tell us about π?



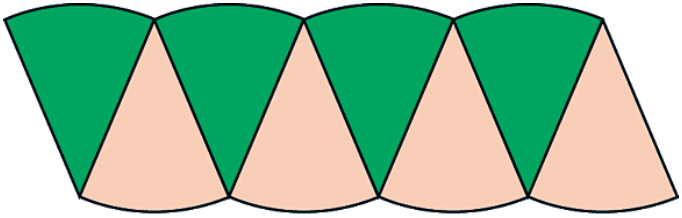
This problem would be more meaningful if we did not already have an approximation at π. Archimedes approximated π this way using a polygon with 96 sides rather than 6 (sort of). One side of the square is 1 $unit$ long, so all four sides give a perimeter of 4 $units$. The circumference of the circle is $C = \pi D = \pi \times 1 = \pi \ units$. The hexagon requires a bit more work. Each black line segment from the center of the hexagon to a vertex of the hexagon is a radius of our circle and has length $\frac{1}{2} unit$. The middle angle of each triangle is $360° \div 6 = 60°$. Each triangle is an isosceles triangle, so the other two angles have to be equal to each other. Since all three angles in any triangle add to 180°, this implies that every angle in each of these triangles is 60° and that each triangle is an equilateral triangle. Finally, this means that every green line segment on this hexagon has length $\frac{1}{2} unit$, and the perimeter of the hexagon is $6 \times \frac{1}{2} = 3 \ units$. So what does this tell us about π? Since the circle is between the hexagon and the square, and since it is closer to the hexagon than it is to the square, π must be between 3 and 4 but closer to 3. (this is something we already knew.)

Area of a Circle

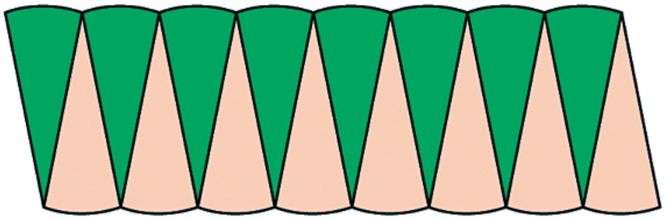
Consider this circle:



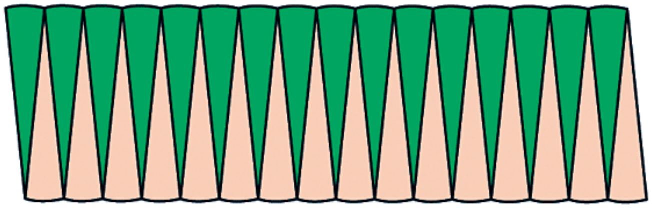
Suppose that we cut the circle into 8 equal size pieces (called sectors) and arrange them alternating pointing up and pointing down. The picture would look like this:



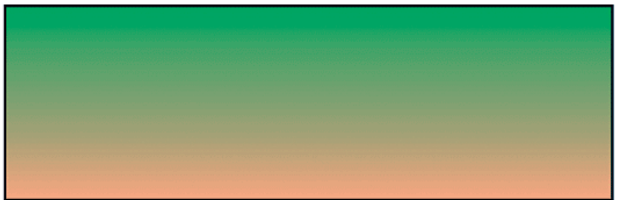
Note that the area of this shape is the same as the area of the original circle. Suppose that we cut each of these sectors in half and arrange the new pieces with points alternating up and down. The new picture would look like this:



This shape, again, has the same area as the original circle. Suppose we do it again:



Again, this shape has the same area. Each time we cut the sectors in half, we get a shape which has the same area and which is closer and closer to being a rectangle. The limiting result is this rectangle:



This rectangle has the same area as the circle. The height of this rectangle used to be the length of the edge of a sector. This is the radius of the circle. The width of this rectangle, the distance along the top, is the distance around the green part of the circle. This is half of the circumference of the circle. Call the radius of the circle R and the circumference C. The height of this rectangle is R, and the width is half of the circumference, $C = 2\pi R$. The area of the rectangle (which is the same as the area of the circle) is $Area = \frac{1}{2} \times C \times R = \frac{1}{2} \times 2 \times \pi \times R \times R = \pi R^2$. Thus we have a formula for the area of a circle.

Area of a Circle: The area A of a circle of radius R is $A = \pi R^2$.

Note: This is not really the area of the circle. It is the area of the *region inside the circle*, which is called a disk. The circle is the edge of the disk. However, we follow the standard custom of calling this the area of the circle.

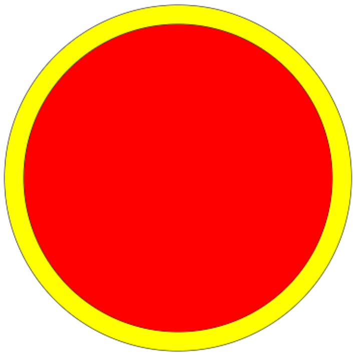
Problem: Find the area of a circle with radius 5 inches.

The area of a circle of radius R is $A = \pi R^2$, so the area of our circle is

$$A = \pi R^2 = 3.14 \times 5 \times 5 = 78.5\ in^2.$$

Problem: A circular pizza has a dimeter of 18 inches. Toppings on the pizza are spread to within one inch of the edge of the pizza. What is the area of the crust of the pizza (the portion without toppings)?

Our pizza looks something like the picture below. The red represents toppings, and the yellow represents crust. Notice that the toppings fill a circle in the center of the pizza circle.



Since the diameter of the pizza is 18 in, the radius is $R = 9\ in$. Since the toppings are spread to within one inch of the edge, the toppings circle has a radius of $r = 8\ in$. To find the crust area (the yellow) we will find the area of the entire pizza and subtract the area of the toppings.

$$Area\ of\ pizza = \pi R^2 = 3.14 \times 9 \times 9 = 254.34\ in^2.$$
$$Area\ of\ toppings = \pi r^2 = 3.14 \times 8 \times 8 = 200.96\ in^2.$$
$$Area\ of\ crust = (Area\ of\ pizza) - (Area\ of\ toppings) = 254.34 - 200.96 = 55.38\ in^2.$$

Problem: A circular pizza has a dimeter of 18 inches. Toppings on the pizza are spread to within one inch of the edge of the pizza. Another pizza is a 15 inch square. The square pizza is crustless, which means that toppings are spread to the edge of the pizza. Which pizza is larger? Which pizza has more toppings?

For the question of which pizza is larger, we will compare area. The circular pizza has a radius of 9 *inches*, and we found its area above.

$$Area\ of\ round\ pizza = \pi R^2 = 3.14 \times 9 \times 9 = 254.34\ in^2.$$

The area of the square pizza is:

$$Area\ of\ square\ pizza = 15 \times 15 = 225\ in^2.$$

The round pizza has more area.

To see which pizza has more toppings, we compare the area of the toppings. The area of the toppings on the square pizza is the same as the area of the pizza, 225 in^2, because there is no crust. We found the area of the toppings of the round pizza above as the area of a circle.

$$Area\ of\ toppings\ on\ the\ round\ pizza = \pi r^2 = 3.14 \times 8 \times 8 = 200.96\ in^2.$$

Therefore, the square pizza has more toppings even though it is the smaller pizza. (This may be the reason that a popular pizza chain that used to sell only square , crustless pizzas now markets round pizzas.)

Problem: Below is a square with edge length one inch. Inside the square, in yellow, is a quarter of a circle of radius one inch centered at the top corner of the square. Also drawn is a line connecting two corners of the square. Find the area of the green region, the blue region, and the red region.

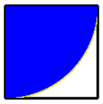
The green region is easiest. This is a triangle base and height 1 inch. The area of the green triangle is:

$$Area\ of\ green\ triangle = \frac{1}{2} \times 1 \times 1 = \frac{1}{2} in^2.$$

To find the blue area, we have to realize that the blue area is the difference between a quarter circle and the green triangle:

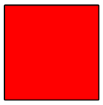
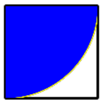
$Area\ of\ blue =$  $-$ 

The area of the quarter circle is on fourth of the area of a circle of radius 1

 $= \frac{1}{4} \times 3.14 \times 1^2 = 0.785\ in^2.$

So the area of the blue is $0.785 - 0.5 = 0.285\ in^2$.

Finally, the red area is the difference between the entire square and the quarter circle.

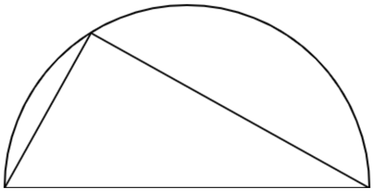
$Area\ of\ red =$  $-$ 

The area of the square is 1 square inch, and the area of the quarter circle we found above to be $0.785\ in^2$, so the area of the red is $1 - 0.785 = 0.215\ in^2$.

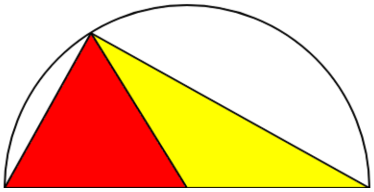
Thale's Theorem

Recall that the axiomatic approach to geometry (stating a few assumptions and rigorously proving theorems based on those assumptions) was first introduced by Thales of Miletus. One of the first theorems proven by Thales was:

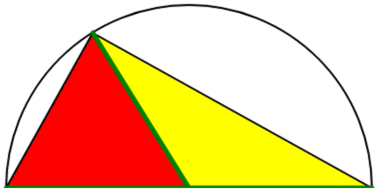
Thales' Theorem: When a triangle is inscribed in a semicircle with one side being the base of the semicircle, that triangle is a right triangle.



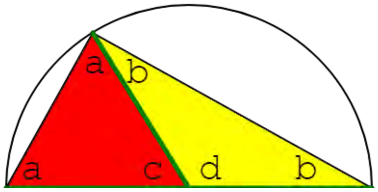
We are going to use what we know about triangles to prove Thales' Theorem. The important facts are that the angles in a triangle add to 180° and that the base angles in an isosceles triangle are equal. First, we draw a radius of our circle to the vertex of the triangle which lies on the circle. This divides our rectangle into two rectangles, which we color red and yellow.



The triangle edges colored green in this image



are all radii of the circle, so they are all the same length. This makes both triangles isosceles triangles. We now name the angles in our two triangles. Since they are isosceles triangles, we label the base angles the same in each triangle.



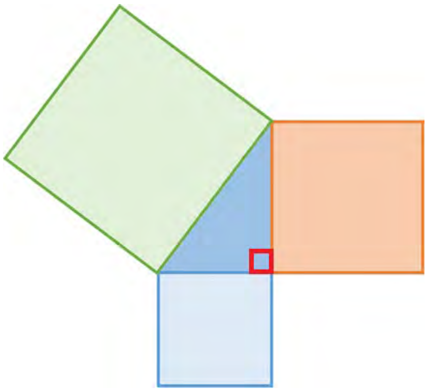
Since the sum of the angles in a triangle is 180°, we know that $a + a + c = 180°$ and $b + b + d = 180°$. This implies that $2a + 2b + c + d = 360°$. However, c and d combine to form a straight angle, so $a + c = 180°$. Thus $2a + 2b + 180° = 360°$. We can now subtract 180° and divide by 2 to find that $a + b = 90°$. This is enough to conclude that our original triangle is a right triangle.

The Pythagorean Theorem

One of the most famous theorems in mathematics is the Pythagorean Theorem. This theorem is attributed to Pythagoras, who lived around 500 BC. However, there is clear evidence on clay tablets that the Babylonians knew of this theorem as early as 2000 BC. Chinese mathematicians knew of this theorem at least by 200 BC.

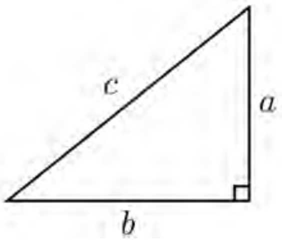
Pythagorean Theorem: In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

For the Pythagoreans (who did not have algebra), the interpretation is that the area of the green square in this figure is equal to sum of the areas of the blue and orange squares.



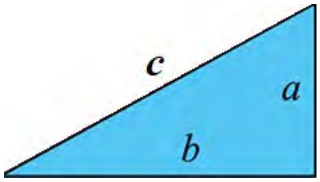
The theorem is easier to state with the use of algebra and variables.

Pythagorean Theorem: If a and b are the lengths of the legs of a right triangle, and if c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.

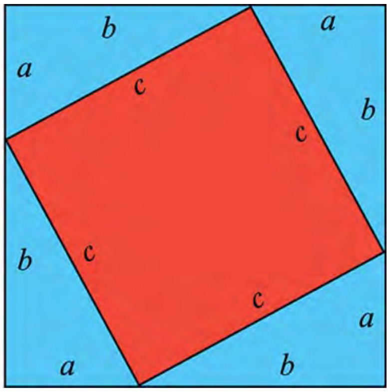


Proof of the Pythagorean Theorem

Entire books have been written exploring different proofs of the Pythagorean Theorem. We demonstrate one here. Consider this right triangle. We will demonstrate that $a^2 + b^2 = c^2$. Before we begin, note that the two angles opposite a and b (the two non-right angles) must add to 90°.



We draw four copies of our triangle arranged in a square:



This figure has a red quadrilateral in the center. All four edges of the red quadrilateral have length c, so this is at least a rhombus. Consider one of the points where one red angle and two blue angles come together. These three angles make a straight angle adding to $180°$. The two blue angles add to $90°$, so the red angle must also be $90°$. Thus, the red shape is actually a square with edge length c.

Now, we will find the area of the large square two ways. First, this is a square with edge length $(a+b)$, so its area is

$$Area\ of\ large\ square = (a+b)^2 = a^2 + 2ab + b^2.$$

We can also find the area of the large square by adding up the areas of four blue triangles and one red square. These areas are

$$Area\ of\ blue\ triangle = \frac{1}{2}ab$$
$$Area\ of\ red\ square = c^2.$$

So the total area of the large square is

$$\begin{aligned} Area\ of\ large\ square &= 4 \times (blue\ triangle) + (red\ triangle) \\ &= 4 \times \frac{1}{2}ab + c^2 \\ &= 2ab + c^2. \end{aligned}$$

We now have two expressions for the area of the large square. Since the area is the same, the expressions have to be equal.

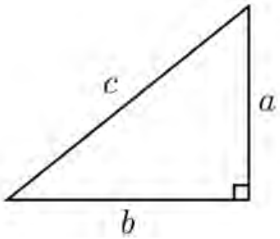
$$a^2 + 2ab + b^2 = 2ab + c^2.$$

Subtracting $2ab$ now gives

$$a^2 + b^2 = c^2.$$

Formulas

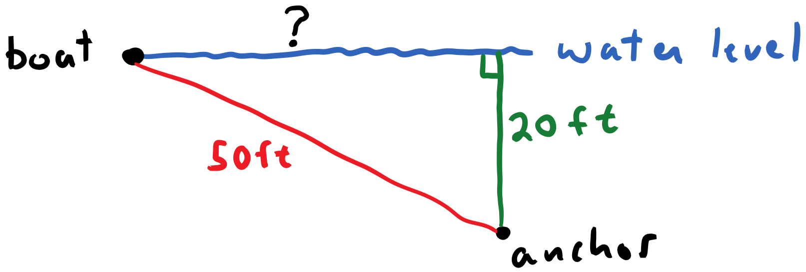
When we first encountered the Pythagorean Theorem, we may not have known much (if any) algebra. At that point, we may have simply been given formulas. In a right triangle like this



If we know the legs (a and b) then $c = \sqrt{a^2 + b^2}$. If we know the hypotenuse (c) and one leg (say, a), then $b = \sqrt{c^2 - a^2}$.

Problem: A boat's anchor is on a line that is 50 feet long. The anchor is dropped in water that is 20 feet deep. How far will the boat be able to drift from the spot on the water's surface directly above the anchor? (Round to the nearest integer.)

We can stretch the line out horizontally from the anchor to the boat and then draw a line straight up from the anchor to the water. These lines form two sides of a right triangle.

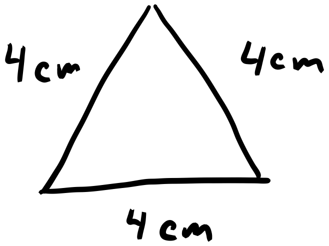


We are looking for the third side of the right triangle. Since we know the hypotenuse of this triangle, we will use the formula with subtraction to find the third side.

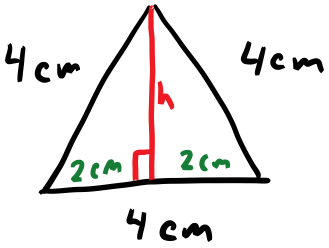
$$? = \sqrt{50^2 - 20^2} = 45.826 \approx 46\ ft.$$

Problem: Find the area of an equilateral triangle whose edge length is $2\ cm$. (Round to two decimal places.)

First, we draw such a triangle.



To find the area of the triangle, we need the length of a base (which we have) and the height (which we do not have). We draw a height for this triangle and note that it cuts the base into two equal length segments (because this is an equilateral triangle).



The height now divides the equilateral triangle into two right triangles with hypotenuse $4\ cm$, one leg $2\ cm$, and one leg h. We can use the subtraction form of the Pythagorean Theorem to find h.

$$h = \sqrt{4^2 - 2^2} = 3.4641\ cm.$$

(Note that our rounding directions called for two decimal places. We keep twice that until we are done with our arithmetic.) The area is now

$$Area\ of\ triange = \frac{1}{2} \times (base\ length) \times height = \frac{1}{2} \times 4 \times 3.4641 = 6.93\ cm^2.$$

Pythagorean Triples

Some right triangles are *nice* in the sense that all three of their sides have integer length. For example, if the legs of a right triangle measure 3 and 4, then the hypotenuse has length 5. The 3-4-5 right triangle is one of the most commonly known right triangles. It is regularly used in construction to test whether or not a corner is actually a right angle.

Three integers a, b, and c are a **Pythagorean triple** if $a^2 + b^2 = c^2$. This means that they could be the lengths of three sides of a right triangle. Lists of Pythagorean triples appear on Babylonian clay tablets dating to 2000 BC. The numbers 3-4-5 form a Pythagorean triple. Some other small Pythagorean triples are:

3-4-5, 5-12-13, 8-15-17, 7-24-25, and 9-40-41.

If we take any Pythagorean triple and multiply all three numbers by the same integer, we get another Pythagorean triple. For example, since 3-4-5 is a Pythagorean triple, so are 6-8-10 and 9-12-15.

One of the items included in Euclid's *Elements* is a way to generate Pythagorean triples. If $m > n > 0$ are integers and if

$$a = m^2 - n^2\ and\ b = 2mn\ and\ c = m^2 + n^2$$

then a, b, and c form a Pythagorean triple. For example, if $m = 5$ and $n = 3$ then

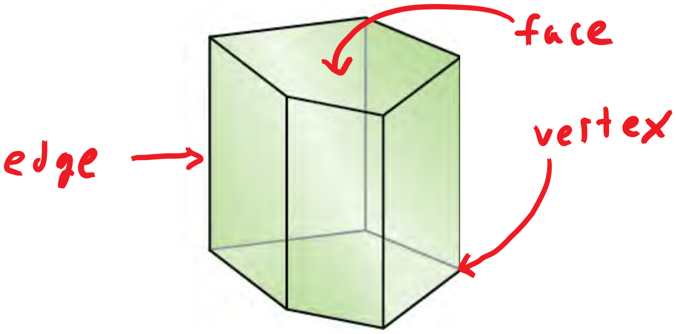
$$a = 5^2 - 3^2 = 16$$
$$b = 2 \times 5 \times 3 = 30\ and$$
$$c = 34$$

form a Pythagorean triple.

Polyhedra

Recall that a polygon is a closed shape in the plane made of finitely many line segments (with some conditions on intersections). Here we extend this idea to three dimensional shapes.

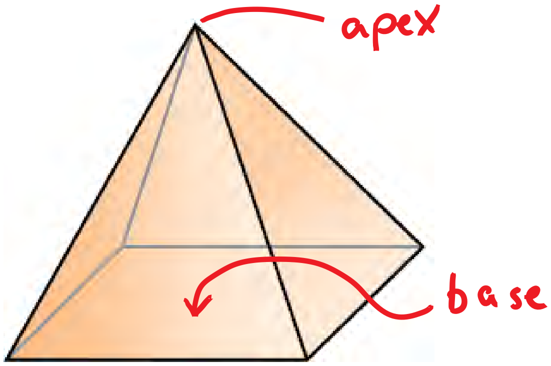
A **polyhedron** is a closed shape in space whose outside surfaces are polygons. The polygons are called the **faces** of the polyhedron. The line segments where the polygon intersect are the **edges** of the polyhedron, and any point where edges intersect is a **vertex** of the polyhedron.



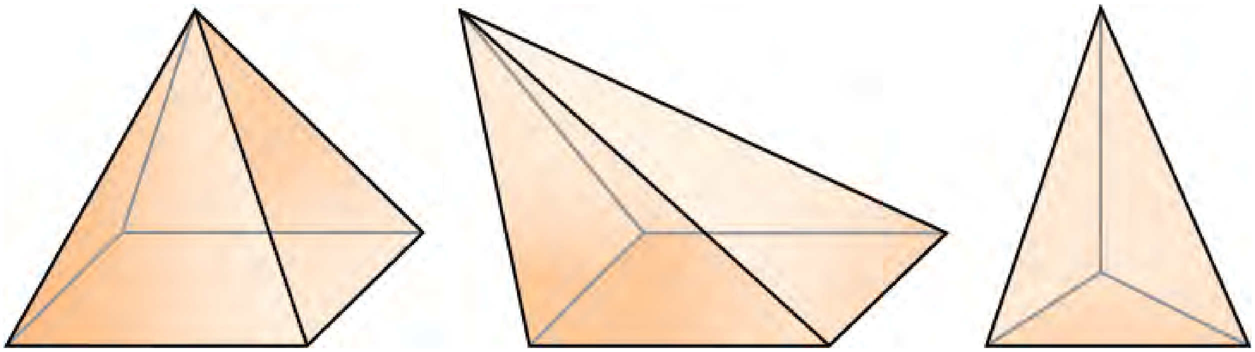
The plural of polyhedron is polyhedra.

Pyramids

Given any polygon, locate a point which is not in the plane containing that polygon and draw line segments from the vertices of the polygon to that point. The resulting polyhedron is a **pyramid**. The original polygon is the **base** of the pyramid. The point not on the plane of the base is the **apex** of the pyramid.



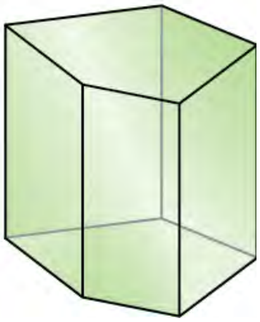
Pyramids are classified by the shapes of their bases if the base is a square, the pyramid is called a square pyramid. If the base is a hexagon, the pyramid is a hexagonal pyramid. Pyramids are also classified by the location of the apex. If the apex is directly over the center of the base, then the pyramid is a **right pyramid**. If the apex is not directly over the center of the base, then the pyramid is an **oblique pyramid**.



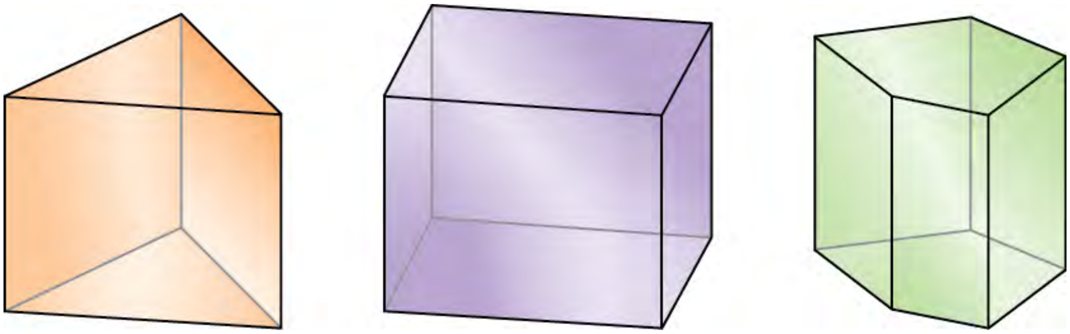
In this image, the pyramid on the left is a right square pyramid. The pyramid in the middle is an oblique square pyramid, and the pyramid on the right is a right triangular pyramid.

Prisms

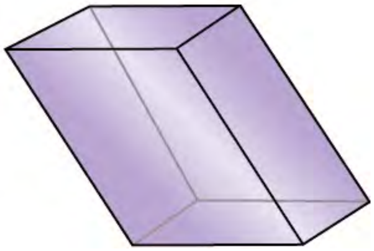
A **prism** consists of two identical, parallel polygons with the same rotation and with the corresponding vertices connected by line segments. The two parallel faces are called the **bases** of the prism. You can form a prims by taking a polygon, dragging it vertically without rotation, and then connecting the vertices at the new location to the vertices at the original location.



This is a picture of a prism whose bases are both pentagons. We would call this a pentagonal prism. If the vertices of one base are oriented directly above the corresponding vertices of the other base, then a prism is a **right prism** otherwise, the prism is an **oblique prism**. Here are some right prisms:

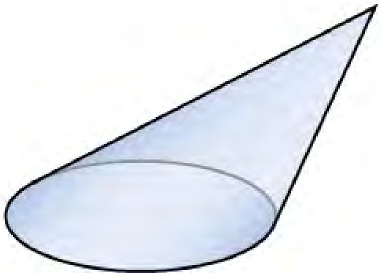


The prism on the left is a triangular prism because its bases are triangles. The prism in the middle is a rectangular prism (or box). The prism on the right is a pentagonal prism. Here is a picture of an oblique rectangular prism:

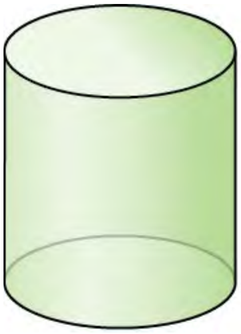


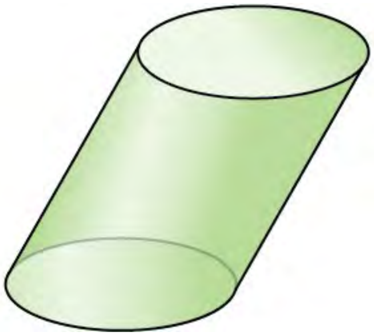
Not Polyhedra

A three dimensional shape similar to a pyramid but with a round or curved base is called a **cone**.



Cones are not polyhedra. A three dimensional shape similar to a prism with a round or curved base is a **cylinder**.





Cylinders are not polyhedra.

Platonic Solids

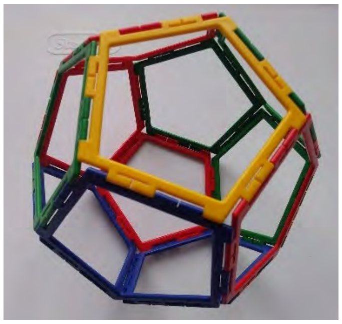
Recall that a regular polygon is a polygon in which all edges have the same length and all angles have the same measure. The corresponding notion for polyhera. is a Platonic solid. A Platonic solid is a convex polyhedron in which every face is an identical regular polygon and the same number of faces come together at each vertex. There are exactly five Platonic solids:

- Tetrahedron – 4 faces which are all equilateral triangles
- Cube (a.k.a. hexahedron) – 6 faces which are all squares
- Octahedron – 8 faces which are all equilateral triangles
- Dodecahedron – 12 faces which are all regular pentagons
- Icosahedron – 20 faces which are all equilateral triangles

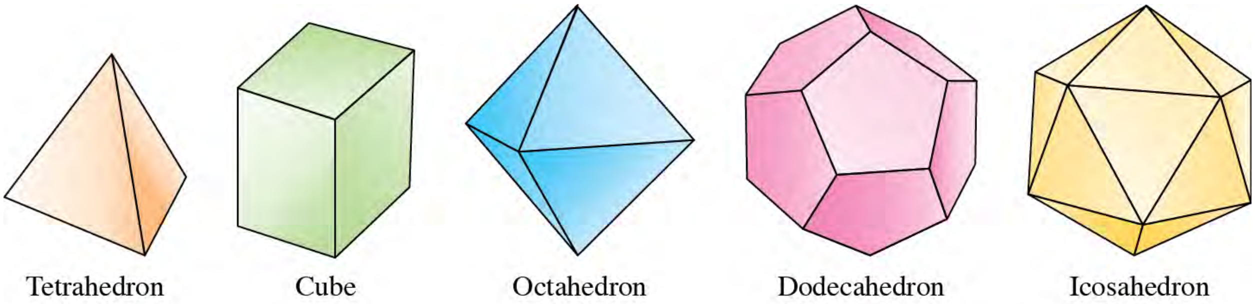
Here is a picture of a tetrahedron (lower left), an octahedron (lower right), and an icosahedron:

Here is a cube:

And here is a dodecahedron:



Here are drawings of all five Platonic solids:



Euler's Formula

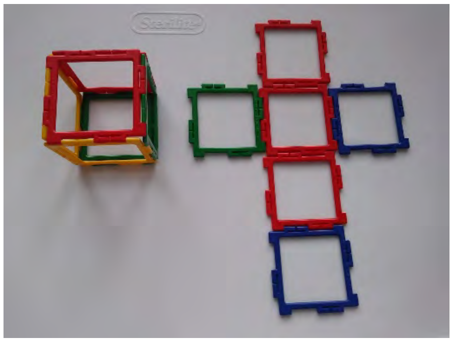
There is a relationship between the numbers of vertices, faces, and edges of a polyhedron. Consider this table which lists the numbers of vertices, faces, and edges for the Platonic solids.

Solid	**Vertices**	**Faces**	**Edges**
Tetrahedron	4	4	6
Cube	8	6	12
Octahedron	6	8	12
Dodecahedron	20	12	30
Icosahedron	12	20	30

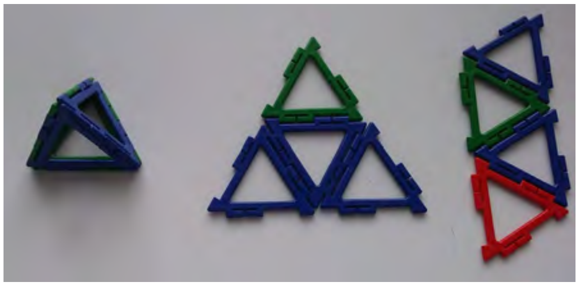
If we add the number of vertices and faces in each row, we miss the number of edges by 2. Euler's formula says that if a convex polyhedron has V vertices, F faces, and E edges, then $V + F - E = 2$.

Nets and Surface Area

Here is a picture of a cube made from children's snap-together toys and a picture of the same cube unfolded.

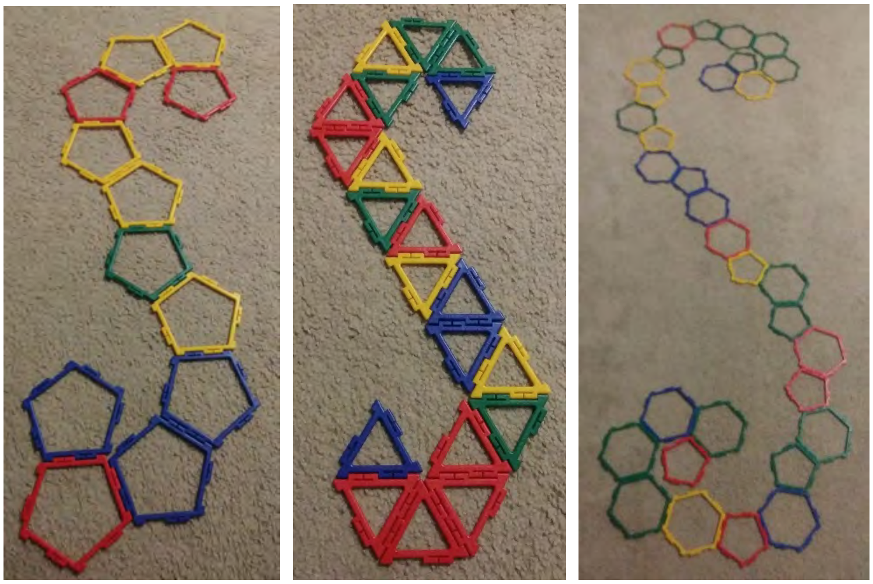


The unfolded cube is called a **net** for the cube. This is a pattern which can be folded up to make the cube. A standard way to make a model of a polyhedron is to print a net on paper, cut it out, fold it, and glue or tape along the edges. Polyhedra typically have multiple nets. Here are two ways to unfold a tetrahedron:

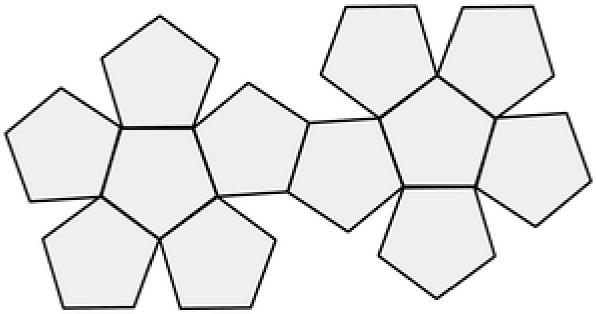


Here is a hexagonal prism along with its net:

Here are nets for a dodecahedron, an icosahedron, and a truncated icosahedron (a soccer ball):

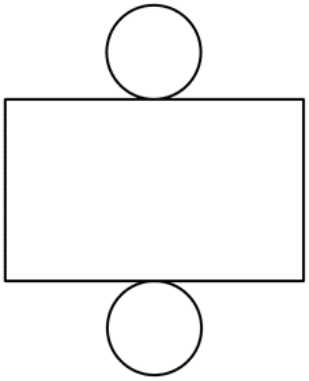


The truncated icosahedron is made by cutting the corners off of an icosahedron. It consists of twenty hexagons and twelve pentagons. Here is another net for a dodecahedron:

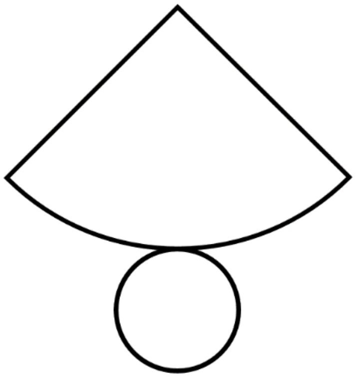


Cylinders and Cones

While cylinders and cones are not polyhedra, we can still draw nets for them. Here is a net for a cylinder:



The net consists of circles for the bases of the cylinder along with a rectangle which is the unfolded “side” of the cylinder. Note that the height of the rectangle is the same as the height of the cylinder. The width of the rectangle is the same as the circumference of the bases. Here is a net for a cone:

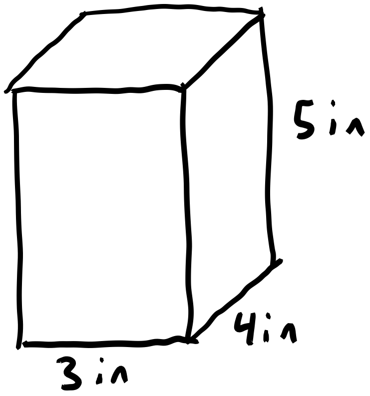


Surface Area

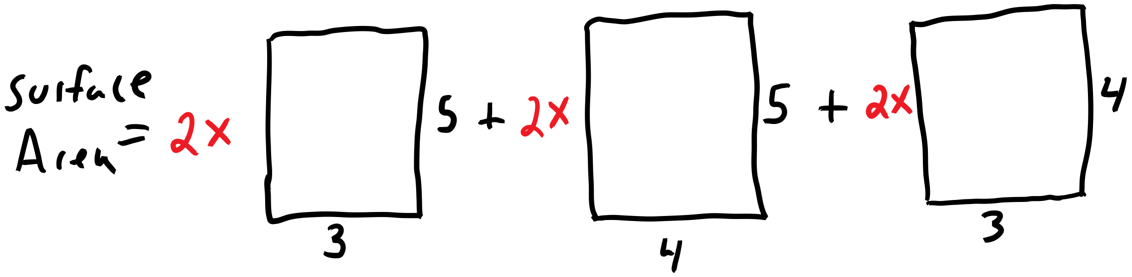
The **surface area** of a polyhedron is the combined area of all of the polygons that form the polyhedron. To find the surface area of a polyhedron, we typically draw (or imagine) each side of the polyhedron (possibly as a net), find the area of each side, and then add up the areas of the sides.

Problem: A rectangular prism (box) measures 3 inches by 4 inches by 5 inches. Find its surface area.

We first draw the box.



The front of this box is a 3 inch by 5 inch rectangle. There are two sides which have this shape (front and back). The right side of the box is a 4 inch by 5 inch rectangle. There are two sides this shape (right and left). The top of the box is a 3 inch by 5 inch rectangle. There are two sides this shape (top and bottom). To find the total surface area, we simply find the areas of each of these rectangles and add up the results.



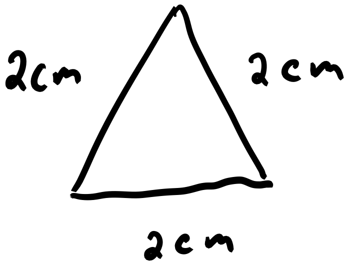
$$= 2\times 3\times 5 + 2\times 4\times 5 + 2\times 3\times 4$$

$$= 30 + 40 + 24 = 94 in^2$$

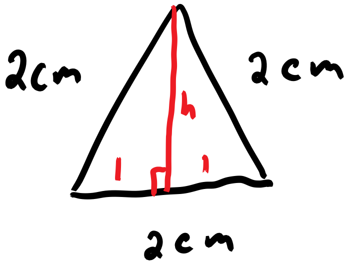
The surface area is $94\ in^2$.

Problem: Every edge of a regular tetrahedron has length 2 centimeters. Find the surface area of the tetrahedron. Round your answer to two decimal places.

The tetrahedron has four sides, which are all equilateral triangles with edge length 2 centimeters. All we have to do is find the area of one of these triangles and multiply by 4. Here is one of the triangles.



To find the area of this triangle, we need the height. When we draw the height, it bisects the base of the triangle and divides the entire triangle into two right triangles with base $1\ cm$, hypotenuse $2\ cm$, and height h.



We can find h using the Pythagorean theorem: $h = \sqrt{2^2 - 1^2} = \sqrt{3} \approx 1.7321cm$. (We keep twice the number of decimal places that we want until we are done with our arithmetic.) The area of one of these triangles is

$$Area\ of\ one\ triangle = \frac{1}{2} \times 2 \times 1.7321 = 1.7231\ cm^2.$$

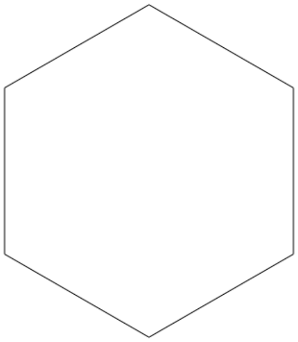
The total surface area of the tetrahedron is four times this

$$Surface\ area = 4 \times 1.7231 = 6.8294\ cm^2.$$

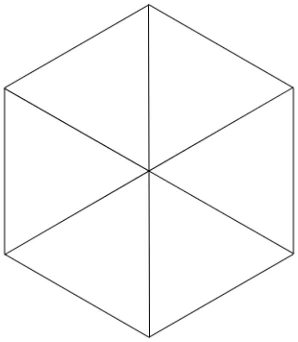
Our surface area is 6.83 cm^2.

Problem: Every edge of a right, regular, hexagonal prism has length 2 cm. Find the surface area of the prism. Round to the nearest integer.

The prism consists of two bases which are both regular hexagons and six "sides" which are each squares. The lengths of all of the edges of the hexagons and squares are 2 cm. The area of one of the squares is $2 \times 2 = 4\ cm^2$. The hexagons require a bit more work. First we draw one.



Now we draw line segments from the center of the hexagon to each of its vertices.



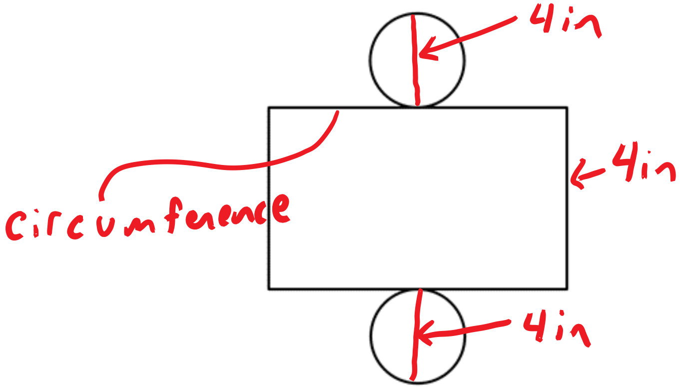
Each of these triangles is an equilateral triangle with edge length 2 cm. Luckily for us, we found the area of one of these triangles above. The area of each triangle is 1.7231 cm^2. Since each hexagon contains six of these triangles, the area of each hexagon is $6 \times 1.7231 = 10.3926\ cm^2$. Now we can find our total surface area:

$$\begin{aligned} Surface\ area &= 2 \times (area\ of\ hexagon) + 6 \times (area\ of\ square) \\ &= 2 \times 10.3926 + 6 \times 4 \\ &= 44.7852\ cm^2 \end{aligned}$$

Rounded to the nearest integer, our surface area is 45 cm^2.

Problem: The height and base diameter of a circular cylinder are both 4 inches. Find the surface area of the cylinder. Round to two decimal places.

First, we draw a net with the appropriate distances labeled.



The net consists of two circles with radius 2 inches and a rectangle. The height of the rectangle is the height of the cylinder, 4 inches. The width of the rectangle is the circumference of the circle, which is

$$\begin{aligned} circumference &= \pi \times diameter \\ &= 3.14 \times 4 \\ &= 12.56\ in. \end{aligned}$$

The area of one circle is

$$\begin{aligned} area\ of\ circle &= \pi r^2 \\ &= 3.14 \times 2^2 \\ &= 12.56\ in^2. \end{aligned}$$

The area of the rectangle is

$$area\ of\ rectangle = 12.56 \times 4 = 50.24\ in^2.$$

The total surface area is

$$\begin{aligned} surface\ area &= 2 \times (area\ of\ circle) + (area\ of\ rectangle) \\ &= 2 \times 12.56 + 50.24 \\ &= 75.36\ in^2. \end{aligned}$$

Spheres

The derivation for the formula of the surface area of a sphere is beyond the scope of what we know at this point, but we can still give the formula.

Surface Area of a Sphere: The surface area of a sphere of radius R is $SA = 4\pi R^2$.

Problem: Find the surface area of a spherical snowball of radius 2 inches.

The surface area is

$$Surface\ area = 4\,\pi R^2 = 4 \times 3.14 \times 2^2 = 50.24\ in^2.$$

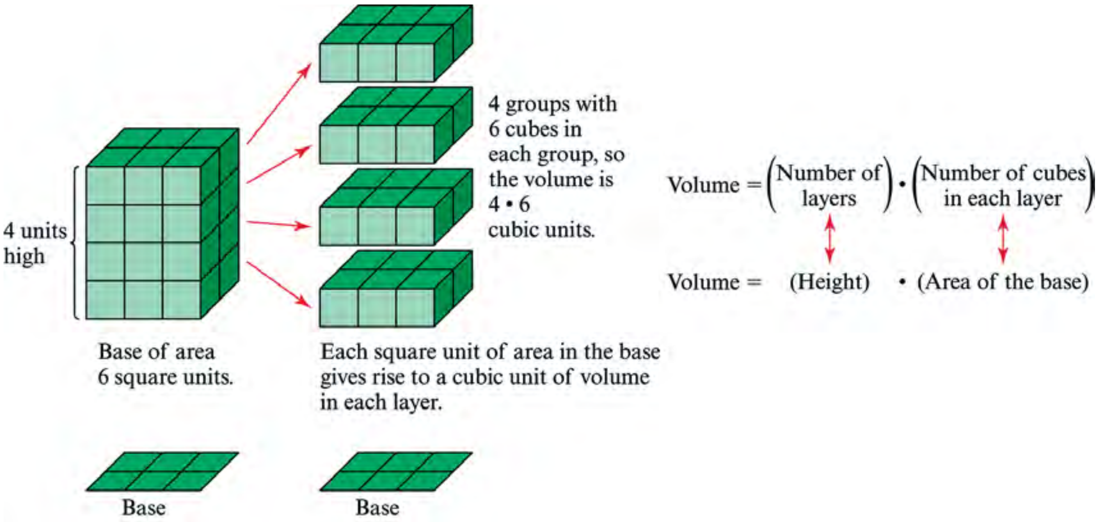
Volume

Recall that when we say the volume of a three dimensional shape is 100 cubic inches, we mean that the shape (or the space it occupies) could be filled with 100 one inch cubes without gaps, without overlaps, but maybe with some cutting. In this section, we address how to find the volume of prisms and pyramids.

Suppose that we have a right prism with height h and volume V. Suppose also that we have filled the prism with V one inch cubes. Since the height of the prism is h, we could slice the prism horizontally into h layers, each with height $1\ unit$. This means that each of the h layers is one cube tall. Now consider the bottom layer that sits on top of the base of the prism. The bottoms of the cubes in this layer exactly cover the base of the prism. However, the bottoms of these cubes are one inch squares. Therefore, the number of cubes in a layer is the number of one inch squares necessary to cover the base of the prism. This is exactly the area of the base of the prism. We have now divided the cubes filling the prism into h levels, where the number of cubes in each level is the area of the base of the prism. The number of cubes is, therefore, equal to h (the height of the prism) times the area of the base.

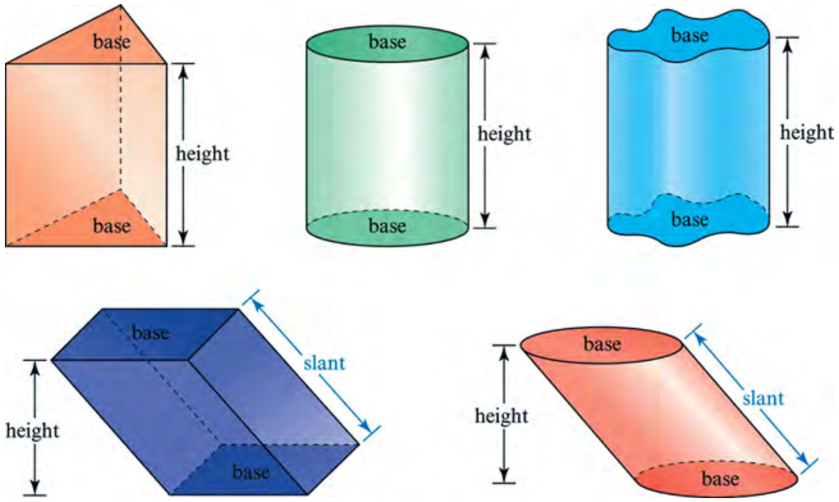
Volume of a Prism: The volume of a prism is $V = (area\ of\ base) \times height$.

Here is an illustration of this process using a rectangular prism:



The cubes are divided into 4 levels because the height of the prism is 4. Each level has 6 cubes in it because the bottom layer exactly covers the base which has an area of 6 square units. The total number of cuves is then 6×4 square units.

This same formula for volume of prisms works for right prisms and oblique prisms along with right and oblique cylinders. Finding the height of an oblique prism is similar to finding the height of a triangle.



Problem: A rectangular prism (a box) measures 3 inches by 4 inches by 5 inches. Find its volume.

We declare that the base of the prism is the rectangle measuring 3 inches by 4 inches (this is almost arbitrary). This means that the area of the base is $3 \times 4 = 12\ in^2$. The volume of the box is now

$$Volume = (area\ of\ base) \times height = 12 \times 5 = 60\ in^3.$$

Problem: Every edge of a right, regular, hexagonal prism has length 2 centimeters. Find the volume of the prism.

The volume is the area of the base times the height. The height of this prism is 2 centimeters. The base is a regular hexagon with edge length 2 centimeters. We found the area of such a hexagon in the section on surface area. It is $10.3926\ cm^2$. The volume is

$$Volume = (area\ of\ base) \times height = 10.3926 \times 2 = 20.7852\ cm^3.$$

Problem: The height of a cylinder is $10\ cm$. The base radius is $2\ cm$. Find the volume.

The volume is the area of the base times the height. The height is $10\ cm$. The area of the base is

$$Area\ of\ base = \pi r^2 = 3.14 \times 2 \times 2 = 12.56\ cm^2.$$

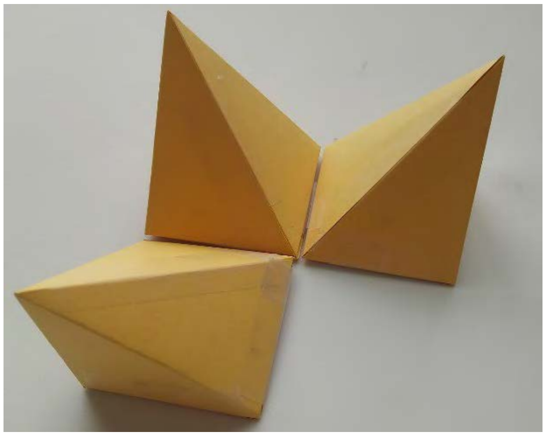
The volume is

$$Volume = (area\ of\ base) \times height = 12.56 \times 10 = 125.6\ cm^3.$$

Pyramids

Here is a picture of a cube made out of paper:

The cube is made of three identical pieces, each of which is called a yangma. If we unfold the cube to see the three pieces, they look like this:



Each of these is a square pyramid whose height is the same as the original cube and whose base is identical to the original cube. This is supposed to indicate that a pyramid may be one third of a prism. In fact, the volume of a pyramid is one third of the volume of a prism with the same height and base.

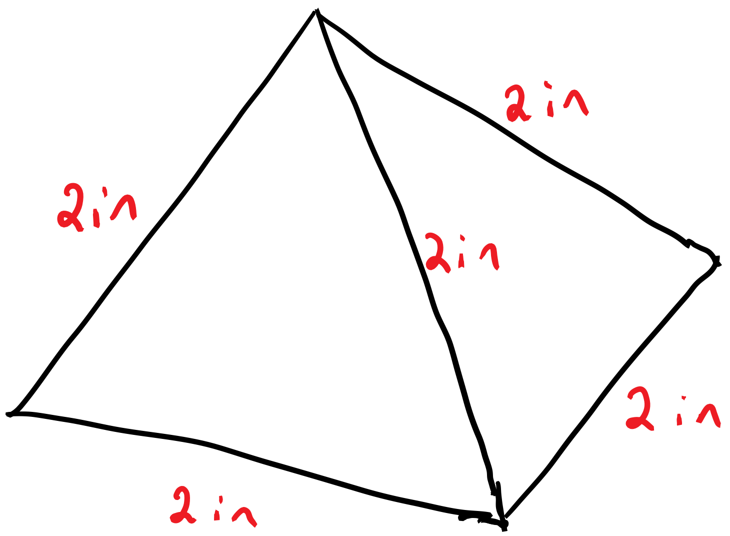
Volume of a Pyramid: The volume of a pyramid is $V = \frac{1}{3} \times (area\ of\ base) \times height$.

Problem: The base of a pyramid is a right triangle with legs of length 3 inches and 4 inches. The height of the pyramid is 6 inches. Find the volume.

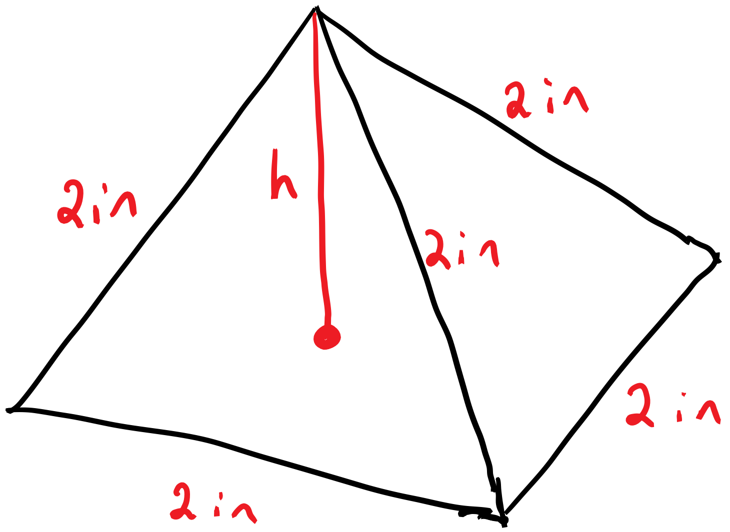
To find the volume, we need the area of the base and the height. Since the base is a right triangle with legs 3 inches and 4 inches, the area of the base is $3 \times 4 = 12\ in^2$. The height is $6\ in$, so the volume is

$$V = \frac{1}{3} \times (area\ of\ base) \times height = \frac{1}{3} \times 12 \times 6 = 24\ in^3.$$

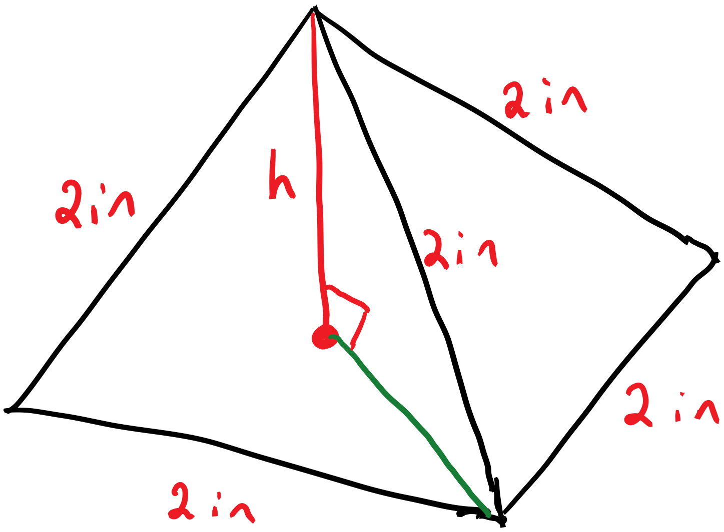
Problem: Every edge of a right square pyramid has length $2\ in$. Find the volume of the pyramid.



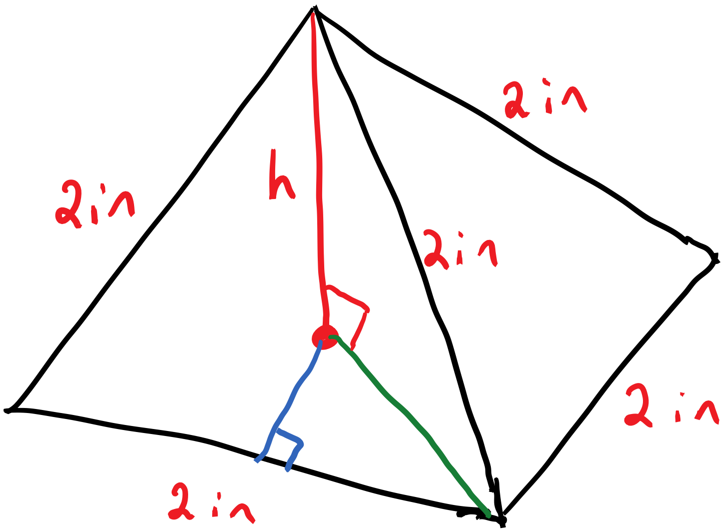
To find the volume of this pyramid, we need the area of the base (which is a square) and the height. We have indicated the height in this picture. The point at the bottom of the height is in the center of the square base.



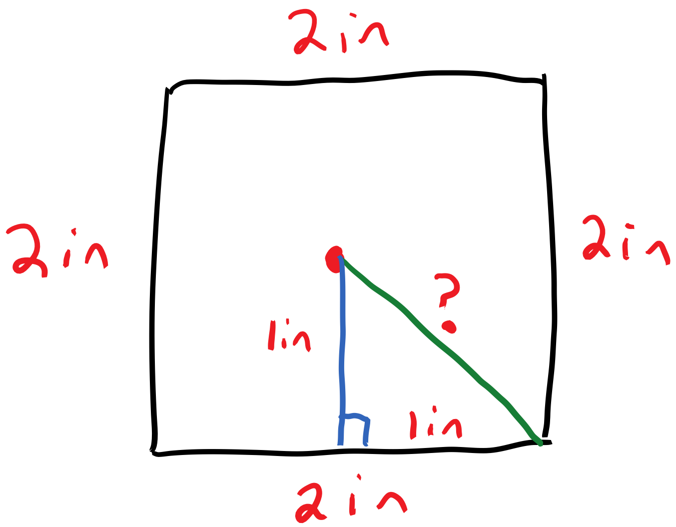
In the picture below, we have drawn a green line segment from the point at the bottom of the height to the nearest vertex. If we could find this distance, then we could use the Pythagorean theorem to find the height.



To find the green distance, we draw another (blue) line segment from the red point to the center of the near side.



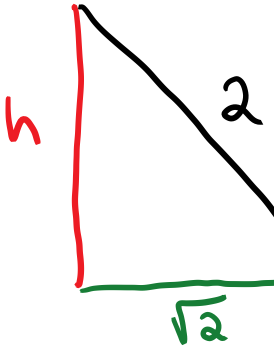
The green segment is now the hypotenuse of the right triangle with blue, black, and green sides. This triangle lies entirely in the base of the pyramid, so we redraw the base of the pyramid, looking straight down, showing the triangle.



We know that the legs of the triangle are each $1in$ because the red point is at the center of the square (because this was a right square pyramid). We can now use the Pythagorean theorem to find the green edge

$$? = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Now the triangle involving the height looks like this:



We can use the Pythagorean theorem to find h.

$$h = \sqrt{2^2 - \left(\sqrt{2}\right)^2} = \sqrt{3} = 1.7321\ in.$$

This is the height of our pyramid. Now we just need the area of the base. The base is a square with edge length $2\ in$, so its area is $2^2 = 4\ in^2$. Our volume is now

$$Volume = \frac{1}{3} \times (area\ of\ base) \times height = \frac{1}{3} \times 4 \times 1.7321 = 2.31 in^3.$$

Spheres

The derivation for the formula for the volume of a sphere is beyond the scope of what we know so far, but we can go ahead and state it.

Volume of a Sphere: The volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$.

Problem: Find the volume of a snowball with radius 2 inches.

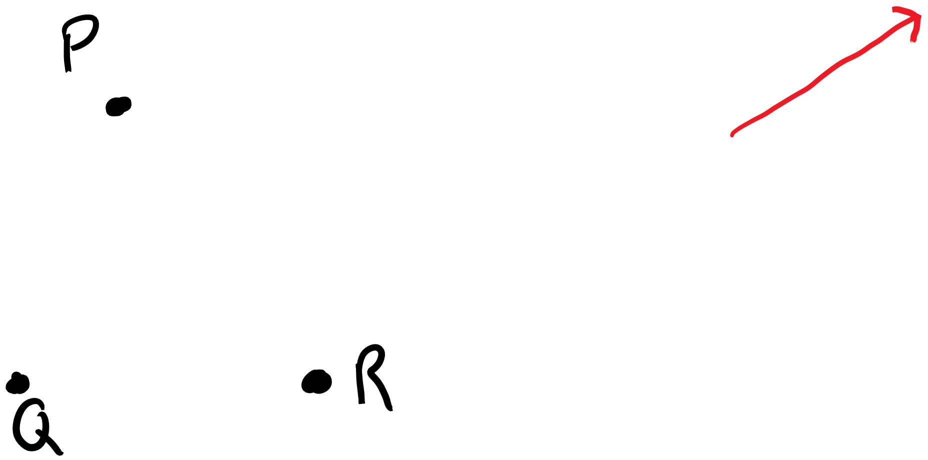
The volume is $V = \frac{4}{3}\pi R^3 = \frac{4}{3} \times 3.14 \times 2^3 = 33.49\ in^3$.

Transformations and Symmetry

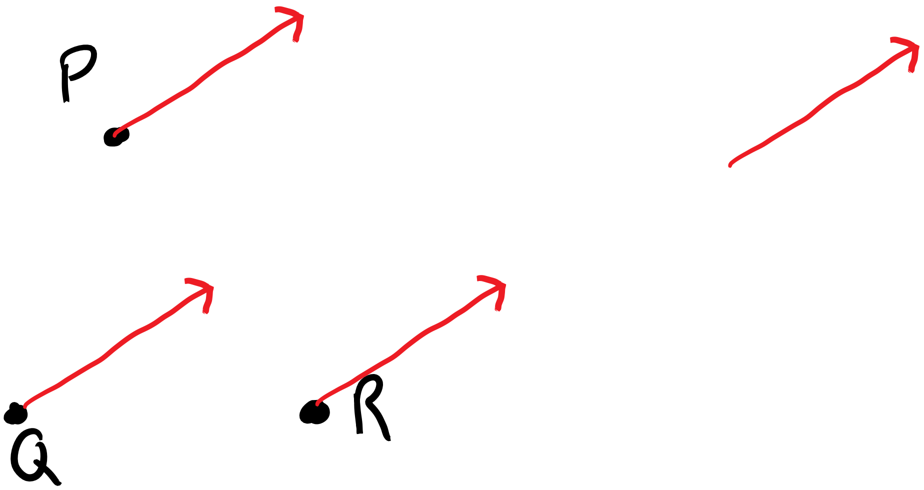
A **transformation** of the plane is a function which maps every point on the plane to a point on the plane. We can think of transformations as picking the plane up, stretching it, twisting it, squishing it, rotating it, flipping it over, and putting it back down. When a transformation is applied to a point P in the plane, then P is moved to another location P' on the plane. The point P' is called the image of P under the transformation. The fundamental transformations we will care about are translations, reflections, and rotations.

Translations

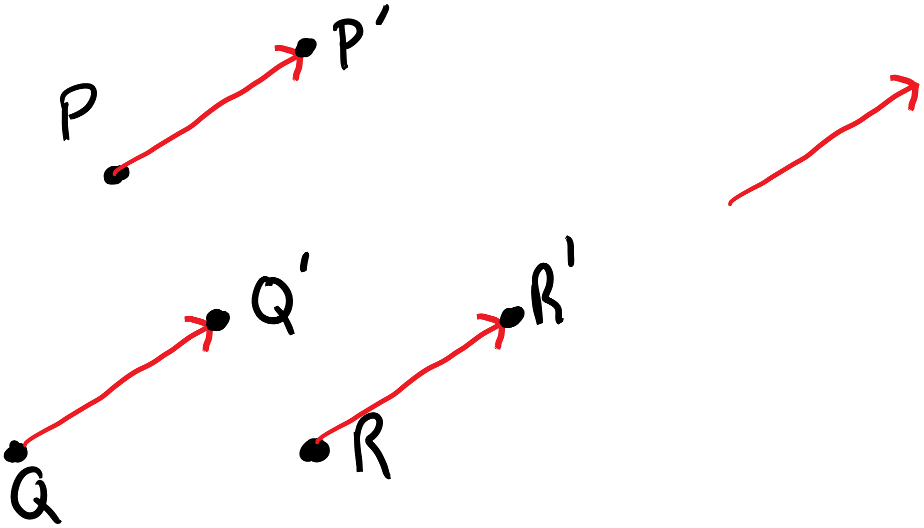
A **translation** is a transformation of the plane in which every point of the plane is moved the same distance in the same direction. The distance and direction of a translation are usually communicated by drawing an arrow which shows the desired direction and distance. Suppose that we want to draw the result of translating each of these points in the direction and distance indicated by this arrow.



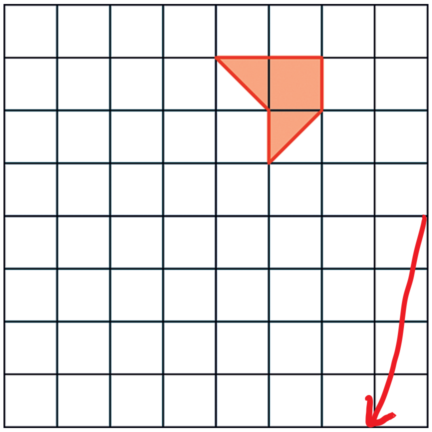
To do so, we first copy the arrow so that the tail rests on each point like so:



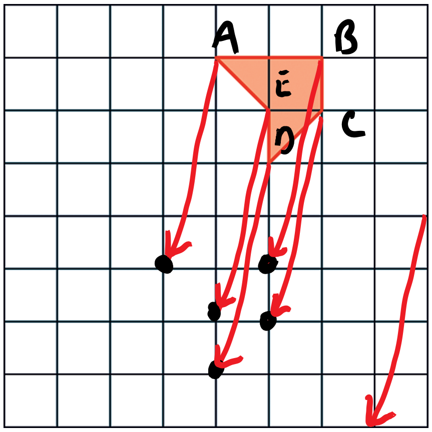
Then we draw the image points at the ends of the arrows.



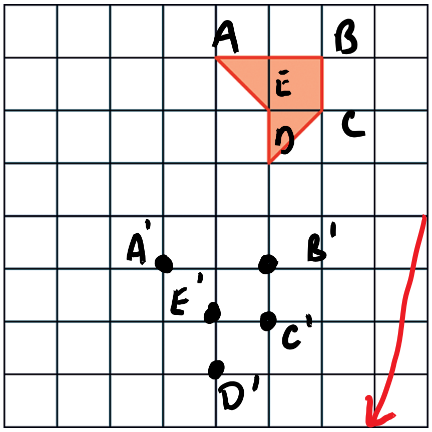
Problem: Translate the shape below according to the arrow.



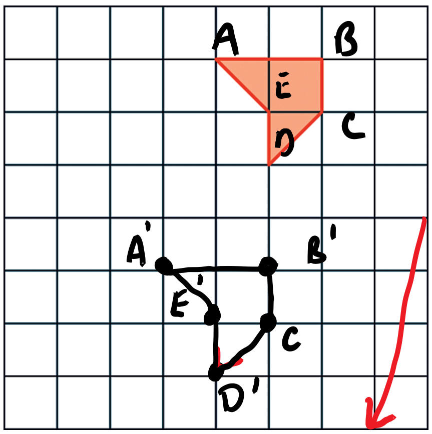
To transform any polygonal object like this, it is usually easiest to name the vertices, transform them, and then connect the dots. We name the vertices, copy the arrow to each vertex, and place a new point at the head of each arrow.



We name the new points appropriately and delete the arrows.

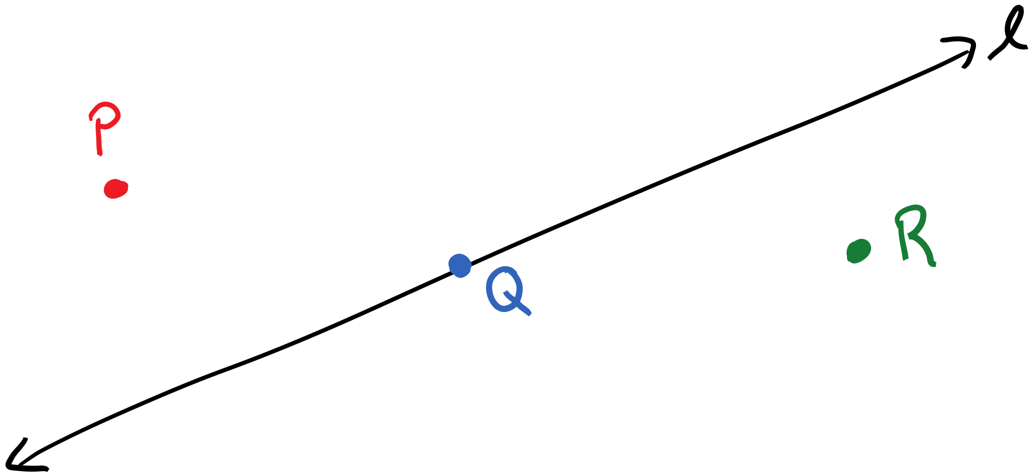


Then we simply connect the dots in order.

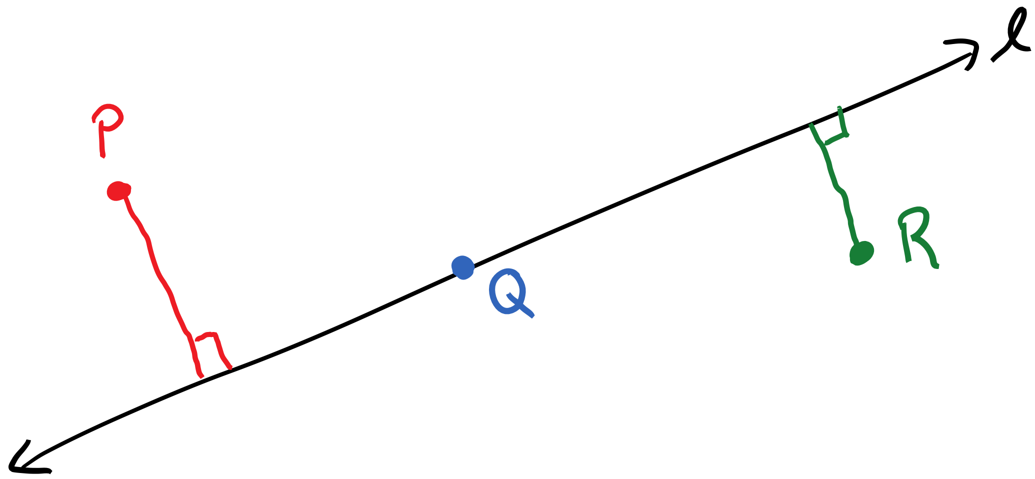


Reflection

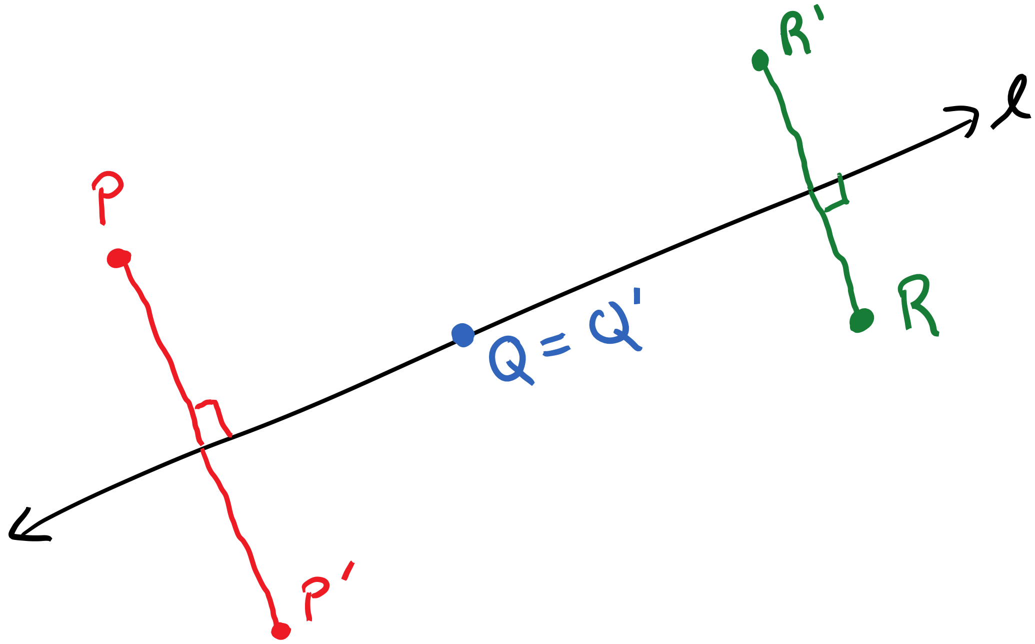
A reflection through a line l maps every point P in the plane to a point P' which is the same distance from l on the opposite side from P. Consider this line and these three points



For points P and R, we draw perpendiculars to the line.

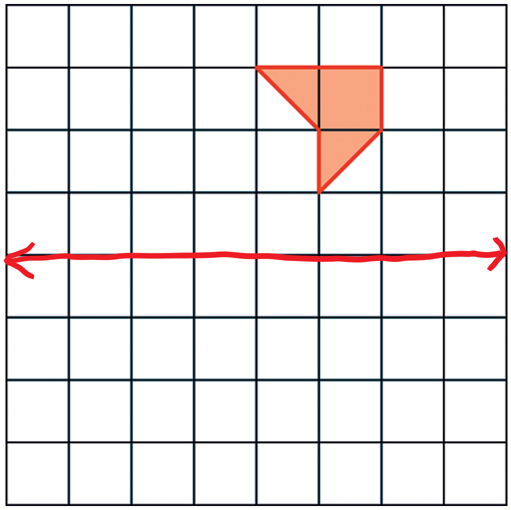


We then extend the perpendiculars past the line and locate P' the same distance from the line as P and R' the same distance from the line as R.

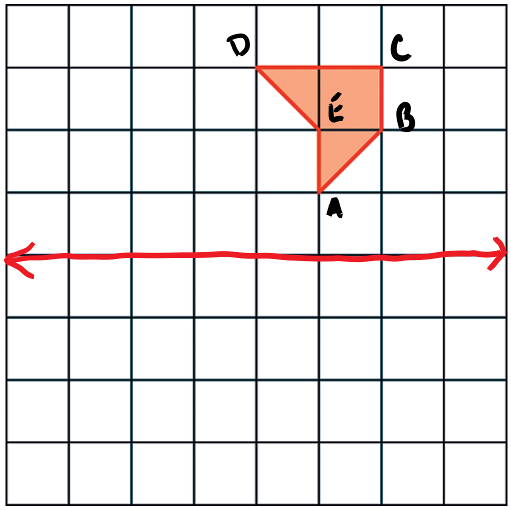


Since Q is on the line, the reflection Q' of Q is the same as Q.

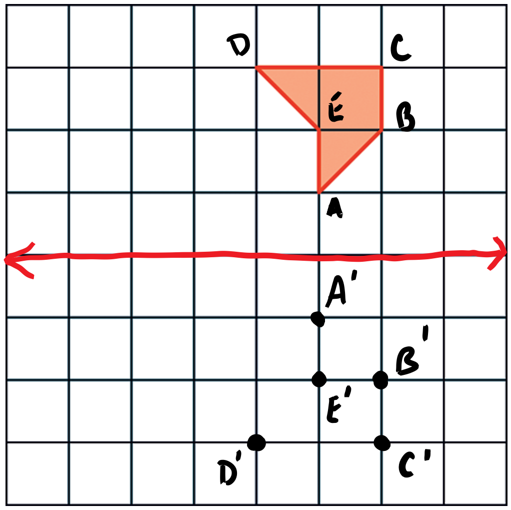
Problem: Reflect the shape below across the given red line.



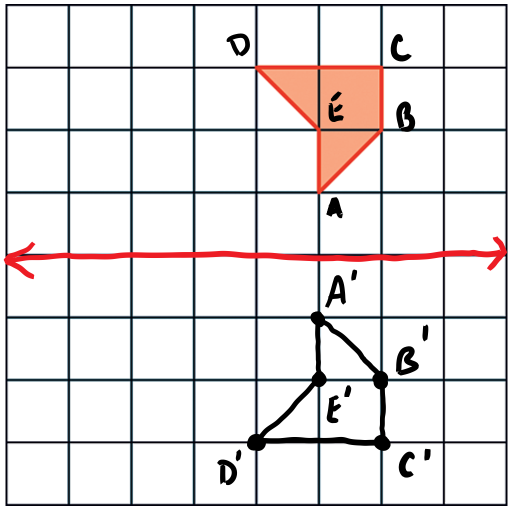
We first name the vertices and move them one at a time.



Reflecting across horizontal lines like this one or vertical lines is not too bad. The point A is one unit above the line, so the point A' will be one unit below the line, directly beneath A. Since B is two units above the line, B' is two units below the line. We perform this process for each point.

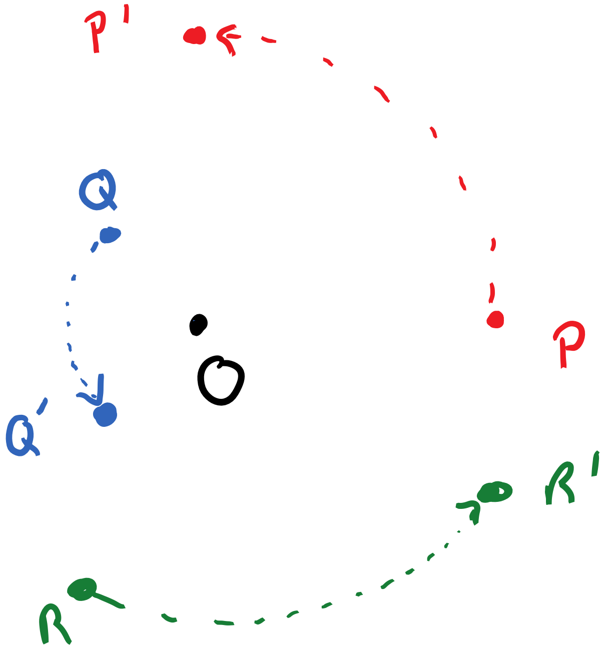


Then we connect the dots in order.

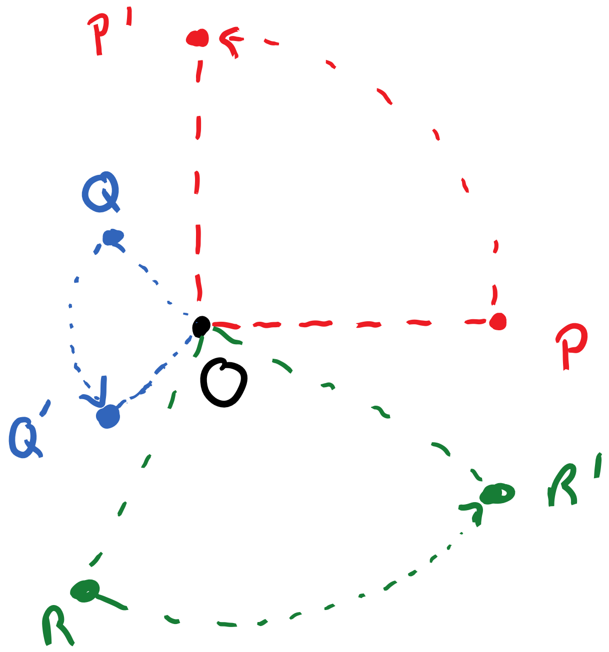


Rotation

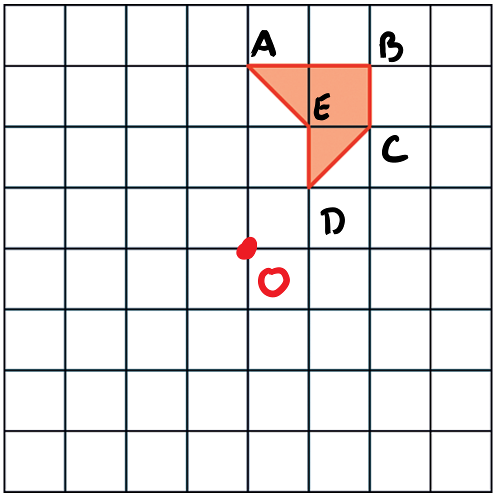
A **rotation** rotates every point on the plane the same angle around a fixed point called the **center of rotation**. In the picture below, we rotate P, Q, and R $90°$ counter clockwise around the point O.



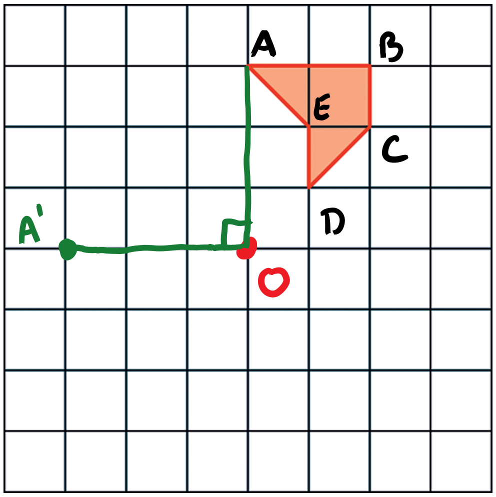
Drawing line segments from the center of rotation to each point can make it easier to see the angle of rotation.



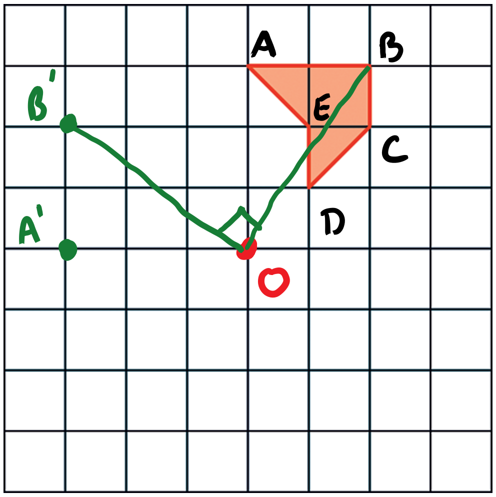
Problem: Rotate the shape below 90° counter clockwise around the point O.



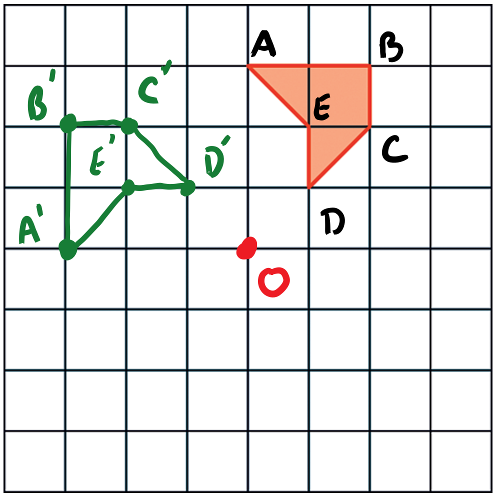
We first draw a line segment to A and then another line segment the same length to make a 90° degree angle. A' is at the end of this new line segment.



We then do the same thing for B. There are a couple of other ways to find B'. Since B is two steps right of A, the point B' should be two units above A'. Alternatively, since we can get from O to B by moving two units right and three units up, we get to B' by moving three units left and two units up.

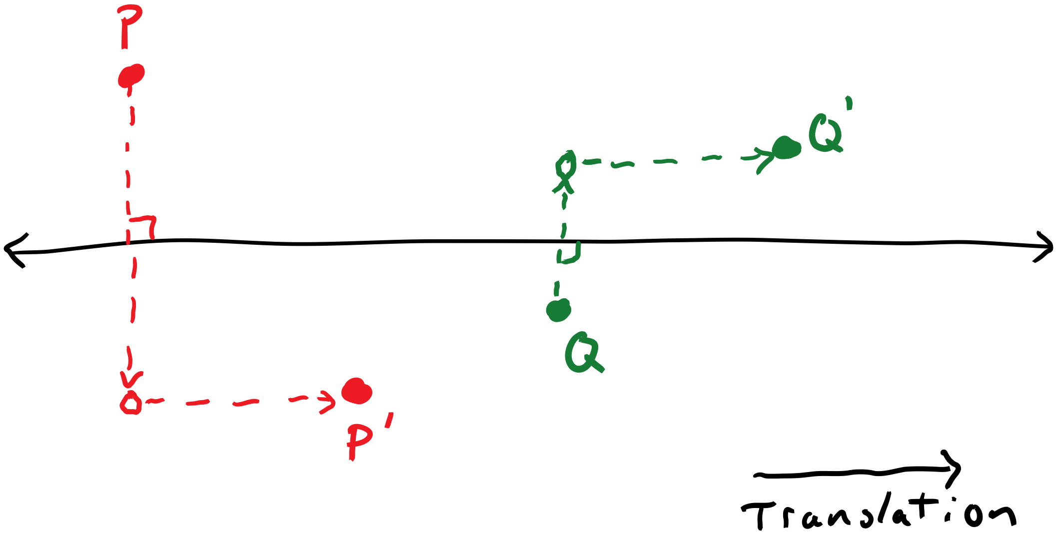


We repeat this process for C, D, and E and then connect the dots.



Glide-Reflection

Performing a reflection followed by a translation parallel to the line of reflection is called a **glide-reflection**.



Isometries

Translations, reflections, rotations, and glide-reflections have the property that they do not change distances. If P and Q are two points that are translated, reflected, rotated, or glide-reflected to points P' and Q', then the distance between P' and Q' is the same as the distance between P and Q. Any transformation with this property is called an **isometry**. It happens to be that:

Isometry Theorem: Every isometry is a sequence of translations, reflections, and rotations.

Many modern textbooks on geometry build concepts beginning with the idea of congruence (which we will discuss below) and define congruence based on isometries. Therefore, understanding translations, reflections, and rotations is essential to understanding the building blocks of geometry.

Symmetry

A shape on the plane is **symmetric** or has **symmetry** if there is a nontrivial isometry which maps the shape onto itself. In particular:

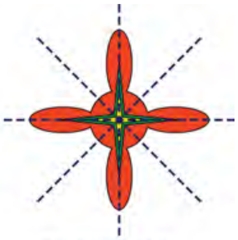
- A shape has **reflection symmetry** if there is a reflection which maps the shape onto itself. In this case, the line of reflection is a **line of symmetry**.


A shape


A shape and its line of symmetry

Shapes can have more than one line of symmetry.


A design


The design and its four lines of symmetry

- A shape has **rotation symmetry** if there is a rotation which maps the shape onto itself.


2-fold rotation symmetry


3-fold rotation symmetry


4-fold rotation symmetry

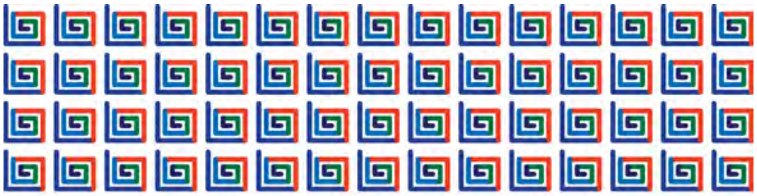

6-fold rotation symmetry

When a shape has rotation symmetry, there is a certain number of times the rotation can be repeated before every point of the shape ends up where it started. If 2 rotations returns every point to its original location, then the shape has 2-fold rotation symmetry. If 3 rotations returns every point to its original location, then the shape has 3-fold rotation symmetry. If a shape has n-fold rotation symmetry, then rotation by $\frac{360}{n}$ degrees will map the shape onto itself.

- A shape has **translation symmetry** if there is a translation which maps the shape onto itself. For a shape to have translation symmetry, the shape will have to "continue forever" in more than one direction. The shape below has translation symmetry because it can be translated to the right or to the left and land on itself.


The pattern continues forever to the right and to the left.

The shape below has many translation symmetries. Every spiral pattern can be translated to land on top of every other spiral pattern.



The pattern continues forever in all directions.

Problem: Find all rotation, reflection, and translation symmetries of the pattern below.

Translation symmetry is easy. There is no translation symmetry since this shape does not “continue forever.” For rotation symmetry, we focus on the heart facing to the right. There is a rotation which takes this heart to the nearby heart above it. If this rotation is repeated 6 times, the heart will land where it started, so this shape has 6-fold rotation symmetry.

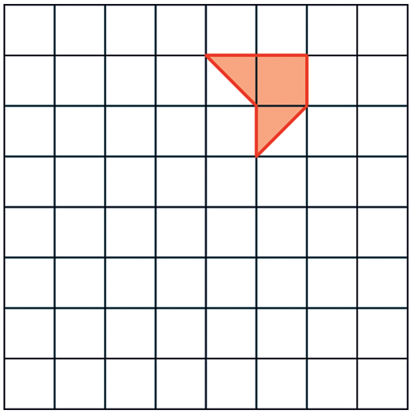
The pattern has many lines of symmetry which are all shown in the picture below.

This shape happens to have 6-fold rotation symmetry and 6 lines of symmetry. This is (almost) a coincidence. The curious student who loses sleep over it might want to investigate the relationship between rotation symmetry and the number of lines of symmetry.

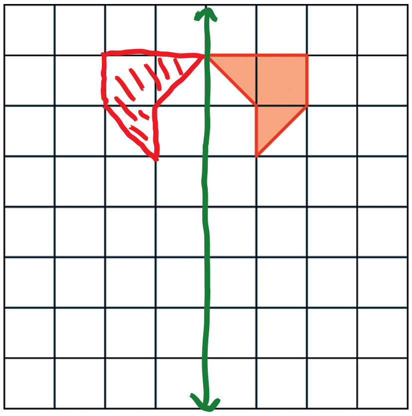
Creating Patterns with Symmetry

If we start with any pattern and repeatedly apply a rotation or reflections, we can create new patterns with symmetry.

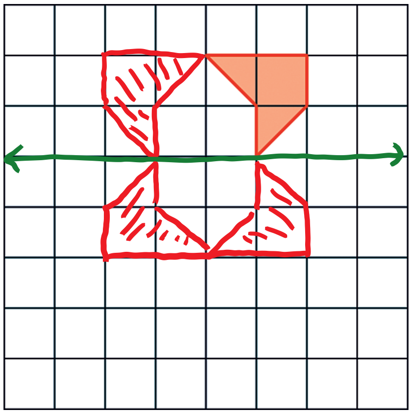
Problem: Apply reflections to the pattern below to create a new pattern with a horizontal line of symmetry and a vertical line of symmetry.



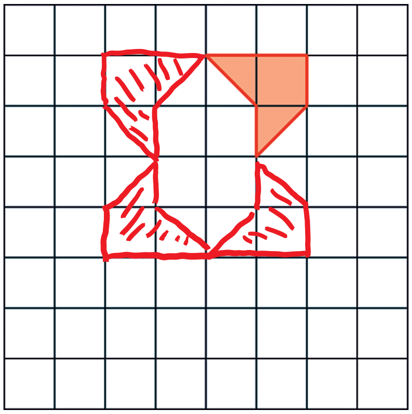
First we select a vertical line and reflect the shape across that vertical line.



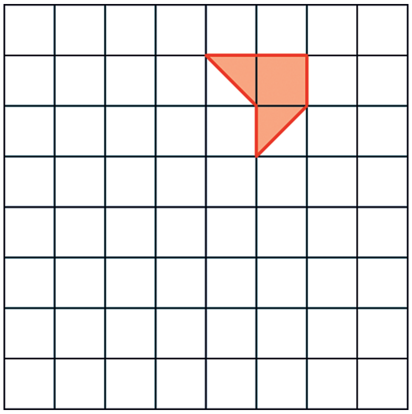
At this point, we already have a vertical line of symmetry. Now we select a horizontal line and reflect the entire shape across it.



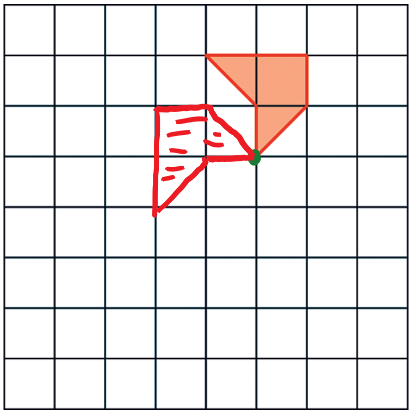
Deleting the line of symmetry now gives us our shape.



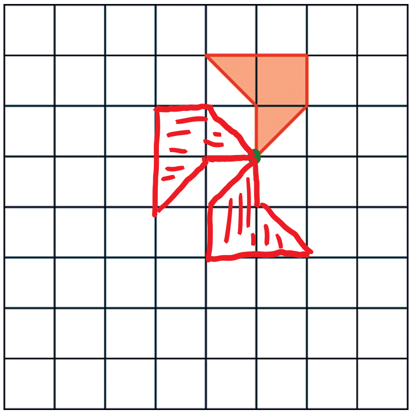
Problem: Apply rotations to this shape to create a pattern with rotation symmetry.



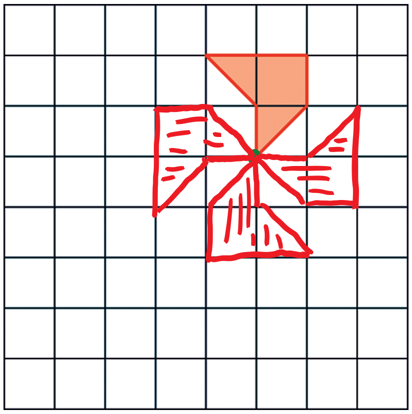
First we select a point to rotate about, and then we rotation 90°. (We choose 90° just to make it easier).



We then rotate 90° again.



And then again.

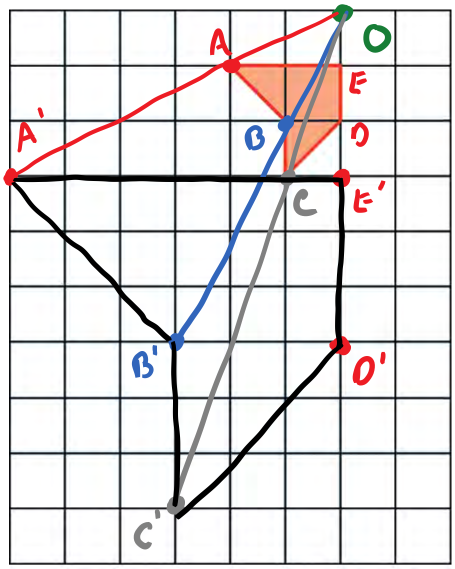


We now have a shape with rotation symmetry.

Dilations

Not all transformations of the plane are isometries. If O is any point and k is a positive number then the **dilation** by k around O works this way: For any point P, draw the raw with endpoint O passing through P. Then P is mapped to the point P' on this ray so that the distance from O to P' is k times the distance from O to P. This dilation multiplies all distances by k.

The picture below illustrates the dilation of this shape by 3 around the point O. The dilation is outlined in black.



Coordinates can be used to make drawing dilations easier. The point A is down 1 unit and left 2 units from O. To dilate by 3, we simply multiple these distances by 3. The point A' is down 3 units and left 6 units from O. Dilations around the origin are simple. We just have to multiply the coordinates of every point by k.

Congruence and Similarity

Suppose that we have two polygonal toys and want to know if they are exactly the same shape. We can take one of them, pick it up, spin it, flip it over, and try to lay it directly on top of the other one. If we are successful, the shapes are the same. Each of these motions is an isometry of the plane. Remember that all isometries reduce to rotations, translations, and reflections. The mathematical term for saying that two shapes are exactly the same is congruence. Thus, two objects in the plane are **congruent** if there is a sequence of translations, reflections, and rotations that maps one of them directly on top of the other. Informally, two objects in the plane are congruent if they are the same shape and the same size.

Triangle Congruence

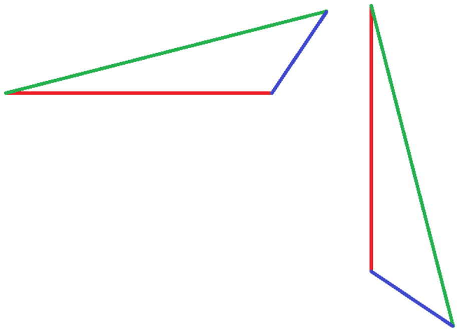
A triangle with vertices A, B, and C is often referred to as ΔABC (read "triangle A, B, C"). If we say that a triangle ΔABC is congruent to a triangle ΔDEF then the order of the points matters. We are saying that there is a sequence of translations, rotations, and reflections that maps A to D, B to E, and C to F. Note that one consequence of this is that the distance from A to B is the same as the distance from D to E because these transformations are isometries. Also, the distance from B to C is the same as the distance from E to F, and the distance from C to A is the same as the distance from F to D. This hints at another characterization of congruence.

Point Correspondence Characterization of Congruence: Two shapes are congruent if there is a correspondence of points on the first shape with points on the second shape so that the distance between every two points on the second shape is the same as the distance between the corresponding points on the first shape.

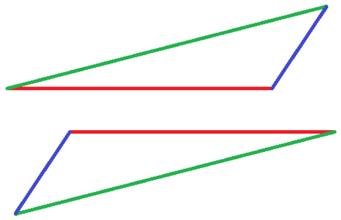
This characterization is a bit wordier, but it emphasizes the distance-preserving nature of correspondence. There are a number of ways to tell if two triangles are congruent. Each of these is called a *congruence criterion*.

Side-Side-Side (SSS) Congruence Criterion: If the lengths of the sides of one triangle are equal to the lengths of the sides of another triangle, then the two triangles are congruent.

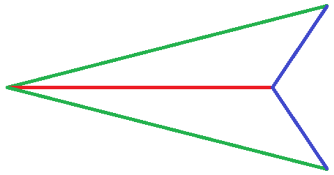
If the sides in these two triangles with the same colors have the same lengths, then the two triangles are congruent.



To see why this is so, we can first rotate the triangle on the right 90°.



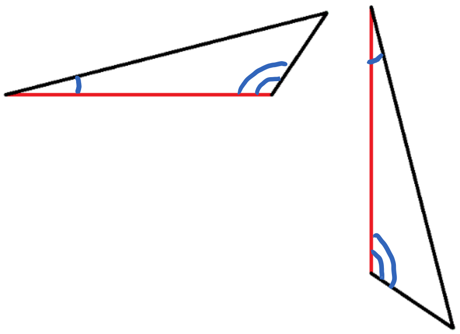
Then, we can reflect the bottom triangle around a vertical line and translate it until the red line segments coincide.



At this point, the two triangles are mirror images of each other, so another reflection will map one on top of the other, making them congruent.

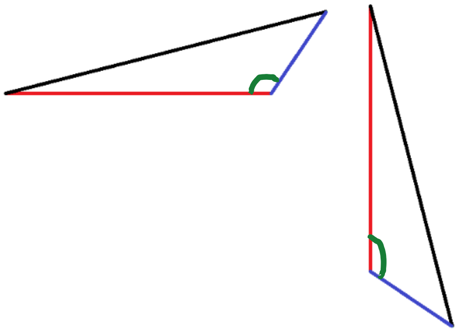
Angle-Side-Angle (ASA) Congruence Criterion: Suppose that two angles in one triangle have the same measure as two angles in another triangle. If the length of the side between those two angles in the first triangle is equal to the length of the side between the two angles in the second triangle, then the triangles are congruent.

If the red sides in these triangles have the same length, and if the indicated angles have the same measure, then the triangles are congruent.



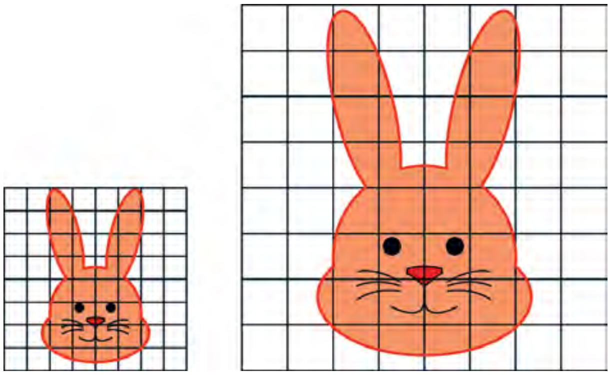
Side-Angle-Side (SAS) Congruence Criterion: Suppose that two sides of one triangle have the same lengths as two sides of another triangle and that the angle between the two sides in the first triangle has the same measure as the angle between the two sides in the second triangle. Then these two triangles are congruent.

If the red sides have the same length, and if the blue sides have the same length, and if the indicated angles have the same measure, then the two triangles are congruent.

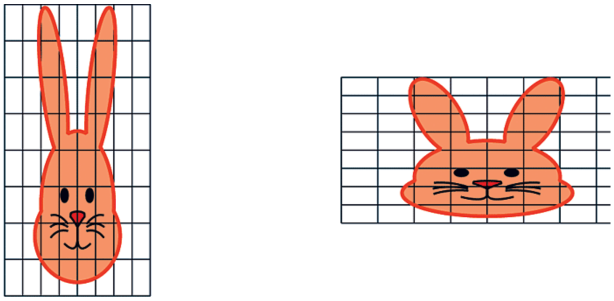


Similarity

Sometimes, objects have the same shape, but a different size. In this case, we will say that the objects are *similar*. Our definition of similar is specifically chosen to mimic our definition of congruent. Two objects are congruent if there is a sequence of isometries that maps one onto the other. For similarity, we want to include a transformation that changes sizes. For us, that would be a dilation. Two objects in the plane are **similar** if there is a sequence of *dilations*, translations, reflections, and rotations that maps one of them directly on top of the other. Informally, two objects in the plane are similar if they are the same shape but *possibly different sizes*. The only changes here from the definition of congruent are in italics. These two bunnies are similar:



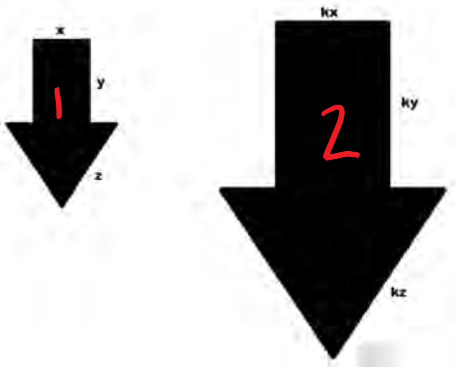
These two bunnies are not similar:



We have a corresponding points characterization of similarity which is analogous to the corresponding points characterization of congruence.

Point Correspondence Characterization of Similarity: Two shapes are *similar* if there is *a number* k *and* a correspondence of points on the first shape with points on the second shape so that the distance between every two points on the second shape is k *times* the distance between the corresponding points on the first shape.

Again, the only changes from the characterization of congruence are in italics. The number k in this characterization is called the **scale factor**. Here are two similar arrows with a scale factor of k.



Notice that to find distances on arrow number 2 we simply multiply distances from arrow number 1 by the scale factor k. If we take any distance from arrow 2 (say, kx) and divide it by the distance from arrow 1 (which would be x), we get the same ratio k. This is to be expected from the Point Correspondence Characterization. Suppose we consider ratios of lengths within each arrow. Consider the ratio of the top measurement over the right hand measurement. On arrow number 1, this is

$$\frac{top}{right} = \frac{x}{y}.$$

On arrow number 2, this measurement is

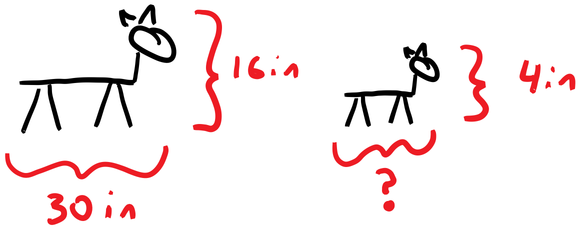
$$\frac{top}{right} = \frac{kx}{ky} = \frac{x}{y}.$$

The ratios are the same.

Internal Factors: If two shapes are similar, then the ratio of any two lengths on one shape is equal to the ratio of the corresponding lengths on the other shape. These ratios are called **internal factors**.

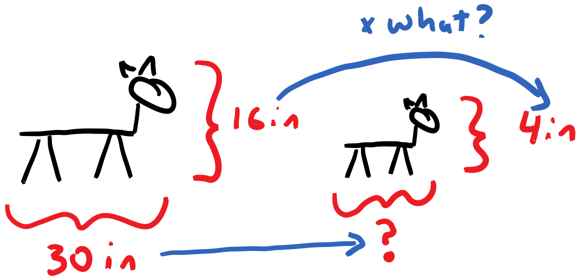
Scale factors and internal factors are the basis for four different ways to solve similarity problems.

Problem: Frank's dog Fido is 16 inches tall and 30 inches long. Frank wants to draw Fido 4 inches tall. How long should Fido be in the picture?



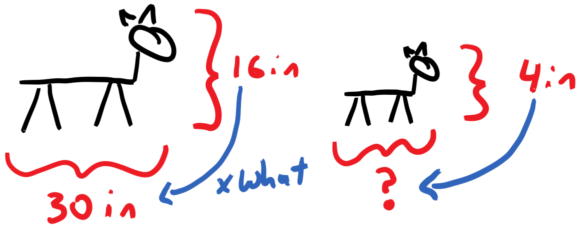
We will solve this problem four different ways – scale (external) factors, internal factors, external proportions, and internal proportions.

Scale (External) Factors: We know that we can convert lengths from the original dog to lengths on the drawing by multiplying by a scale factor. We just have to find that scale factor. To find it, we notice that we know the height for both dogs, so we ask ourselves, "What do we have to multiply the original height (16 inches) by to get the drawing height (4 inches)?" Once we have that answer, we multiply 30 inches by the same factor.



The answer to this question is $\frac{4}{16} = \frac{1}{4}$, so we multiply the original length (30 inches) by the same number. Therefore, the drawing should be $30 \times \frac{1}{4} = 7.5\ in$ long.

Internal Factors: The external factors approach involved multiplying measurements from one object to get measurements in the other object. The internal approach involves multiplying measurements on one object to get other measurements on the same object. Since we know the height of both dogs, and since we know the length of one, but we want the length of the other, we ask ourselves, "What do we multiply the height (16 inches) of the original dog by to get the length (30 inches)?" Once we have that answer, we will multiply 4 inches by the same factor.



The answer to this question is $\frac{30}{16} = \frac{15}{8}$, so we multiply the height of the drawing (4 inches) by $\frac{15}{8}$. Therefore, the drawing should be $4 \times \frac{15}{8} = 7.5\ in$ long.

If we are willing to use a little algebra, we can also solve these problems by setting up equal proportions.

External Proportions: Here we set up a fraction where half of the fraction comes from the original dog and the other half comes from the drawing. Our work will always be easier if we place our question mark on the top of the fraction. The question mark here is on the drawing, so our fractions will have measurements from the drawing on top and measurements from the original dog on bottom. Remember the ratio $\frac{drawing}{original}$ should be the same for every measurement. These are our fractions:

$$\frac{?}{30} = \frac{drawing\ length}{original\ length} = \frac{drawing\ height}{original\ height} = \frac{4}{16}.$$

To solve for the ? here, we just multiply by 30 to get $? = \frac{4}{16} \times 30 = 7.5\ in.$

Internal Proportions: Internal proportions will work similarly to external proportions, but our fractions will have different measurements from the same dog in one fraction. Since we are looking for the length of the drawing, and since we know the height, we will make the fraction $\frac{length}{height}$ for both dogs. These fractions must be equal.

$$\frac{?}{4} = \frac{drawing\ length}{drawing\ height} = \frac{original\ length}{original\ height} = \frac{30}{16}.$$

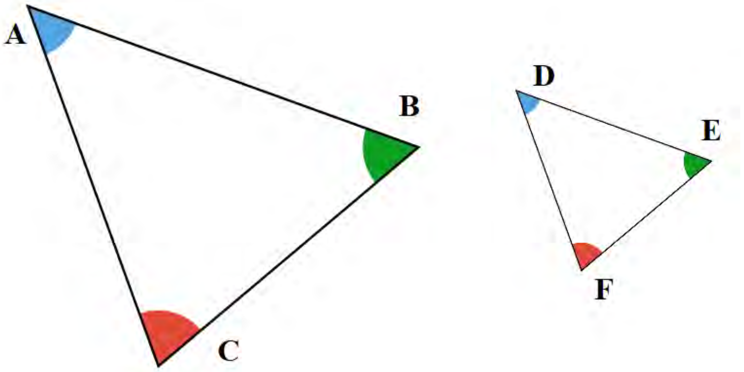
To solve for ? we now just multiply by 4 to get $? = \frac{30}{16} \times 4 = 7.5\ in.$

Notice that we got the same answer in all four approaches. Also notice that the arithmetic was the same for External Factors and External Proportions. Arithmetic was also the same for Internal Factors and Internal Proportions. Also notice that when we used proportions we always put the unknown value on top of the fraction. This is a step that requires little preparation that will always simplify algebra. Doing this also makes it possible to teach proportion methods to students who know almost no algebra.

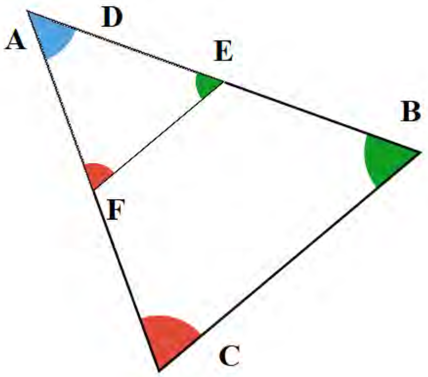
Which method we choose to solve similarity problems is largely up to personal choice. Sometimes one method will have simpler arithmetic. However, **if we want to solve similarity problems involving area, volume, or weight, then we have to use the external factor (scale factor) approach**.

Similar Triangles

Consider these two triangles which have the same angle measures (indicated by color).



We can translate (and rotate or reflect, if necessary) the small triangle so that the point D lies on top of the point A and so that the line segment from D to E lies on the line segment from A to B. Since the blue angles are equal, the line segment from D to F will then lie on the line segment from A to C. Our diagram then looks like this.

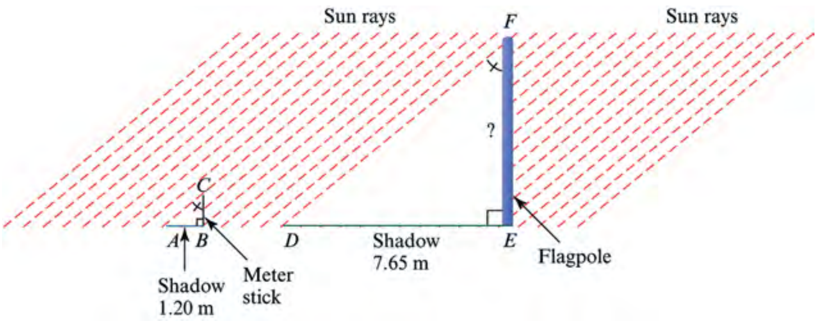


We can now scale (dilate) the small triangle so that point E lies on top of point B. When we do so, ASA congruence will force the new triangle to lie completely on top of the original large triangle. We have thus moved the small triangle through translations, rotations, reflections, and dilations on top of the large triangle. This implies that the two triangles are similar.

Angle-Angle-Angle (AAA) Similarity Criterion: If two triangles have the same angles, then the triangles are similar.

This is a theorem proven by Thales. He supposedly used this theorem and similarity to measure the heights of the pyramids. We will use it to measure a flagpole.

Problem: On a sunny day, the shadow cast by a flagpole is 7.65 meters long. A nearby meter stick held perpendicular to the ground casts a shadow that is 1.2 meters long. How tall is the flagpole?



The key here is that the rays of light from the sun are (more or less) parallel to each other. This implies that the two triangles here have the same top angle by the Parallel Postulate. The flagpole and meter stick make right angles with the ground. Since the angles in a triangle add to $180°$, the bottom angles are also equal. By AAA, these two triangles are similar. We will use external proportions to solve for the question mark. First, we set up our proportions

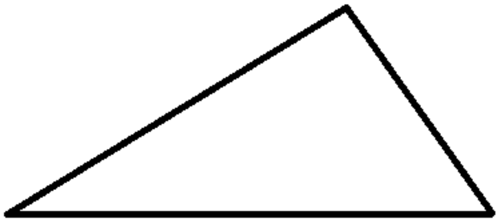
$$\frac{?}{1} = \frac{flagpole\ height}{meter\ stick\ height} = \frac{flagpole\ shadow}{meter\ stick\ shadow} = \frac{7.65}{1.2}.$$

Multiplying by 1 (that is why we chose external proportions) gives

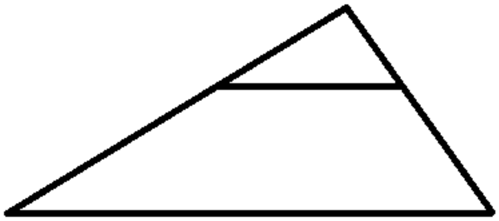
$$? = \frac{7.65}{1.2} \times 1 = 6.375\ m.$$

The flagpole is 6.375 meters tall. Thales supposedly performed a computation such as this to calculate the height of the pyramids.

Suppose that we have a triangle such as this one:



And suppose that we draw a line across the triangle parallel to one of the sides like so:

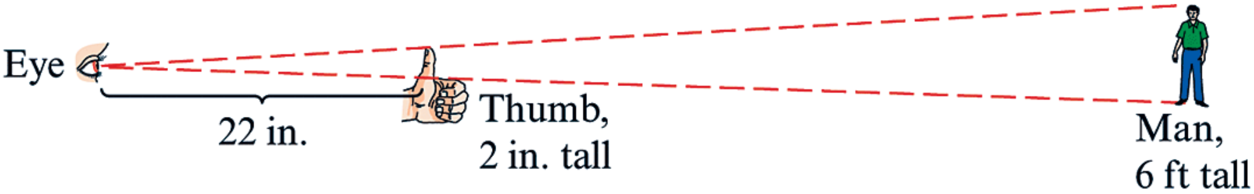


We now have a smaller triangle inside of a larger triangle. In this case, the Parallel Postulate tells us that the angles in the small triangle are equal to the angles in the larger triangle. By AAA, the two triangles are similar.

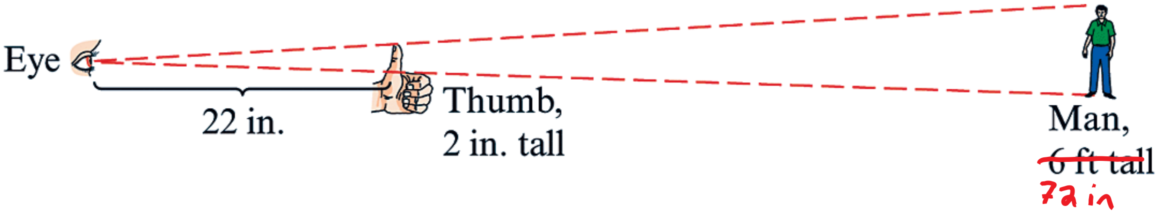
Triangle within a Triangle Theorem: If a line is drawn across a triangle parallel to one side, then the smaller triangle formed is similar to the original triangle.

This theorem was also proven by Thales, and he used it to measure distances to ships and sizes of objects in the distance.

Problem: Sue's brother Bob is 6 feet tall. Sue sees Bob in the distance. When she raises her thumb up between her eye and Bob, the thumb and Bob appear to be the same height. If Sue's thumb is 2 inches long, and if her thumb is 22 inches from her eye, how far away is Bob?



Before we even think about how to approach the problem, we convert the man's height to inches so that all of our units are the same. (The board student who wonders about it should attempt to work the problem without this conversion. Such a student should get the correct answer and should wonder if that always works and why.)



We should now see our triangle-within-a-triangle arrangement where the eye is one vertex of both triangles. The base of the smaller triangle is the thumb. The base of the larger triangle is the man. We will use similar triangles and the internal proportion approach to find the distance from the eye to the man.

$$\frac{?}{72} = \frac{distance\ to\ man}{height\ of\ man} = \frac{distance\ to\ thumb}{height\ of\ thumb} = \frac{22}{2}.$$

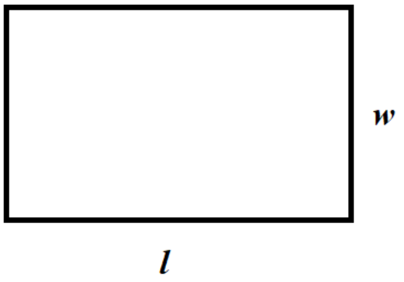
We can solve for the ? by multiplying by 72

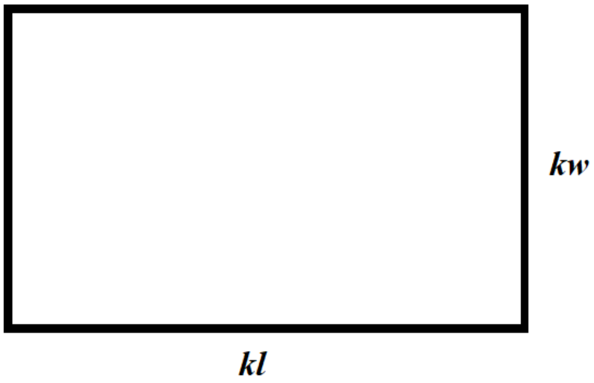
$$? = \frac{22}{2} \times 72 = 792\ in.$$

The man is 792 inches (or 66 feet) away.

Area, Volume, and Weight

Suppose that we have a rectangle with length l and width w and that we scale the rectangle by a constant k.





The area of the original rectangle is lw while the area of the scaled rectangle is $(kl) \times (kw) = k^2lw$. Notice that while we have multiplied the lengths by k, the area gets scaled by k^2. Similarly, suppose that we scale a box with dimensions $l \times w \times h$ by a constant k. The new box will have dimensions $kl \times kw \times kh$. The original volume is lwh while the new volume is $(kl) \times (kw) \times (kh) = k^3lwh$. We have multiplied the volume by k^3.

If we scale an object by a constant k then

- Lengths are multiplied by k.
- Areas are multiplied by k^2.
- Volumes are multiplied by k^3.
- Since weight is usually assumed to be proportional to volume, weights are also multiplied by k^3.

Problem: Erwin made a scale model of his house. The door on the scale model is 2 inches tall. The door on the actual house is 84 inches tall. If the actual house has a floor area of 3200 square feet, then what is the floor area of the model in square inches? Round to the nearest square inch.

We can use the information about the doors to find the scale factor k. The scale factor k is the number we multiply $84\ in$ by to get $2\ in$. That is $k = \frac{2}{84} = \frac{1}{42}$. (We are looking for floor area of the model, so the model's door height goes on top.) To find the floor area of the model, we multiply the floor area of the original house by k^2.

$$floor\ area\ of\ model = 3200\ ft^2 \times \frac{1}{42} = 76.1905\ ft^2.$$

Note that this area is in square feet. We have to convert that to square inches.

$$floor\ area\ of\ model = 76.1905\ ft^2 \times \frac{12\ in}{1\ ft} \times \frac{12\ in}{1\ ft} = 10971.432\ in^2.$$

The model has a floor area of about $10971\ in^2$.

Problem: A sculptor has made a scale model of a granite sculpture. The model is 3 feet tall and weighs 300 pounds. If the actual sculpture is to be 12 feet tall, how much will it weigh?

We can use the information about height to find the scale factor k. The scale factor k is the number we multiply $3\ ft$ by to get $12\ ft$. This is $k = \frac{12}{3} = 4$. (Note that we put the height of the actual sculpture on top since we want information about the actual sculpture.) To find the weight of the actual sculpture, we scale the model weight by k^3.

$$actual\ weight = 300 \times 4^3 = 19200\ lb.$$

The actual sculpture (if made of the same granite) will weigh 19200 lb.

Data and Statistics

Statistics is the science of decision making. It is a discipline related to mathematics which provides a set of tools for

- Collecting data,
- Analyzing data, and
- Making decisions based on that data.

Almost every realm of our lives involves decisions based on statistics. Big box stores use statistics to decide what products to carry, how many of each product to stock, and what price to charge for those products. Colleges use statistics to decide what majors and classes to offer, what to charge for tuition, and how many meals to make in the dining hall. Car manufacturers use statistics to decide what types of cars to make, how tall to make the seats in those cars, and how to engineer the cars to be safe. Educators use statistics to try to decide which students are on track and what to do about those that are not on track. Any time we make decisions in our lives in which we decide that one option is better than another, we are at least informally using some form of statistics.

Data

Data are collections of observations made in studies and experiments. For our purposes, data come in two flavors.

- **Numerical data** consists of numbers that are actual counts or measurements. Some examples of numerical data are: number of siblings, height, and pandemic infection rates. An example of a number which is not numerical data is a zip code. There is no counting or measuring that goes into determining a zip code.
- **Categorical data** consists of names or labels which are not actual counts or measurements. Some examples of categorical data are eye color, home state, and zip codes.

Problem: Determine if each of these is numerical or categorical data: jersey number, eye color, weight, pulse, likert scores.

The number on an athlete's jersey is simply a name or label not related to any measurement or computation. This is categorical data. Eye color is categorical data (unless we are talking about wavelengths). A person's weight is numerical data. We actually measure it. A person's pulse is numerical data. It is determined by counting. Likert scores come from surveys in which we give responses such as "agree" or "strongly disagree" or "neutral." These responses are frequently treated as numbers (such as 1, 2, 3, 4, and 5). However, the responses are just categories. Nothing is being measured. Likert scores are categorical data.

We might wonder why we should care about whether data is categorical or numerical. The reason is that there are computations that we can make with numerical data that simply do not make sense with categorical data such as finding averages and percentiles and standard deviations. We should never perform these computations on categorical data – even if that data looks numerical. For example, likert scores should never be averaged. It makes as much sense to average likert scores as it does to average jersey numbers. However, they are often (and erroneously) averaged. Many types of grades are categorical in nature. These types of grades should never be averaged (but they usually are).

Populations and Samples

In most studies involving statistics, there is a set of all of the individuals being studied. For example, we may want to know what proportion of the adults in the United States are employed. Here, we are concerned with all adults in the country. We might want to know what proportion of squirrels are black (I saw black squirrels at Chalk Falls this weekend.) In that case the set of all of the individual being studied is the set of all squirrels. The set of all individuals being studied is called the **population**. When we collect data from an entire population, we call the process a **census**. The United States attempts to conduct a census of all its citizens every ten years. A measurement based on a census of an entire population is called a **parameter**.

Usually, populations are entirely too big to actually study every individual in the population. It would be impossible to observe every squirrel that exists to determine if it is black or not. Therefore, instead of considering the entire population, we usually consider only a set of some of the individual in a population. A set of some of the individuals in a population is called a **sample**. A measurement based on a sample is called a **statistic**. A sample is a **random sample** if every individual in the population is equally likely to be selected in the sample. A sample is a **representative sample** if the statistics from the sample are close to the parameters for the population.

It is a major result in statistics that sufficiently large, sufficiently random samples are likely to be representative. This allows us to draw conclusions about an entire population based on relatively small random samples. We study what “sufficiently large” and “sufficiently random” mean in classes about statistics.

Problem: There are 100 marbles in a bag. All of the marbles are red or blue. Bob randomly grabs 20 marbles from the bag. Of those 20, 7 are blue. About how many blue marbles are in the bag?

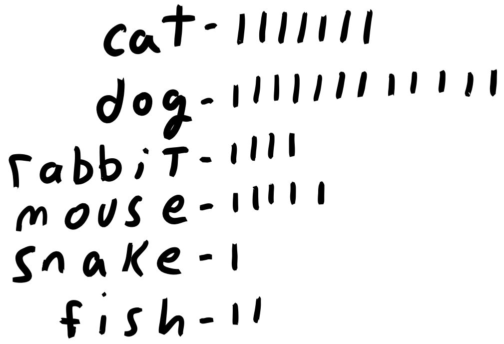
The proportion of marbles which are blue in Bob’s sample is $\frac{7}{20}$. Bob’s sample of 20 marbles is random and is likely to be representative, therefore, we assume that the proportion of all of the marbles which are blue is close to $\frac{7}{20}$. Therefore, the proportion of marbles which are blue is probably about

$$\frac{7}{20} \times 100 = 35.$$

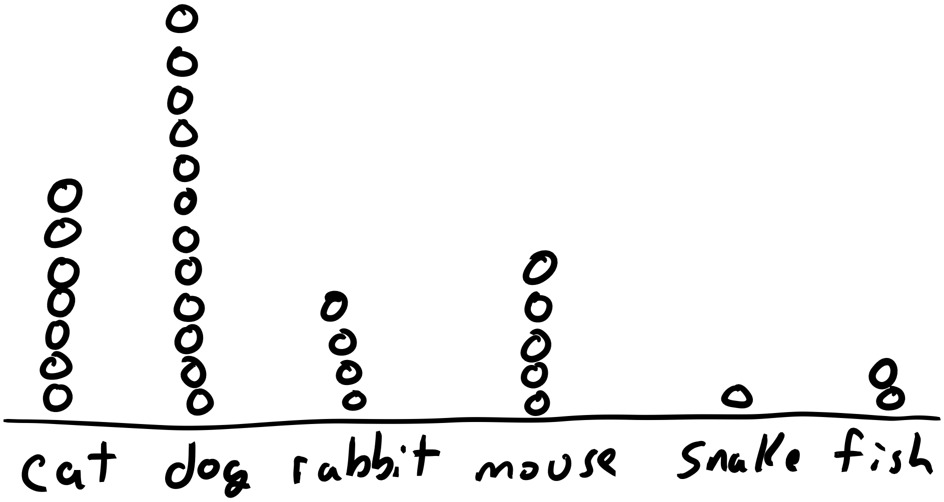
We think that there are about 35 blue marbles in the bag.

Displaying Data

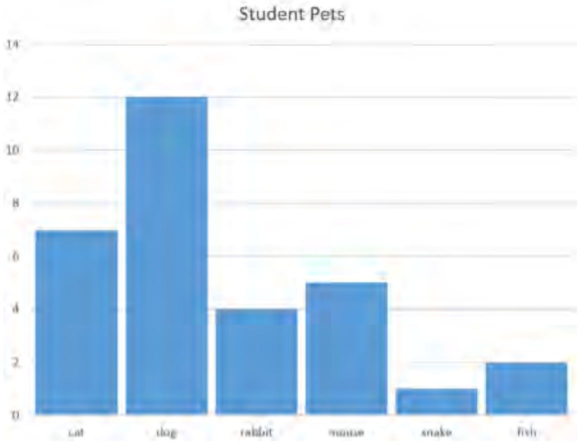
A classroom of students was asked to write down the types of pets they had. These are the results: dog, rabbit, snake, dog, cat, mouse, cat, rabbit, fish, rabbit, dog, dog, mouse, mouse, dog, cat, dog, cat, mouse, cat, dog, fish, dog, dog, dog, dog, cat, rabbit, dog, mouse, cat. Listing the results like this does not do much for communication. We can draw a picture of the data which will make it much easier to make observations about the data. One way to do this is to work through the animals and tally how many of each animal there are. That would look like this:



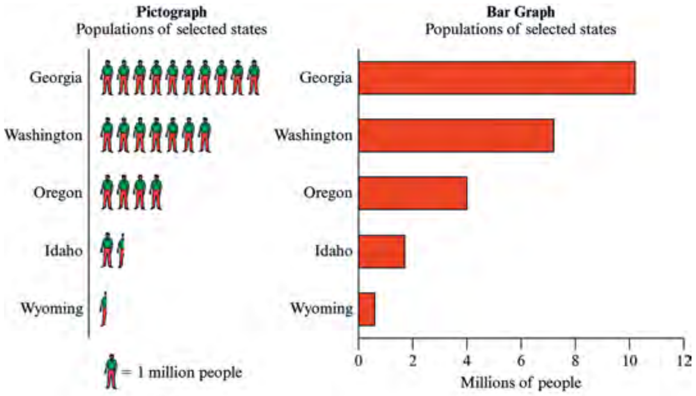
Some people prefer to use dots (or circles) rather than tallies so that it is easier to keep the spacing uniform. This is called a **dot plot**.



If the dots or tallies are covered by bars or boxes, we get a **bar graph**.



Bar graphs can be drawn sideways like the tallies above. Also, instead of dots or tallies, we could use pictures of other shapes. These result in **pictographs**.



(a) (b)

Pictographs are generally discouraged because they can be confusing or misleading. For example, if the pictograph above only included an arm, then we may not know what fraction of one million people is represented. The arm is a tiny bit of the area of the person, so we may interpret it as a tiny part of one million. However, the arm is about a quarter of the way across the small person, so it may represent a quarter million. Also, the right arm (the one on the left) sticks out more than the other. Does it represent more people?

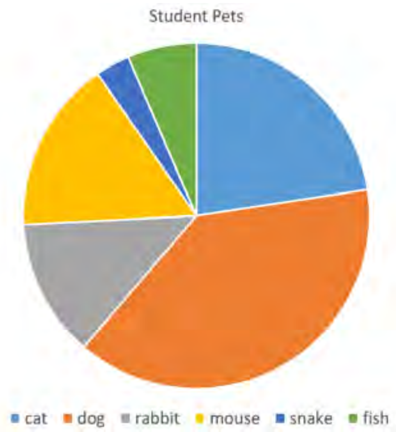
Questions about Data

Once we have a picture of our data, it is easier to answer questions about the data. Questions generally come in three flavors:

- **Questions about the data**: The simplest type of question is a question about the data. This type of question simply asks us to read the data. An example related to the pet data would be: How many students had pet rabbits? To answer this question, we simply look at one of the graphs above and read (or count) four rabbits.
- **Between the data**: The next simplest type of question asks us for comparisons within the data. These are questions between the data. Two examples (whose answers are easily seen in the graphs) are: Which type of pet was most popular? and Where there more mice or snakes?
- **Beyond the data**: Questions beyond the data ask us to think beyond the given numbers and often ask for explanations. For example: Why are dogs the most popular pet? or Why are mice more common than fish?

Pie Charts

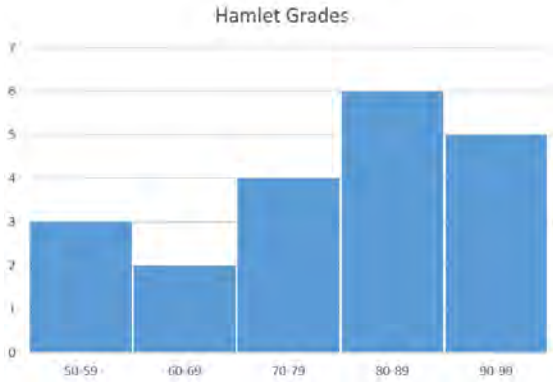
When data are divided into different categories and we just want a depiction of how large the categories are related to each other, we can draw a pie chart. For our pet data, a pie chart would like this.



Each category gets a different color of slice of the pie. The larger the category, the larger the slice. Mathematicians and statisticians tend to avoid pie charts. They tend to be confusing because of how they incorporate area and angles, and it is sometimes difficult to compare slices that are not adjacent. For comparing categories, simple bar charts are the best.

Histograms

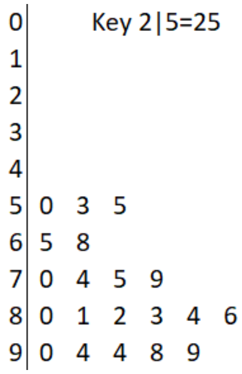
Here are the grades of twenty students on a quiz over *Hamlet*: 94, 82, 74, 68, 99, 53, 70, 83, 80, 55, 50, 81, 94, 98, 90, 84, 86, 75, 79, 65. We are going to draw a bar chart for these grades. To draw a bar chart, we need to decide what the categories will be. It is convenient to use ranges of numbers such as 50-59, 60-69, 70-79, 80-89, 90-99. This results in the following bar chart.



A bar chart where the categories are equal width ranges of numbers is called a **histogram**. Histograms are the most common type of chart that we might encounter in a course on statistics.

Stem and Leaf Plots

Another way to display the Hamlet grade data is with a stem and leaf plot. A stem and leaf plot divides every data value into two parts, a stem and a leaf. For example, a value of 45 might be considered to have a stem of 4 and a leaf of 5. The plot has a row for each stem, and the leaf for each data value is placed in the row corresponding to the stem of that data value. For the value 45, we would place a 5 in the row for the stem 4.



0			Key 2|5=25				
1							
2							
3							
4							
5	0	3	5				
6	5	8					
7	0	4	5	9			
8	0	1	2	3	4	6	
9	0	4	4	8	9		

The stem and leaf plot conveys all of the information in the bar chart. However, it also allows us to completely recover the data if we like. For example, from the bar chart, we know that there are 5 scores in the range 90-99. From the stem and leaf plot, we know that two of these are equal to 94.

Summarizing and Comparing Data

Once we have a collection of data from a sample (or even from a population) we may want to summarize the data. Two things we might look for in the data are a central, representative value and a measure of how the data are spread out.

Measures of Center

There are four measures of center that we might use as a single, central, representative value of a data set. They all start with the letter m:

- **Mean (or average)**: The **mean** is the sum of the data values divided by the number of data values. This should only be used with numerical data.
- **Median**: The **median** is the middle value when data values are placed in order. If there is no middle value (because there are an even number of values), then the median is the average of the two middle values.
- **Mode**: The **mode** is the most common data value. If no data value is repeated, then there is no mode. If there is a tie between two data values for the most common, then the data is **bimodal**. Data with a three-way tie is **trimodal**. After that, we simply call the data **multimodal**. The mode is the only measure of center than can be used with categorical data.
- **Midrange**: The **midrange** is the average of the highest and lowest data values.

Problem: Calculate the mean, median, mode, and midrange for this data: 10, 23, 45, 67, 75, 75, 81, 81, 81, 92.

First, note that the data is already in order. If it were not sorted, then sorting the data would make the median, mode, and midrange easier. We start with the mean. There are ten data values, so we add them up and divide by 10.

$$mean = \frac{10 + 23 + 45 + 67 + 75 + 75 + 81 + 81 + 81 + 92}{10} = \frac{630}{10} = 63.$$

The average is 63. For the median, we cross out numbers on the left and right until we end up with one or two in the middle.

$$\cancel{10},\ \cancel{23},\ \cancel{45},\ \cancel{67},\ \boxed{75,\ 75,}\ \cancel{81},\ \cancel{81},\ \cancel{81},\ \cancel{92}$$

Since there are two numbers in the middle, we average them to find the median.

$$median = \frac{75 + 75}{2} = 75.$$

The median is 75. For mode, we count how many occurrences there are of each value. The most common is 81 (with 3 repetitions). The mode is 3. Finally for midrange, we simply average the highest and lowest.

$$midrange = \frac{10 + 92}{2} = 51.$$

The midrange is 51.

Problem: Calculate the median of this data: 12, 23, 45, 56, 65, 67, 87, 88, 89.

The data is already sorted, so we just cross out on the left and right until we are left with one or two values in the middle.

~~12~~, ~~23~~, ~~45~~, ~~56~~, 65, ~~67~~, ~~87~~, ~~88~~, ~~89~~

We are left with one number, 65, in the middle, so the median is 65.

Problem: Bob has three test grades of 30, 50, and 76. What grade would have to get on a fourth test to have an average of 80?

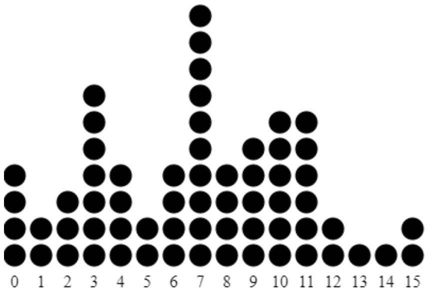
We focus on points earned for this problem. If Bob's four grades are to average to 80, then the sum of the four grades divided by 4 should be 80.

$$\frac{sum}{4} = 80.$$

This means that the sum should be $sum = 4 \times 80 = 320$. Bob needs to accumulate 320 points to get an 80. So far, he has $30 + 50 + 76 = 156$. That means he still needs $320 - 156 = 164$ points to average an 80. If Bob's tests are on a standard 100 point scale, then this is unlikely to happen.

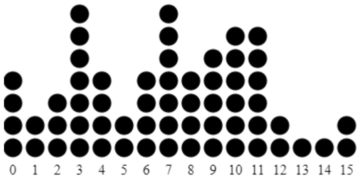
Mode in Graphs

Modes are often easy to spot in dot plots and bar graphs **if each category is a single data value**. In this case, the mode is represented by the tallest column.



The mode in this data is 7.

Problem: Find the mode in the data depicted in the dot plot below.



There is a tie here for the tallest column of dots between 3 and 7, so this data is bimodal.

Weighted Averages

Suppose that two-thirds of a class are female and that one-third of the class is male. Suppose also that the average height of the females is 64 inches and the average height of the males is 70 inches. What is the average height of the class? We cannot simply average 64 and 70 to get 67 inches, because this ignores the fact that there are more females in the class. We need to *weigh* the 64 inches more than the 70 inches. One way to do this is to focus on the fractions two-thirds and one third. We could imagine that we have three students. Two of them are females with a height of 64 inches and one of them is male with a height of 70 inches. (Note, the actual height of the females does not matter as long as their average is 64 inches. The easiest way to do this is to make them both 64 inches tall.) Now, we just average the heights of these three people.

$$\frac{64+64+70}{3}.$$

We would like to rearrange this a bit before we compute to see a simpler way to perform the computations when there are more numbers.

$$\frac{64+64+70}{3}=\frac{64+64}{3}+\frac{70}{3}=\frac{2\times 64}{3}+\frac{70}{3}=\frac{2}{3}\times 64+\frac{1}{3}\times 70.$$

In this last expression, we are just multiplying the 64 inches by the fraction of the population that has an average of 64 inches, and we are multiplying the 70 inches by the fraction with that average. The average height is

$$average\ height=\frac{2}{3}\times 64+\frac{1}{3}\times 70=66.$$

The average height in this class is 66 inches. This last computation is a weighted average of 64 and 70 where 64 accounts for two-thirds of the average and 70 accounts for one-third. The fractions $\frac{2}{3}$ and $\frac{1}{3}$ are called the weights. To compute a weighted average of a collection of numbers, we simply multiply each number by its weight and add the resulting products.

Problem: In a certain class, test account for 75% of the grade, quizzes for 15%, and homework for 10%. Sue has a test average of 87, a quiz average of 92, and a homework average of 95. What is her grade in the class?

We multiply each score by its weight and add.

$$grade=0.75\times 87+0.15\times 92+0.10\times 95=88.55.$$

Sue's grade is 88.55.

Quartiles and the Five Number Summary

We now shift to trying to summarize how data is spread out. **Quartiles** are three numbers Q_1, Q_2, and Q_3 which separate a data set into four sets which each contain about 25% of the data. About 25% of the data is below Q_1. About 50% of the data is below Q_2, and about 75% of the data is below Q_3. For calculations, Q_2 is the same as the median. Q_1 is the median of the lower half of the data, and Q_3 is the median of the top half of the data.

Problem: Find the three quartiles of this set of data: 12, 13, 15, 20, 24, 29, 39, 48, 49, 50, 67, 71, 75, 78, 87, 98.

Note that the data is already in order. If it were not, then we would have to sort it. We first identify the middle of the data.

12, 13, 15, 20, 24, 29, 39, 48, | 49, 50, 67, 71, 75, 78, 87, 98

$Q_2 = \text{median}$
$= 48.5$

Since there are two numbers in the middle, the median is the average of these two numbers. The second quartile Q_2 is equal to the median, which is 48.5. Now we identify the middle of the lower half of the numbers and the middle of the upper half of the numbers. The medians of these two halves are the first and third quartiles.

12, 13, 15, 20, | 24, 29, 39, 48, | 49, 50, 67, 71, | 75, 78, 87, 98

$Q_1 = \frac{20+24}{2} = 22$

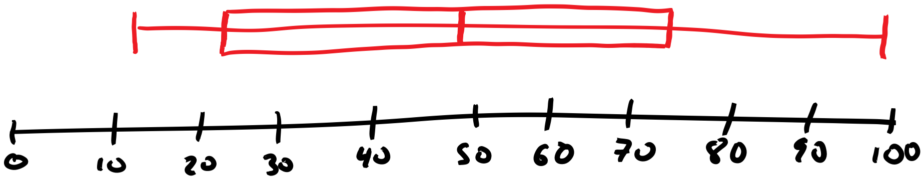
$Q_2 = \text{median} = 48.5$

$Q_3 = \frac{71+75}{2} = 73$

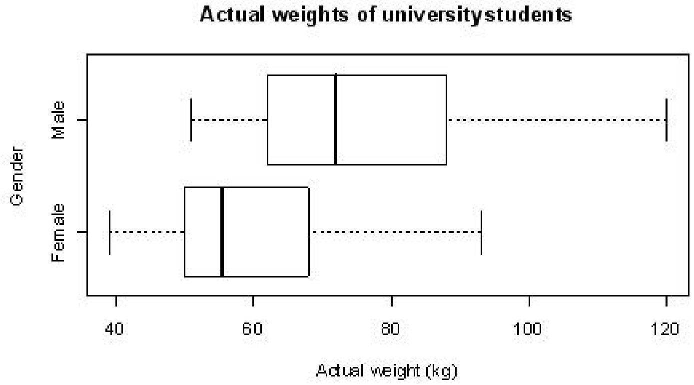
The **five number summary** of a set of data consists of the minimum value, Q_1, Q_2, Q_3, and the maximum value. These five numbers separate the data into five intervals which each contain about 25% of the data. The five number summary of the data in the last problem is

$$12, 22, 48.5, 73, 98$$

One way to graphically display a five number summary is a box plot. This is a diagram above a number line that has vertical lines for each number in the five number summary, a horizontal line from the minimum to the maximum, and horizontal connectors from Q_1 to Q_3. The box plot of the data in the last problem would look like this.



What we can see from this box plot is that the bottom quarter of the data is not very spread out at all compared to the other three quarters. When box plots for two sets of data are drawn side by side, then it is possible to compare the distributions. For example, here are box plots of weights of female and male college students.



We can immediately see that more than half of the males are heavier than three quarters of the females because the median for the males is higher than the third quartile for the females. We can also see that the males weights are a little more spread out than the female weights, and a few females weigh more than the third quartile of the males.

Percentiles

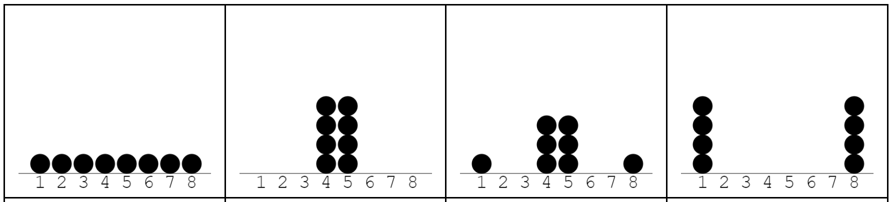
The median separates data values into two groups that are each about 50% of the data. Quartiles are three numbers Q_1, Q_2, and Q_3 that separate data into four groups that are each about 25% of the data. **Percentiles** are 99 numbers $P_1, P_2, \ldots P_{99}$ that separate data into 100 groups that are each about 1% of the data. Percentiles are often used to report student scores on standardized tests. Saying that a student scored at the 73^{rd} percentile means that the student scored the same as or better than about 73% of the students taking the test.

Problem: What is the difference between saying that a student scored 80% on a test and saying the student scored at the 80^{th} percentile. Why might we want one form of scoring over the other?

Saying that a student scored 80% on a test means that the student got about 80% of the questions correct (sort of). Saying that the student scored at the 80^{th} percentile means that the student scored as well as or better than 80% of the students taking the test. We would use a percent score if we are concerned about what percent of the material that a student has mastered and want to identify if that student is progressing or not. We would use a percentile score if we are concerned about how well the student is doing compared to everyone else.

Variation and Relative Standing

Consider the four sets of data represented by these four dot plots.



Data set 1 | Data set 2 | Data set 3 | Data set 4

The means and medians of these data sets are all the same (4.5), but the data sets are very different from each other because of how they are spread out or how much variation they have. Here, we introduce four ways to measure the variation in a data set.

Our first way to measure variation is range. The **range** of a set of data is the difference between the highest data value and the lowest data value. The ranges of these data sets are:

- Data set 1: $high - low = 8 - 1 = 7$
- Data set 2: $high - low = 5 - 4 = 1$
- Data set 3: $high - low = 8 - 1 = 7$
- Data set 4: $high - low = 8 - 1 = 7$

As you can see, the range clearly indicates that data set 2 is less spread out than the other data sets. However, the range does not pick up on the differences between data sets 1, 3, and 4. A positive feature of the range is that it is easy to calculate. A negative feature of the range is that it is extremely sensitive to outliers. Suppose our data are home values in a small town. If one very wealthy person moves to town, they might build a single home that drastically changes the range. Two data sets might be identical except for one data value and might have extremely different ranges.

Our next measure of variation is the interquartile range. The **interquartile range** or **IQR** is the difference between the third quartile and the first quartile. This is the range of the middle half of the data. The idea behind the interquartile range is that it will measure how spread out the middle core of the data is while ignoring the effects of outliers. To find the IQR, we have to first find the quartiles.

- Data set 1: The data values separated into quarters are: 12|34|56|78. The quartiles are numbers that fit where the dividers are: $Q_1 = \frac{2+3}{2} = 2.5$, $Q_2 = \frac{4+5}{2} = 4.5$, and $Q_3 = \frac{6+7}{2} = 6.5$. The IQR is $IQR = Q_3 - Q_1 = 6.5 - 2.5 = 4$.
- Data set 2: The data values separated into quarters are: 44|44|55|55, so the quartiles are $Q_1 = 4$, $Q_2 = 4.5$, and $Q_3 = 5$. The IQR is $IQR = Q_3 - Q_1 = 5 - 4 = 1$.
- Data set 3: The data values are: 14|44|55|58, so the quartiles are $Q_1 = 4$, $Q_2 = 4.5$, and $Q_3 = 5$. The IQR is $IQR = Q_3 - Q_1 = 5 - 4 = 1$.
- Data set 4: The data values are 11|11|88|88, so the quartiles are $Q_1 = 1, Q_2 = 4.5,$ and $Q_3 = 8$. The IQR is $IQR = Q_3 - Q_1 = 8 - 1 = 7$.

The IQR detects that data sets 4 and 1 are more spread out than data sets 2 and 3. However, it (intentionally) does not capture the difference between the highest and lowest data values in data set 3.

Our next measure of variation is the mean absolute deviation. The **mean absolute deviation** or **MAD** is the average distance of data values from the data's mean. To calculate the MAD, we must calculate the data mean, calculate the distance of each data value from the mean, and then average these distances. Here, the distance between two numbers is the absolute value of their difference. Each of the data sets we are looking at has a mean of 4.5. For each data value x, we will calculate $|x - 4.5|$ and then average all of these values.

Data set 1

x	$\lvert x - mean \rvert$	
1	3.5	
2	2.5	
3	1.5	
4	0.5	
5	0.5	
6	1.5	
7	2.5	
8	3.5	
Sum:	16	IQR (average): 2

Data set 2

x	$\lvert x - mean \rvert$	
4	0.5	
4	0.5	
4	0.5	
4	0.5	
5	0.5	
5	0.5	
5	0.5	
5	0.5	
Sum:	4	IQR (average): 0.5

Data set 3

x	$\lvert x - mean \rvert$	
1	3.5	
4	0.5	
4	0.5	
4	0.5	
5	0.5	
5	0.5	
5	0.5	
8	3.5	
Sum:	10	IQR (average): 1.25

Data set 4

x	$\lvert x - mean \rvert$	
1	3.5	
1	3.5	
1	3.5	

1	3.5	
8	3.5	
8	3.5	
8	3.5	
8	3.5	
Sum:	28	IQR (average): 3.5

Our IQRs are

- Data set 1: IQR=2
- Data set 2: IQR=0.5
- Data set 3: IQR=1.25
- Data set 4: IQR=3.5

Notice that the more spread out the data is, the higher the IQR is.

Our final measure of variation is standard deviation, SD. MAD captures the variation of data well, but MAD is not well-behaved mathematically speaking. This means (among other things) that the MAD for samples does not tend to approximate the MAD for populations well. Instead of MAD, statisticians use a related measure called standard deviation. MAD is the average of values of the form $|x - mean|$ where x ranges over the data values. **Standard deviation** is the square root of the average of values of the form $(x - mean)^2$. When we hear "standard deviation" we can think "average distance from the mean" although this is not technically correct. That is MAD. A higher standard deviation means that data is more spread out. The standard deviations of our data sets are:

- Data set 1: $SD = 2.45$
- Data set 2: $SD = 0.53$
- Data set 3: $SD = 1.93$
- Data set 4: $SD = 3.74$

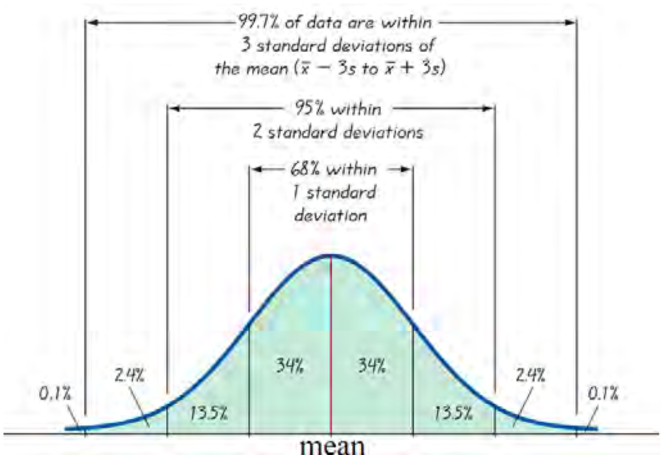
Again, the higher standard deviation goes with the data that is more spread out.

Range Rule of Thumb

The histograms of many types of data closely fit a special, bell-shaped curve called a normal distribution. It follows from a theorem in statistics called the Central Limit Theorem that when enough variables affect data values, those values tend to be close to normally distributed. For example, a person's height is affected by genetics, nutrition, health, and the environment. The combined effects of these influences cause height to be normally distributed. For data that are normally distributed, about 95% of data values are within two standard deviations of the mean. This means that about 95% of data values are between $mean - 2 \times SD$ and $mean + 2 \times SD$. This is known as the Range Rule of Thumb.

Range Rule of Thumb: For normally distributed data, about 95% of data values are within two standard deviations of the mean.

Similarly, about 68% are within one standard deviation of the mean, and about 99.7% are within three standard deviations of the mean. This is summarized in the following diagram.



This knowledge helps to identify a *usual range* of data values for different types of data, as indicated in the next problem.

Problem: Infant birth weights in the United States are normally distributed with a mean of 7.5 pounds and a standard deviation of 1.1 pounds. Find two weights that include the middle 95% of weights of newborn infants in the United States.

First, note that we might want to have this information if we worked in a hospital. Perhaps the lightest and heaviest babies need special attention, so we can use this information to identify the usual range of 95% of the baby weights. The Range Rule of Thumb declares that 95% of the weights will be between these two numbers:

$$mean - 2 \times SD = 7.5 - 2 \times 1.1 = 5.3\ lb$$

and

$$mean + 2 \times SD = 7.5 + \times 1.1 = 9.7\ lb.$$

Therefore, about 95% of newborns weigh between 5.3 pounds and 9.7 pounds.

Problem: Adult males in the United States have heights that are normally distributed with a mean of 69.6 inches and a standard deviation of 3.2 inches. Find two heights which include the middle 95% of adult heights.

We might want a solution to a problem such as this if we are designing a car. Perhaps we want the seat in the car to accommodate 95% of the population. The Range Rule of Thumb declares that 95% of the heights are between

$$mean - 2 \times SD = 69.6 - 2 \times 3.2 = 63.2\ in$$

and

$$mean + 2 \times SD = 69.6 + 2 \times 3.2 = 76\ in.$$

Therefore, 95% of adult males in this country are between 5 feet 3.2 inches tall and 6 feet 4 inches tall.

Z-Scores

The Range Rule of Thumb indicates that the *number of standard deviations from the mean* might be used to determine how unusual data values are and to compare data values. Based on this, we define the z-score of a data value x to be:

$$z = \frac{x - mean}{SD}.$$

We can use z-scores for basic evaluation and comparison of data values:

- A z-score near 0 indicates that a data value is close to the mean.
- The farther from 0 a z-score is, the more unusual or exceptional that data value is.
- A z-score above 0 indicates a data value above average.
- A z-score below 0 indicates a data value below average.
- According to the Range Rule of Thumb, for normal distributions, about 95% of z-scores should be between -2 and 2. Any data value with a z-score less than -2 or greater than 2 should be considered exceptional or unusual.

Problem: Sierra is six feet tall and has an IQ of 135. Which is more exceptional, her height or her IQ? Female heights are normally distributed with a mean of 64 inches and a standard deviation of 2.5 inches. IQs are normally distributed with a mean of 100 and a standard deviation of 15.

Here, we are basically asking if Sierra is smarter than she is tall. The concept of z-scores allows us to compare things that seem incomparable. We first calculate z-scores for Sierra's height and her IQ. Note that six feet is 72 inches.

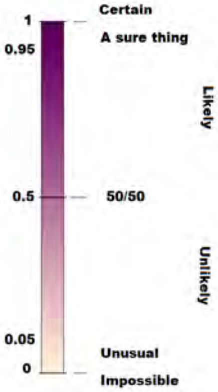
$$z - score\ for\ height = \frac{height - mean}{SD} = \frac{72 - 64}{2.5} = 3.2$$

$$z - score\ for\ IQ = \frac{IQ - mean}{SD} = \frac{135 - 100}{15} = 2.3.$$

Note that both z-scores are positive since both values are larger than their respective means. Since Sierra's z-score for height is farther from 0, her height is more exceptional than her IQ. Sierra is taller than she is smart.

Probability

The **probability** of an event is a measure of the likelihood of the event. Probabilities can be as small as 0 or as large as 1. An event with probability 0 is considered impossible (though, oddly enough, it may occur), and an event with probability 1 is considered certain (though, again oddly enough, such events sometimes do not occur). Events with probabilities above 0.5 are considered **likely**, and events with probabilities below 0.5 are considered **unlikely**.



An **event** is an outcome of a *repeatable experiment*. For example, an experiment might be rolling a six-sided die (singular of dice). An outcome of this experiment might be rolling a 3. Another outcome might be rolling an even number. Another experiment is flipping two coins. One event is flipping two heads. Another is flipping at least one head. Notice that we only discuss outcomes of *repeatable experiments*. It makes no sense to talk about the probability that life exists on planets other than Earth. Either it does exist, or it does not. There is no repeatable experiment here. Similarly, it makes no sense to talk about the probability that God exists. Either God exists, or he does not. There is no repeatable experiment involved.

A **simple event** is a single outcome of an experiment that cannot be broken down into smaller outcomes. For example, in the experiment of rolling a die, the simple events are rolling 1, 2, 3, 4, 5, or 6. The set of all simple events in an experiment is called the **sample space** of the experiment. The sample space of the experiment of rolling a die is $\{1, 2, 3, 4, 5, 6\}$. Mathematicians like to list sample spaces within braces to indicate that the sample space is a *set*. A **compound event** is an event that is made up of simple events. For the experiment of rolling a die, one compound event is rolling an even number. This event consists of the events 2, 4, and 6.

Problem: Find the sample space for the experiment of flipping two coins. Also list at least three different compound events.

To find the sample space means to list all simple events. We will abbreviate "heads" as H and "tails" as T, and we will use ordered pairs of Hs and Ts to indicate an outcome. For example, HT means the first flip was an H while the second was a T. All simple events are

$$\{HH, HT, TH, TT\}.$$

Some compound events related to this experiment are:

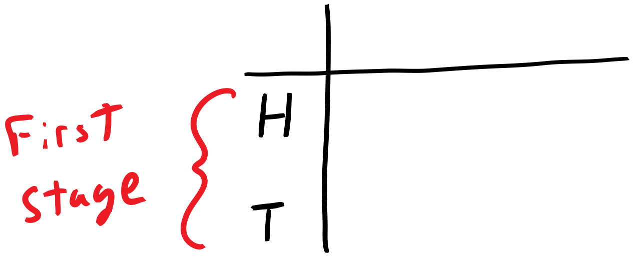
- Flipping at least one H (which can happen three ways).

- Flipping at least one T (which can also happen three ways).
- Flipping the same side twice (which can happen two ways).

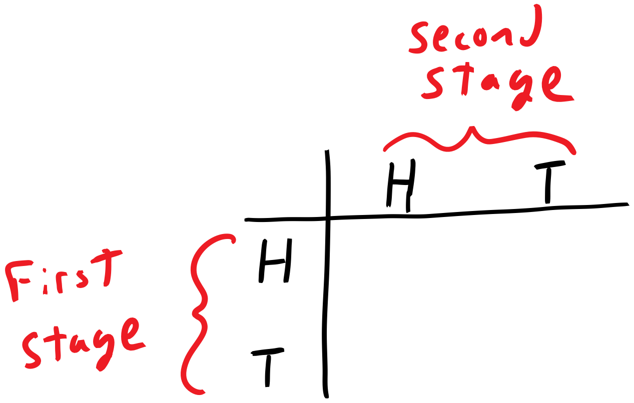
Arrays and Trees

The experiment in the last problem actually involved two actions – flipping the first coin and flipping the second coin. There are two common ways of listing the outcomes of two-stage experiments such as this, arrays and trees.

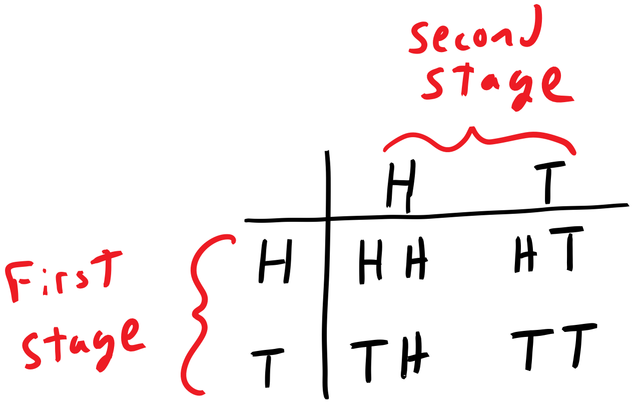
When using an array to list the outcomes of an experiment, we label the rows of an array with the outcomes of the first experiment.



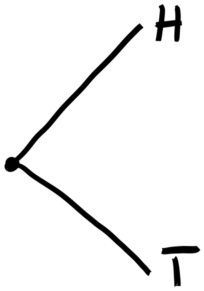
We label the columns of the array with the outcomes of the second experiment.



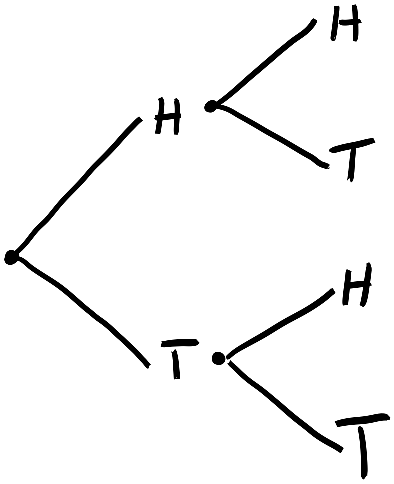
Then we fill in each entry of the array with an ordered pair. The first element of the pair is the label of the row, and the second element of the pair is the label of the column. This is just like an addition table or multiplication table.



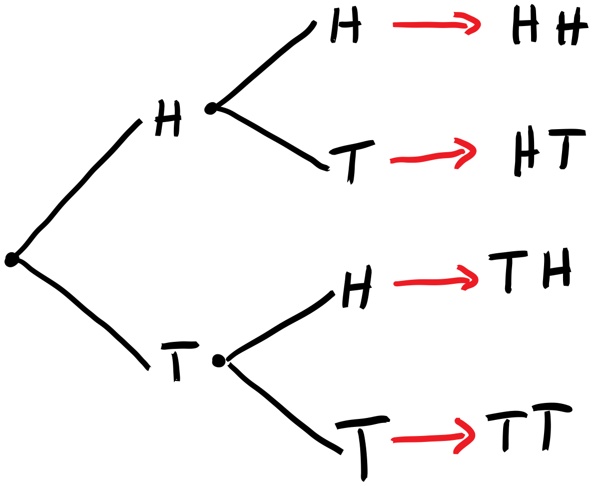
We can also use a picture called a tree to list the outcomes of a two-stage experiment. We first list the outcomes of the first stage at the end of "branches" coming off of a single point.



The single point is the root of the tree. For each outcome of the first stage, we draw branches labeled with the outcomes of the second stage.



These outcomes are located at "leaves" of the tree. We can now list all outcomes of the experiment by following branches from the root of the tree to the leaves.



Arrays are often easier to draw for two-stage experiments, and they are adaptable to certain conditions we will add to experiments later. However, trees are useful if experiments involve more than two stages. To add a third stage, we can simply branch off of every outcome at the second stage.

Empirical Approach to Probability

We use capital letters such as A, B, C ... to name probabilities. The probability of an event A is denoted as $P(A)$. Suppose that A is a possible outcome of an experiment. Also suppose that we repeat the experiment a number of times and count how often the outcome A occurs. Then the empirical approximation or relative frequency approximation of the probability of A is

$$P(A) \approx \frac{number\ times\ A\ occurred}{number\ of\ times\ experiment\ performed}.$$

Problem: A tack was flipped 200 times and landed pointing up 127 times. Use this information to approximate the probability that this tack lands point up when it is flipped.

The relative frequency or empirical approximation of the probability that the tack lands pointing up is

$$P(tack\ lands\ pointing\ up) \approx \frac{number\ times\ tack\ landed\ pointing\ up}{number\ of\ times\ tack\ flipped} = \frac{127}{200} = 0.635.$$

Whenever we calculate a probability, we should pause to interpret it. A probability of 0.635 means that landing up is likely, but it is definitely not a sure thing.

A natural question to ask is how good of an approximation this is. Luckily for us, statistics provides tools to determine how good the approximation probably is. Here is a table that compares the number of times an experiment is repeated with the likely worst-case-scenario margin of error of a relative frequency approximation.

Number of Repetitions	Likely Margin of Error (Worst Case)
100	10%
200	7%
400	5%
600	4%
1100	3%

What this means is that if we repeat an experiment 100 times to approximate a probability, then that approximation will be within 10% of the actual probability 95% of the time. In our case, the margin of error is (at worst) 7% or 0.07. This means the actual probability this tack lands pointing up is probably between .565 and .705.

Notice that the more times an experiment is repeated, the smaller the margin of error is likely to be. This is something known as the Law of Large Numbers

Law of Large Numbers: If an experiment is performed a large number of times, then an empirical approximation of a probability based on that experiment should be close to the actual probability. As the experiment is performed more and more times, the empirical approximations will tend to become closer and closer to the actual probability.

Probability and Simple Events

If we can list all simple events in an experiment, and if all of the simple events are equally likely, then it is easy to calculate the actual probability of an event. If all simple events are equally likely, then the probability of an event A is

$$P(A) = \frac{number\ of\ ways\ that\ A\ can\ occur}{number\ of\ simple\ events}.$$

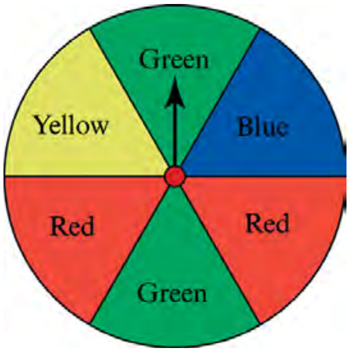
Problem: Two coins are flipped. What is the probability that at least one of the coins lands on H?

We listed all simple events in this experiment earlier. There are four of them, $\{HH, HT, TH, TT\}$. There are three ways in which at least one coin lands on H, {HH, HT, TH}. Therefore, the probability of at least one H is

$$P(at\ least\ one\ H) = \frac{number\ of\ ways\ to\ get\ at\ least\ one\ H}{number\ of\ outcomes} = \frac{3}{4}.$$

Therefore, it is likely that when two coins are flipped at least one of them lands on H.

Problem: The spinner below is spun. What is the probability that the color spun is green?



There are six "slices" to this spinner. These are our simple events, and they all seem equally likely. Two of the slices are green, so the probability of green is

$$P(green) = \frac{number\ of\ green\ slices}{number\ of\ slices} = \frac{2}{6} = \frac{1}{3}.$$

Oddly enough, it is unlikely that this spinner lands on green with our definition of unlikely. However, no other color is more likely.

Problem: The spinner above is spun. What is the probability that the color spun is not green?

We can solve this again by counting.

$$P(not\ green) = \frac{number\ of\ slices\ that\ are\ not\ green}{number\ of\ slices} = \frac{4}{6} = \frac{2}{3}.$$

A probability of $\frac{2}{3}$ falls into the category of likely. The events in the last problem, "green" and "not green" are called complements. Note that their probabilities add to 1. This always happens. The complement of an event A is the event that A does not occur (or $not\ A$). It happens to be that

$$P(A) + P(not\ A) = 1$$

or

$$P(not\ A) = 1 - P(A).$$

Problem: Two dice are rolled. What is the probability that the sum of the two dice is 6?

We use an array to list every outcome when two dice are rolled.

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

An entry here such as 2,3 means that the first die was a 2 while the second was a 3. We now add the rolls of the dice.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

We have shaded those rolls which sum to 6. There are 5 of them, and there are 36 total outcomes of this experiment, so the probability the sum is 6 is

$$P(sum = 6) = \frac{5}{36} = 0.139.$$

Multistage Experiments and Counting

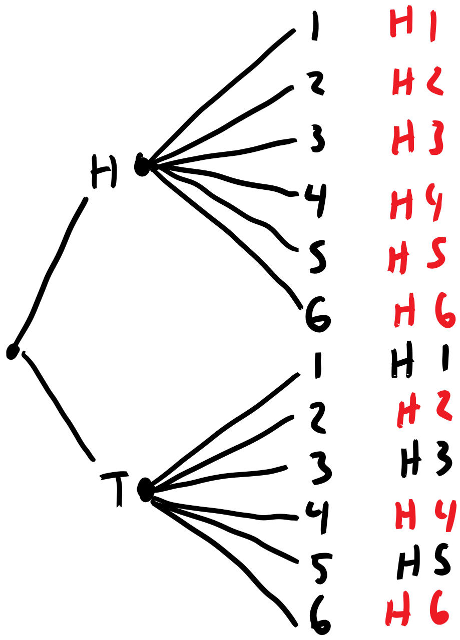
Many experiments consist of two smaller experiments or stages.

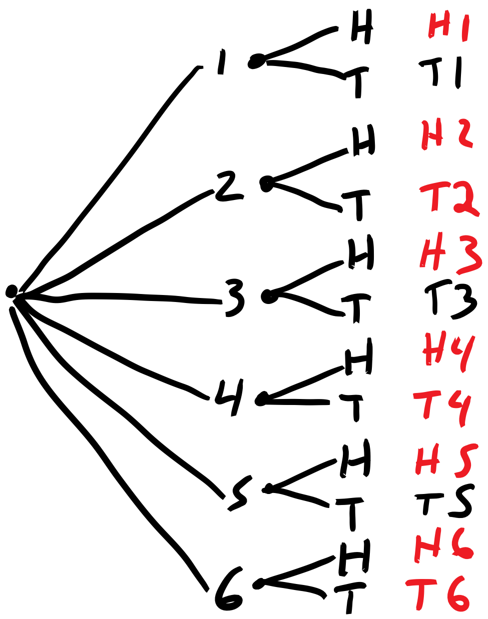
Problem: An experiment consists of flipping a coin and rolling a die. What is the probability that the coin lands on H or the die roll is even?

We start by listing all of the possible outcomes. We could use an array.

	1	2	3	4	5	6
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T3	T4	T5	T6

Here, we have labeled the rows with the coin flip and the columns with the dice roll. We have also highlighted those outcomes which include an H or an even number. We could also use a tree. There are two ways to draw the tree. We can branch for the die first or for the coin first.





Whichever way we choose to draw the outcomes, there are twelve total outcomes, and we have an H or an even number in nine of them. Therefore

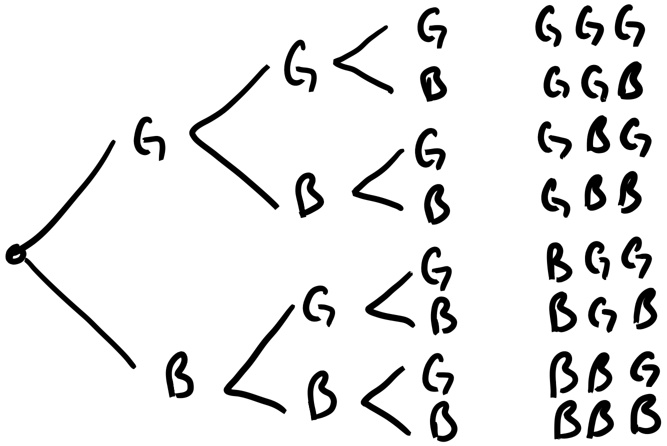
$$P(H\ or\ even) = \frac{9}{12} = \frac{3}{4}.$$

Problem: A family has three children. Find the probability of each of these events.

1. There is at least one girl.
2. There are no girls.

3. The children are all girls.
4. The second child is a girl.

We start by listing all of the possible outcomes. We will use G to represent girl and B to represent boy. An array does not help here since arrays can only accommodate two stage experiments (unless you are good at drawing three dimensional arrays). Therefore, we use a tree.



The outcomes are {GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB}. That gives 8 possible outcomes, so all of our probabilities will be fractions with denominator 8.

1. The events in which there is at least one G are {GGG, GGB, GBG, GBB, BGG, BGB, BBG}. There are 7 events here, so the probability of at least one girl is $\frac{7}{8}$.
2. There is only one event in which there are no girls, BBB, so the probability of no girls is $\frac{1}{8}$.
3. There is only one event in which all of the children are girls, GGG, so the probability of all girls is $\frac{1}{8}$.
4. The events in which the second child is a girl are {GGG, GGB, BGG, BGB}. There are four events here, so the probability that the second child is a girl is $\frac{4}{8} = \frac{1}{2}$.

Problem: A bowl contains two red marbles and three green marbles. A marble is selected randomly from the bowl. Its color is noted, and it is put back. Then a second marble is selected from the bowl. Its color is noted. Find the probability of each of these events.

1. At least one marble is red.
2. The marbles are the same color.
3. The marbles are different colors.

Selection such as this, when we put the first marble back in the bowl, is called selection **with replacement**. We start by using an array to list the outcomes. This is the way we should draw the array. Notice how we use subscripts to indicate which red and which green marble are chosen.

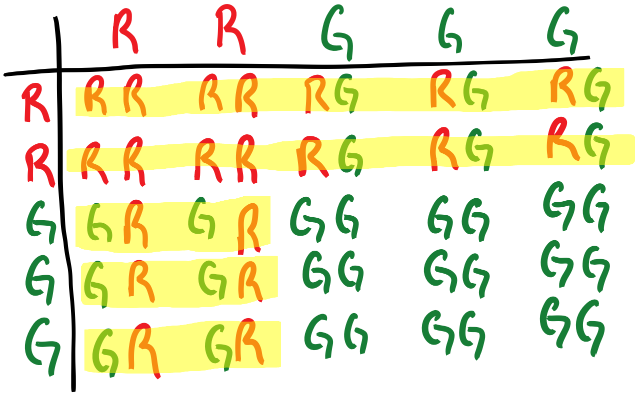
	R_1	R_2	G_1	G_2	G_3
R_1	R_1R_1	R_1R_2	R_1G_1	R_1G_2	R_1G_3
R_2	R_2R_1	R_2R_2	R_2G_1	R_2G_2	R_2G_3
G_1	G_1R_1	G_1R_2	G_1G_1	G_1G_2	G_1G_3
G_2	G_2R_1	G_2R_2	G_2G_1	G_2G_2	G_2G_3
G_3	G_3R_1	G_3R_2	G_3G_1	G_3G_2	G_3G_3

If we are careful, we can also draw the array this way, avoiding the subscripts. We have to be careful about which marble is being drawn though.

	R	R	G	G	G
R	RR	RR	RG	RG	RG
R	RR	RR	RG	RG	RG
G	GR	GR	GG	GG	GG
G	GR	GR	GG	GG	GG
G	GR	GR	GG	GG	GG

Note that there are 25 outcomes here. We can now address our probabilities

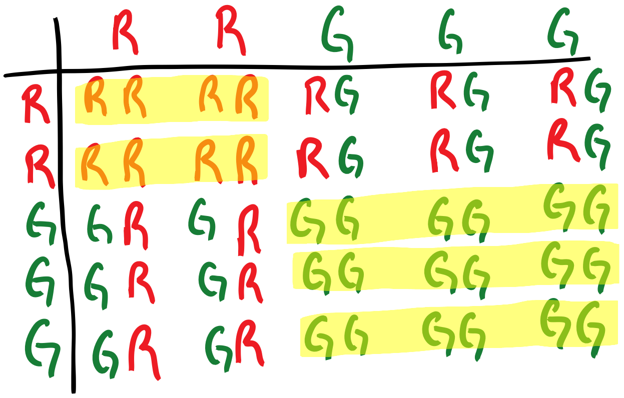
1. First, we locate all of the outcomes in which there is at least one R.



	R	R	G	G	G
R	RR	RR	RG	RG	RG
R	RR	RR	RG	RG	RG
G	GR	GR	GG	GG	GG
G	GR	GR	GG	GG	GG
G	GR	GR	GG	GG	GG

   There are 16 outcomes in which at least one marble is red, so the probability of at least one red is $\frac{16}{25}$.

2. Here, we begin by highlighting those outcomes where the marbles are the same color.



	R	R	G	G	G
R	RR	RR	RG	RG	RG
R	RR	RR	RG	RG	RG
G	GR	GR	GG	GG	GG
G	GR	GR	GG	GG	GG
G	GR	GR	GG	GG	GG

There are 13 outcomes in which the marbles are the same color, so the probability that the marbles are the same color is $\frac{13}{25}$.

3. The outcomes in which the marbles are different colors are exactly those outcomes in the last question which are not highlighted. There are 12 outcomes, so the probability that the marbles are different colors is $\frac{12}{25}$.

Note that on number 3 would could have used complements:

$$P(different\ colors) = P(not\ same\ color) = 1 - P(same\ color) = 1 - \frac{13}{25} = \frac{12}{25}.$$

Problem: A bowl contains two red marbles and three green marbles. A marble is selected randomly from the bowl. Its color is noted, and it is **not** put back. Then a second marble is selected from the bowl. Its color is noted. Find the probability of each of these events.

1. At least one marble is red.
2. The marbles are the same color.
3. The marbles are different colors.

Selection such as this, when the first marble is not returned to the bowl is called selection **without replacement**. When we do not put the first marble back, then the state of the bowl is different when the second marble is chosen. This affects our probabilities. For the approach we use here, it affects how we draw our array. We can use the array that we drew above for the outcomes of this experiment. However, since we do not put the first marble back, we cannot draw the same marble twice. This means we cannot have outcomes such as R_1R_1. All such outcomes are on the diagonal of our array, so we simple delete the diagonal and proceed as in the last problem. This array contains our outcomes.

	R	R	G	G	G
R	[redacted]	RR	RG	RG	RG
R	RR	[redacted]	RG	RG	RG
G	GR	GR	[redacted]	GG	GG
G	GR	GR	GG	[redacted]	GG
G	GR	GR	GG	GG	[redacted]

Note that there are now 20 outcomes.

1. We highlight those outcomes in which at least one marble is red.

	R	R	G	G	G
R	[redacted]	RR	RG	RG	RG
R	RR	[redacted]	RG	RG	RG
G	GR	GR	[redacted]	GG	GG
G	GR	GR	GG	[redacted]	GG
G	GR	GR	GG	GG	[redacted]

   There are 14 highlighted outcomes, so the probability of at least one red marble is now $\frac{14}{20} = \frac{7}{10}$.

2. For the probability that the marbles are the same color, we mark all those outcomes of the form RR or GG.

	R	R	G	G	G
R	[redacted]	RR	RG	RG	RG
R	RR	[redacted]	RG	RG	RG
G	GR	GR	[redacted]	GG	GG
G	GR	GR	GG	[redacted]	GG
G	GR	GR	GG	GG	[redacted]

   There are 8 outcomes marked, so the probability that the marbles are the same color is $\frac{8}{20} = \frac{2}{5}$.

3. For the probability the marbles are different colors, we note that there are 12 outcomes in the last question not marked. These are exactly those outcomes in which the marbles are different colors, so the probability that the marbles are different colors is $\frac{12}{20} = \frac{3}{5}$.

Counting

Many, perhaps most, computations in probability reduce to counting. The number of outcomes of multistage experiments are often easy to count using the Fundamental Counting Principle.

Fundamental Counting Principle: If experiment A can end in N different outcomes and experiment B can end in M different outcomes then the performing A and then B can end in $N \times M$ outcomes.

The Fundamental Counting Principle can be seen to be true by considering an array. We can label the rows by the N outcomes of A and the columns by the M outcomes of B. This leads to N rows of M entries, or $N \times M$ outcomes.

Problem: A spinner with 8 colors is spun, and then a six-sided die is rolled. What are the total number of outcomes?

The spinner can end in 8 outcomes. The dice can end with 6 outcomes. The combined experiment can end in $8 \times 6 = 48$ outcomes.

Problem: Two letters are randomly chosen in order. How many possibilities are there? Assume that the same letter can be chosen twice.

There are 26 choices for the first letter and 26 choices for the second letter, so the total number of outcomes is $26 \times 26 = 676$.

Problem: Two letters are randomly chosen in order. How many possibilities are there? Assume that the same letter **cannot** be chosen twice.

There are 26 choices for the first letter. Once that letter has been chosen, there are only 25 choices left for the second letter. The total number of outcomes is $26 \times 25 = 650$.

Problem: If your pin number is 4 digits (which may include repeats), then what is the probability that someone randomly guesses your pin number?

There are ten digits (0, 1 2, 3, 4, 5, 6, 7, 8, 9). This means that there are 10 choices for each digit. We can extend the Fundamental Counting Principle to four experiments, and the total number of pin numbers is $10 \times 10 \times 10 \times 10 = 10000$. You have one pin number, so the probability of guessing it is $\frac{1}{10000}$.

Complements

The **complement** of an event A is the event that A does not occur. We have seen above that

$$P(complement\ of\ A) + P(A) = 1$$

so

$$P(complement\ of\ A) = 1 - P(A).$$

This formula can be useful for calculating probabilities involving “at least one” because the complement of “at least one” is none.

Problem: A family has five children. What is the probability that they have at least one girl?

The complement of having at least one girl is having no girls, which means having all boys. We can easily calculate the probability of having all boys. There is only one way to have all boys (they are all boys). We simply need to know how many possible outcomes there are for the genders of five children. There are two outcomes for the first child (male or female), and two for the second, and two for the third, and two for the fourth, and two for the fifth. To find out how many total outcomes there are, we simply multiply these twos together.

$$number\ of\ possible\ combinations\ of\ genders\ of\ five\ children = 2 \times 2 \times 2 \times 2 \times 2 = 32.$$

The probability that all five children are boys is $\frac{1}{32}$. Therefore, using complements, we can find

$$\begin{aligned} P(at\ least\ one\ girl) &= 1 - P(no\ girls) \\ &= 1 - P(all\ boys) \\ &= 1 - \frac{1}{32} \\ &= \frac{31}{32} \\ &= .96875. \end{aligned}$$

It is very likely that a family of five children will have at least one girl.

Independent Events and the Multiplication Rule

Two events are **independent** if the occurrence of one event does not affect the occurrence of the other. If the occurrence of one event does affect the probability of the occurrence of the other, then the events are **dependent**. When drawing multiple marbles from a bowl like we did earlier, if we use selection *with replacement*, then the colors of the marbles are *independent*. The bowl is exactly the same prior to each selection, so the probabilities do not change. If we use selection *without replacement*, then the colors of the marbles are *dependent*. Removing the first marble changes the numbers of marbles in the bowl and changes the probabilities for the second marble.

If events A and B are independent, then the probability that A and B both occur is the product of the probabilities of A and B. You can (sort of) compare this to drawing an array of outcomes. If the rows and columns are independent of each other, the number of outcomes is the product of the number of rows and columns. If the rows and columns are not independent, some of the outcomes might be missing (or crossed out) from the array.

Multiplication Rule (Simple Form): If A and B are independent events, then

$$P(A\ and\ B) = P(A) \times P(B).$$

Problem: Suppose that the probability that a certain type of battery operated alarm clock works on a given morning is 0.8. Bob has two of these clocks. What is the probability that they both fail on a given morning? What is the probability that at least one works?

First, failing is the complement of working, so the probability of one of these clocks failing should be

$$P(fails) = 1 - P(works) = 1 - 0.8 = 0.2$$

The reason we are looking at battery operated clocks is that (hopefully) whether or not one works is independent of whether or not the other works. Therefore

$$\begin{aligned} P(both\ fail) &= P(first\ fails\ and\ second\ fails) \\ &= P(first\ fails) \times P(second\ fails) \\ &= 0.2 \times 0.2 \\ &= 0.04 \end{aligned}$$

Here, we used the multiplication rule at the second equal sign. Now, for the probability that at least one works, we use complements:

$$\begin{aligned} P(at\ least\ one\ works) &= 1 - P(both\ fail) \\ &= 1 - 0.04 \\ &= 0.94 \end{aligned}$$

Notice how two not-too-reliable clocks can be used together to get a pretty reliable system.

The multiplication rule can be extended to event that are not independent.

Multiplication Rule: If A and B are events, then $P(A\ and\ B) = P(A) \times P(B|A)$ where $P(B|A)$ is the probability that B occurs if we assume that A has already occurred.

Problem: A bowl contains two red marbles and three green marbles. A marble is selected randomly from the bowl. Its color is noted, and it is **not** put back. What is the probability that both marbles are red?

This is the same bowl of marbles from earlier, so we could simply go back to the array for this bowl of marbles. However, we can work the problem without drawing the array using the multiplication rule.

$$\begin{aligned} P(both\ red) &= P(first\ red\ and\ second\ red) \\ &= P(first\ red) \times P(second\ red\ assuming\ the\ first\ was\ red) \end{aligned}$$

Now, at the beginning, there are 2 red marbles and 3 green marbles, so the probability that the first is red is

$$P(first\ red) = \frac{3}{5}.$$

After one red marble is taken out, there is one red marble left and 3 green marbles. This means that the probability the second is red assuming the first is red is

$$P(second\ red\ assuming\ the\ first\ was\ red) = \frac{1}{4}.$$

So

$$\begin{aligned} P(both\ red) &= P(first\ red\ and\ second\ red) \\ &= P(first\ red) \times P(second\ red\ assuming\ the\ first\ was\ red) \\ &= \frac{3}{5} \times \frac{1}{4} \\ &= \frac{3}{20}. \end{aligned}$$