Geometry Theorem Sequence

Fall 2026

October 7, 2025

Contents

1 Incidence Geometry

1.1 Incidence

Primitives. The undefined terms for incidence geometry are *point* and *line*.

Axiom. Lines are sets of points.

Definition. If P is a point and ℓ is a line then each of these means that $P \in \ell$:

- P is on ℓ.
- P lies on ℓ.
- ℓ is on P.
- ℓ lies on P.
- ℓ passes through P.
- P is incident with ℓ.
- ℓ is incident with P.
- ℓ contains P.

Definition. We call the set of all points under consideration *the plane* and denote it $\mathbb{P}$.

Incidence Axiom 1 (IA1)**.** *If P and Q are distinct points, then there is at least one line ℓ so that $P \in \ell$ and $Q \in \ell$.*

Incidence Axiom 2 (IA2)**.** *If P and Q are distinct points, then there is at most one line ℓ so that $P \in \ell$ and $Q \in \ell$.*

Incidence Axiom 3 (IA3)**.** *If ℓ is a line, then there are distinct points P and Q so that $P \in \ell$ and $Q \in \ell$.*

Incidence Axiom 4 (IA4)**.** *There exist three distinct points P, Q, and R so that there is no line ℓ with $P \in \ell$ and $Q \in \ell$ and $R \in \ell$.*

Incidence Axioms 1 and 2 guarantee that any two distinct points are contained in exactly one line. We give a name to this line.

Definition. If P and Q are distinct points then the unique line that contains both P and Q is denoted $\overleftrightarrow{PQ}$.

Suppose that P and Q are distinct points. It follows immediately from this definition that:

- If ℓ is a line that contains P and Q, then $\ell = \overleftrightarrow{PQ}$.
- $\overleftrightarrow{QP} = \overleftrightarrow{PQ}$.
- If ℓ is a line and $P \in \ell$ and $Q \in \ell$, then $\ell = \overleftrightarrow{PQ}$.

Exercise 1.1. Find a model which satisfies Incidence Axioms 1, 2, and 3 but not 4.

Exercise 1.2. Find a model which satisfies Incidence Axioms 1, 2, and 4 but not 3.

Exercise 1.3. Find a model which satisfies Incidence Axioms 1, 3, and 4 but not 2.

Exercise 1.4. Find a model which satisfies Incidence Axioms 2, 3, and 4 but not 1.

These three exercises establish that the incidence axioms are *independent* of each other. No one of them can be proven from the other three. If one of them were provable from the other three, then any model which satisfies the three would also have to satisfy the one.

Exercise 1.5. Find a model of IA1, IA2, IA3, and IA4 which has three or fewer points.

It follows from this exercise that you cannot prove the existence of four points. If you could, the model you found could not exist.

Proposition 1.6. *The following are true.*

1. *There is a point.*
2. *There is a line.*

Proposition 1.7. *If ℓ is a line, then there exist points P and Q so that $\ell = \overleftrightarrow{PQ}$.*

Proposition 1.8. *If ℓ and m are distinct lines, then there is at most one point P which is on both ℓ and m.*

Proposition 1.9 (IA4'). *There is a line ℓ and a point P so that $P \notin \ell$.*

Exercise 1.10. Assume IA1, IA2, IA3, and IA4' and use these assumptions to prove $IA4$.

This last exercise demonstrates that in the presence of the other Incidence Axioms, IA4 and IA4' are *equivalent.*

Definition. Points $P_1, P_2, \ldots, P_n$ are *collinear* if there is a line ℓ so that $P_i \in \ell$ for all i. Points which are not collinear are *noncollinear.* Lines $\ell_1, \ell_2, \ldots, \ell_n$ are *concurrent* if there is a point P so that $P \in \ell_i$ for all i. Lines which are not concurrent are *nonconcurrent.*

Proposition 1.11. *There are three noncollinear points.*

Proposition 1.12. *There are three nonconcurrent lines.*

Proposition 1.13. *If ℓ is any line, then there exists at least one point P so that $P \notin \ell$.*

Proposition 1.14. *If P is any point, then there exist at least two lines ℓ and m so that $P \in \ell$ and $P \in m$.*

Definition. If ℓ and m are lines, and if P is a point so that $P \in \ell$ and $P \in m$, then ℓ and m *intersect* at P.

Proposition 1.15. *If ℓ is any line, then there exist lines m and n so that l, m, and n are distinct and both m and n intersect ℓ.*

Proposition 1.16. *Suppose that A, B, and C are distinct points and that P is a point so that $P \in \overleftrightarrow{AB}$, $P \in \overleftrightarrow{AC}$, and $P \in \overleftrightarrow{BC}$. Then A, B, and C are collinear.*

Proposition 1.17. *If P is any point, then there exists at least one line ℓ such that $P \notin \ell$.*

Proposition 1.18. *If P is any point, then there exist points Q and R so that P, Q, and R are noncollinear.*

Proposition 1.19. *If P and Q are two distinct points, then there exists a point R so that P, Q, and R are noncollinear.*

2 Incidence and Betweenness Geometry

2.1 Betweenness

Primitives. We add a new undefined term, *betweenness.*

Axiom. Betweenness is a ternary (three-place) relation on the set of all points. To denote that points P, Q, and R are in the betweenness relation, we use the notation $P * Q * R$.

Definition. If P, Q, and R are points and the relation $P * Q * R$ holds, we say that Q is *between* P and R.

Betweenness Axiom 1 (BA1)**.** *If A, B, and C are points so that $A * B * C$, then A, B, and C are distinct collinear points and $C * B * A$.*

Betweenness Axiom 2 (BA2)**.** *If B and D are distinct points, then there are points A, C, and E so that $A * B * D$, $B * C * D$, and $B * D * E$.*

Betweenness Axiom 3 (BA3)**.** *If A, B, and C are distinct collinear points, then exactly one of these holds: $A * B * C$ or $A * C * B$ or $B * A * C$.*

Definition. Suppose that A and B are distinct points. The *line segment* $\overline{AB}$ is defined as

$$\overline{AB} = \{P | A * P * B\} \cup \{A, B\}.$$

Definition. If $P = Q$ is any point, then $\overleftrightarrow{PQ} = \{P\} = \{Q\}$. Note that this IS NOT a line. Similarly, if $P = Q$ is any point, then $\overline{PQ} = \{P\} = \{Q\}$. Note that this IS NOT a line segment.

Definition. Suppose that A and B are distinct points. The *ray* $\overrightarrow{AB}$ is defined as

$$\overrightarrow{AB} = \overline{AB} \cup \{P | A * B * P\}.$$

Proposition 2.1. *If A, B, and P are points so that $P * A * B$ or $A * B * P$ then P is not in $\overline{AB}$.*

Proposition 2.2. *If A, B, and P are points so that $P * A * B$ then P is not in $\overrightarrow{AB}$.*

Proposition 2.3. *If A and B are distinct points, then $\overline{AB} = \overline{BA}$.*

Proposition 2.4. *If A and B are distinct points, then $\overrightarrow{AB} \neq \overrightarrow{BA}$.*

Proposition 2.5. *If A and B are distinct points, then $\overline{AB} \subseteq \overleftrightarrow{AB}$.*

Proposition 2.6. *If A and B are distinct points, then $\overline{AB} \subseteq \overrightarrow{AB}$.*

Proposition 2.7. *If A and B are distinct points, then $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$.*

Proposition 2.8. *If A and B are distinct points then $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$.*

Proposition 2.9. *If A and B are distinct points then $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.*

2.2 Plane Separation

Primitives. We add a new undefined term, *half-plane*.

Definition. A set of points S is *convex* if whenever A and C are in S and $A * B * C$ then B is in S also.

Plane Separation Axiom 1. *Suppose ℓ is a line. There are two sets H_1 and H_2 called the* half-planes bounded by ℓ *so that*

1. $\mathbb{P} = H_1 \cup \ell \cup H_2$.
2. $H_1 \neq \emptyset$ *and* $H_2 \neq \emptyset$.
3. $H_1 \cap H_2 = H_1 \cap \ell = H_2 \cap \ell = \emptyset$.
4. H_1 *and* H_2 *are convex.*
5. *If* $P \in H_1$ *and* $Q \in H_2$ *then* $\overline{PQ} \cap \ell \neq \emptyset$.

Proposition 2.10. *If ℓ is a line and A is a point not on ℓ, then there is exactly one half-plane bounded by ℓ which contains A.*

Definition. If ℓ is a line and A is a point not on ℓ, then the half-plane bounded by ℓ containing A is denoted $H_\ell(A)$. The two half-planes bounded by ℓ are called the *sides* of ℓ. If H is a half-plane bounded by ℓ, we will sometimes express $A \in H$ as "A is *in* H" and sometimes as "A is *on* H." We will call the two distinct sides of ℓ the *opposite sides* of ℓ. If A and B are two points not on ℓ and $H_\ell(A) = H_\ell(B)$, then we say that A and B are *on the same side* of ℓ. If $H_\ell(A) \neq H_\ell(B)$, then we say that A and B are *on opposite sides* of ℓ.

Proposition 2.11. *If A and B are distinct points not on a line ℓ, then A and B are on the same side of ℓ if and only if $\overline{AB} \cap \ell = \emptyset$. A and B are on opposite sides of ℓ if and only if $\overline{AB} \cap \ell \neq \emptyset$.*

Theorem 2.12 (Pasch's Theorem)**.** *Suppose that A, B, and C are distinct noncollinear points and that ℓ is a line intersecting $\overline{AB}$.*

- *ℓ intersects $\overline{AC}$ or $\overline{BC}$.*
- *If ℓ does not contain A, B, or C, then ℓ intersects exactly one of $\overline{AC}$ and $\overline{BC}$.*

This theorem is proven through the following claimable lemmas.

Lemma 2.13 (Part of the proof of 2.12)**.** *Suppose that A, B, and C are distinct noncollinear points and that ℓ is a line intersecting $\overline{AB}$. Then ℓ intersects $\overline{AC}$ or $\overline{BC}$.*

Lemma 2.14 (Part of the proof of 2.12)**.** *Suppose that A, B, and C are distinct noncollinear points and that ℓ is a line intersecting $\overline{AB}$. If ℓ does not contain A, B, or C, then ℓ intersects exactly one of $\overline{AC}$ and $\overline{BC}$.*

Proposition 2.15. *Suppose A, B, and C are points not on a line ℓ. If A and B are on opposite sides of ℓ but B and C are on the same side of ℓ, then A and C on opposite sides of ℓ.*

Proposition 2.16 (Ray Theorem)**.** *Suppose that A is a point on a line ℓ and that B is a point not on ℓ. Every point on $\overrightarrow{AB}$ other than A is on the same side of ℓ as B.*

2.3 Betweenness of Four Points

Definition. Let A, B, C, and D be points. The relation $A - B - C - D$ holds if $A * B * C$, $A * B * D$, $A * C * D$, and $B * C * D$.

Proposition 2.17. *Let A, B, C, and D be points. If $A - B - C - D$, then A, B, C, and D are distinct collinear points.*

Proposition 2.18. *Let A, B, C, and D be points. If $A - B - C - D$, then $D - C - B - A$.*

Proposition 2.19 (Four Point Squeezing 1)**.** *Let A, B, C, and D be points. If $A * B * C$ and $A * C * D$ then $A - B - C - D$.*

Proposition 2.20 (Four Point Squeezing 2)**.** *Let A, B, C, and D be points. If $A * B * D$ and $B * C * D$ then $A - B - C - D$.*

Proposition 2.21 (Four Point Overlap)**.** *Let A, B, C, and D be points. If $A * B * C$ and $B * C * D$ then $A - B - C - D$.*

Proposition 2.22. *Let A, B, C, and D be distinct collinear points with $A * B * C$. Then exactly one of these is true: $D - A - B - C$ or $A - D - B - C$ or $A - B - D - C$ or $A - B - C - D$.*

Proposition 2.23. *Let A, B, C, and D be distinct collinear points. There is a permutation X, Y, Z, W of A, B, C, D so that $X - Y - Z - W$.*

Proposition 2.24. *If A, B, and C are points so that $A * B * C$ then $\overline{AB} \subseteq \overline{AC}$.*

Proposition 2.25. *If A, B, and C are points so that $A * B * C$ then $\overline{AB} \cup \overline{BC} = \overline{AC}$.*

Proposition 2.26. *If A, B, and C are points so that $A * B * C$ then $\overline{AB} \cap \overline{BC} = \{B\}$.*

Definition. If A, B, and C are points so that $B * A * C$ then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are *opposite rays.*

Proposition 2.27. *If A, B, and C are points so that $B * A * C$ then $\overrightarrow{AB} \cap \overrightarrow{AC} = \{A\}$.*

Proposition 2.28. *If A, B, and C are points so that $B * A * C$ then $\overrightarrow{AB} \cup \overrightarrow{AC} = \overleftrightarrow{AB}$.*

Proposition 2.29. *If A, B, and C are points so that $A * B * C$ then $\overrightarrow{AB} = \overrightarrow{AC}$.*

Proposition 2.30. *If A, B, and C are distinct points so that $C \in \overrightarrow{AB}$ then $\overrightarrow{AB} = \overrightarrow{AC}$.*

Proposition 2.31. *If X and Y are distinct points on the ray $\overrightarrow{AB}$ then it is not the case that $X * A * Y$.*

Proposition 2.32. *If C is any point on the ray $\overrightarrow{AB}$ and $C \neq A$, then there is a point D on $\overrightarrow{AB}$ so that $A * C * D$.*

Proposition 2.33. *If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are rays so that $\overrightarrow{AB} = \overrightarrow{CD}$ then $A = C$.*

Definition. If $\overrightarrow{AB}$ is a ray, then A is the *vertex* of the ray.

Proposition 2.34. *If X and Y are distinct points on the segment $\overline{AB}$ then it is not the case that $X * A * Y$.*

Proposition 2.35. *If X and Y are distinct points on the segment $\overline{AB}$ then it is not the case that $X * B * Y$.*

Proposition 2.36. *If $\overline{AB}$ and $\overline{CD}$ are line segments so that $\overline{AB} = \overline{CD}$ then $\{A, B\} = \{C, D\}$.*

Definition. If $\overline{AB}$ is a line segment, then A and B are the *endpoints* of the segment.

Proposition 2.37. *Every line contains infinitely many points.*

Proposition 2.38. *Every line segment contains infinitely many points.*

2.4 Angles

Definition. Suppose that A, B, and C are noncollinear points. The *angle* $\angle BAC$ is defined to be $\overrightarrow{AB} \cup \overrightarrow{AC}$.

Proposition 2.39. *Suppose that A, B, and C are noncollinear points. Then $\angle BAC = \angle CAB$.*

Proposition 2.40. *Suppose that A, B, and C are noncollinear points. If $B' \in \overrightarrow{AB}$, $C' \in \overrightarrow{AC}$, $B' \neq A$, and $C' \neq A$, then A, B', and C' are noncollinear and $\angle BAC = \angle B'AC'$.*

Proposition 2.41. *Suppose that A, B, and C are noncollinear points. Then A is the only point on $\angle BAC$ which is not between any two other points of $\angle BAC$.*

Proposition 2.42. *If $\angle BAC$ and $\angle EDF$ are angles and $\angle BAC = \angle EDF$, then $A = D$.*

Proposition 2.43. *If $\angle BAC$ and $\angle DAE$ are angles and $\angle BAC = \angle DAE$ then either $\overrightarrow{AB} = \overrightarrow{AD}$ and $\overrightarrow{AC} = \overrightarrow{AE}$ or $\overrightarrow{AB} = \overrightarrow{AE}$ and $\overrightarrow{AC} = \overrightarrow{AD}$.*

Definition. Suppose that A, B, and C are noncollinear points. Then A is the *vertex* of $\angle BAC$, and $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are the *sides* of $\angle BAC$.

Proposition 2.44. *Suppose that $\angle BAC$ and $\angle B'AC$ are angles and $\angle BAC = \angle B'AC$. Then $H_{\overleftrightarrow{AC}}(B) = H_{\overleftrightarrow{AC}}(B')$.*

Definition. Suppose that A, B, and C are noncollinear points. The *interior* of $\angle BAC$ is $H_{\overleftrightarrow{AC}}(B) \cap H_{\overleftrightarrow{AB}}(C)$. The *exterior* of $\angle BAC$ is the set of all points which are not on $\angle BAC$ and are not in the interior of $\angle BAC$.

Proposition 2.45. *Suppose that A, B, and C are noncollinear points. A point D is in the interior of $\angle BAC$ if and only if D is on the same side of $\overleftrightarrow{AC}$ as B and D is on the same side of $\overleftrightarrow{AB}$ as C.*

Proposition 2.46 (Crossbar Betweenness)**.** *Suppose that A, B, and C are noncollinear points, and suppose D is a point on $\overleftrightarrow{BC}$. Then D is in the interior of $\angle BAC$ if and only if $B * D * C$.*

Proposition 2.47. *Suppose that A, B, and C are noncollinear points, and suppose D is a point in the interior of $\angle BAC$. Then every point on $\overrightarrow{AD}$ other than A is in the interior of $\angle BAC$.*

Definition. Suppose that A, B, C, and D are points. The ray $\overrightarrow{AD}$ is *between* the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ if A, B, and C are noncollinear and D is in the interior of $\angle BAC$.

Proposition 2.48. *Suppose that A, B, and C are noncollinear points and that D is a point different from A. If $\overrightarrow{AD}$ intersects the interior of $\overline{BC}$ then D is in the interior of $\angle BAC$.*

Lemma 2.49 (Z-Lemma)**.** *Suppose that A and B are distinct points on a line ℓ and that C and D are points on opposite sides of ℓ. Then $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are disjoint.*

Theorem 2.50 (Crossbar Theorem)**.** *Suppose A, B, and C are noncollinear points and that D is in the interior of $\angle BAC$. Then $\overrightarrow{AD}$ intersects $\overline{BC}$.*

Proposition 2.51. *Suppose A, B, C, D, and E are distinct points so that $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and that $\overrightarrow{AE}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Then $\overrightarrow{AE}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.*

Proposition 2.52. *Suppose A, B, C, D, and E are distinct points so that $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and that $\overrightarrow{AE}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Then $\overrightarrow{AD}$ is between $\overrightarrow{AE}$ and $\overrightarrow{AC}$.*

Proposition 2.53 (Interior Inclusion)**.** *Suppose that A, B, and C are noncollinear points. If D is in the interior of $\angle BAC$ and E is in the interior of $\angle BAD$, then E is in the interior of $\angle BAC$.*

Proposition 2.54. *If three distinct rays have the same vertex, then at most one of the rays is between the other two.*

Proposition 2.55. *Suppose that A, B, C, and D are four points such that C and D are on the same side of $\overleftrightarrow{AB}$ and D is not on $\overrightarrow{AC}$. Then either C is in the interior of $\angle BAD$ or D is in the interior of $\angle BAC$.*

Exercise 2.56. Draw three rays with a common endpoint so that no one of them is between the other two.

Definition. Suppose A, B, and C are points so that $B * A * C$. Let D be a point not on $\overleftrightarrow{BC}$. The angles $\angle BAD$ and $\angle DAC$ are a *linear pair*.

Proposition 2.57. *Every angle is part of a linear pair.*

Proposition 2.58. *If $C * A * B$ and D is in the interior of $\angle BAE$ then E is in the interior of $\angle DAC$.*

Proposition 2.59 (Supplementary Interiors 1)**.** *The interiors of angles in a linear pair are disjoint.*

Proposition 2.60 (Supplementary Interiors 2)**.** *If $\angle BAD$ and $\angle DAC$ form a linear pair, and if E is on the same side of $\overleftrightarrow{BC}$ as D, then exactly one of these must hold:*

1. *E is on $\overrightarrow{AD}$.*
2. *E is in the interior of $\angle BAD$.*
3. *E is in the interior of $\angle DAC$.*

Proposition 2.61. *Suppose that A, B, C, D, and E are points so that $B * A * C$. If $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AE}$ then $\overrightarrow{AE}$ is between $\overrightarrow{AD}$ and $\overrightarrow{AC}$.*

Definition. Suppose that A, B, and C are noncollinear points. The *triangle* $\triangle ABC$ is defined to be

$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{AC}.$$

Proposition 2.62. *If A, B, and C are noncollinear points then*

$$\triangle ABC = \triangle ACB = \triangle BAC = \triangle BCA = \triangle CAB = \triangle CBA.$$

Proposition 2.63. *Suppose that A, B, and C are noncollinear points. The points A, B, and C are the only points in $\triangle ABC$ which are not between other points on $\triangle ABC$.*

Proposition 2.64. *If $\triangle ABC$ and $\triangle DEF$ are triangles so that $\triangle ABC = \triangle DEF$ then $\{A, B, C\} = \{D, E, F\}$. It follows that*

$$\{\overline{AB}, \overline{BC}, \overline{AC}\} = \{\overline{DE}, \overline{EF}, \overline{DF}\}$$

and that

$$\{\angle ABC, \angle BCA, \angle CAB\} = \{\angle DEF, \angle EFD, \angle FDE\}.$$

Definition. Suppose that A, B, and C are noncollinear points. Then A, B, and C are the *vertices* of $\triangle ABC$. $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are the *sides* of $\triangle ABC$, and the angles $\angle ABC$, $\angle BCA$, and $\angle CAB$ are the *angles* of $\triangle ABC$. The *interior* of $\triangle ABC$ is the intersection of the interiors of the angles of $\triangle ABC$. The *exterior* of $\triangle ABC$ is the set of all points not on $\triangle ABC$ and not in the interior of $\triangle ABC$.

Proposition 2.65. *Suppose that A, B, and C are noncollinear points. The interior of $\triangle ABC$ is the set $H_{\overleftrightarrow{AB}}(C) \cap H_{\overleftrightarrow{BC}}(A) \cap H_{\overleftrightarrow{AC}}(B)$.*

2.5 Convexity

Definition. A set of points S is *convex* if whenever A and C are in S and $A * B * C$ then B is in S also.

Proposition 2.66. *The empty set is convex.*

Proposition 2.67. *If A is a point then $\{A\}$ is convex.*

Proposition 2.68. *Lines are convex.*

Proposition 2.69. *Rays are convex.*

Proposition 2.70. *Half planes are convex.*

Proposition 2.71. *If S and T are convex sets, then $S \cap T$ is a convex set.*

Proposition 2.72. *Line segments are convex.*

Proposition 2.73. *Interiors of angles and triangles are convex.*

Exercise 2.74. Show by means of a picture that angles are not convex. (This might bother you.)

3 Incidence, Betweenness, and Congruence Geometry

3.1 Segment Congruence

Primitives. We add here a new primitive, *segment congruence.*

Axiom. Segment congruence is a binary relation on the set of all segments. If $\overline{AB}$ and $\overline{CD}$ are segments, then $\overline{AB} \cong \overline{CD}$ indicates that the segment congruence relation holds between $\overline{AB}$ and $\overline{CD}$.

Segment Congruence Axiom 1 (SCA1)**.** *Segment congruence is an equivalence relation on the set of all segments.*

Segment Congruence Axiom 2 (SCA2 Point Construction)**.** *Suppose that $\overline{AB}$ is a segment and $\overrightarrow{CD}$ is a ray. There is a unique point E on $\overrightarrow{CD}$ which is distinct from C so that $\overline{AB} \cong \overline{CE}$.*

Segment Congruence Axiom 3 (SCA3)**.** *Suppose that $\overline{AC}$ and $\overline{A'C'}$ are congruent segments. If B is a point so that $A * B * C$ then there is a point B' so that $A' * B' * C'$, $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$.*

Proposition 3.1. *The point B' in SCA3 is unique.*

Proposition 3.2 (Segment Addition)**.** *Suppose that $A * B * C$ and $A' * B' * C'$. If $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$ then $\overline{AC} \cong \overline{A'C'}$.*

Proposition 3.3 (Segment Subtraction)**.** *Suppose that $A * B * C$ and $A' * B' * C'$. If $\overline{AC} \cong \overline{A'C'}$ and $\overline{AB} \cong \overline{A'B'}$ then $\overline{BC} \cong \overline{B'C'}$.*

Definition. Suppose that $\overline{AB}$ and $\overline{CD}$ are segments. We write $\overline{AB} < \overline{CD}$ or $\overline{CD} > \overline{AB}$ if there is a point E so that $C * E * D$ and $\overline{AB} \cong \overline{CE}$.

Proposition 3.4. *Suppose that $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{EF}$. Then $\overline{AB} < \overline{EF}$.*

Proposition 3.5 (Segment Substitution)**.** *If $\overline{AB} < \overline{CD}$, $\overline{CD} \cong \overline{C'D'}$, and $\overline{AB} \cong \overline{A'B'}$ then $\overline{A'B'} < \overline{C'D'}$.*

Proposition 3.6. *If $A * B * C$ then $\overline{AB} < \overline{AC}$.*

Proposition 3.7. *Suppose that $\overrightarrow{AB} = \overrightarrow{AC}$. If $\overline{AB} < \overline{AC}$ then $A * B * C$.*

Lemma 3.8. *If $\overline{AB}$ is a line segment, then it is not the case that $\overline{AB} < \overline{AB}$.*

Proposition 3.9 (Segment Trichotomy)**.** *If $\overline{AB}$ and $\overline{CD}$ are segments, then exactly one of these holds: $\overline{AB} < \overline{CD}$ or $\overline{AB} \cong \overline{CD}$ or $\overline{AB} > \overline{CD}$.*

3.2 Angle Congruence

Primitives. We add here a new primitive, *angle congruence.*

Axiom. Angle congruence is a binary relation on the set of all angles. If $\angle ABC$ and $\angle DEF$ are angles, then $\angle ABC \cong \angle DEF$ indicates that the angle congruence relation holds between $\angle ABC$ and $\angle DEF$.

Definition. By an *oriented triangle* we will mean a triangle with a specified ordering of its vertices. If A, B, and C are noncollinear points, then the triangle with these vertices in the order A, B, then C is denoted $\triangle ABC$.

Note. We are using the same notation for an oriented triangle as for a non-oriented triangle. Generally, if congruence is under consideration, we will mean an oriented triangle.

Definition. Two oriented triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be *congruent* if

- $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, $\overline{AC} \cong \overline{A'C'}$,
- $\angle ABC \cong \angle A'B'C'$, $\angle BCA \cong \angle B'C'A'$, and $\angle CAB \cong \angle C'A'B'$.

Angle Congruence Axiom 1 (ACA1)**.** *Angle congruence is an equivalence relation on the set of all angles.*

Angle Congruence Axiom 2 (ACA2 Angle Construction or Ray Construction)**.** *If $\angle BAC$ is any angle and $\overrightarrow{DE}$ is any ray, then on either side of $\overleftrightarrow{DE}$ there is a unique ray $\overrightarrow{DF}$ so that $\angle BAC \cong \angle EDF$.*

Angle Congruence Axiom 3 (ACA3 Side-Angle-Side Congruence or SAS)**.** *If $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\overline{AB} \cong \overline{A'B'}$, $\angle CAB \cong \angle C'A'B'$, and $\overline{AC} \cong \overline{A'C'}$, then $\triangle ABC \cong \triangle A'B'C'$.*

Proposition 3.10 (Triangle Construction)**.** *Suppose that A, B, and C are noncollinear points and that $\overline{A'B'}$ is a segment with $\overline{AB} \cong \overline{A'B'}$. Then on either side of $\overleftrightarrow{A'B'}$ is a point C' with $\triangle ABC \cong \triangle A'B'C'$.*

Proposition 3.11. *The point C' in Triangle Construction is unique.*

Definition. An *isosceles triangle* is a triangle in which at least two sides are congruent. An *equilateral triangle* is a triangle in which all three sides are congruent.

Theorem 3.12 (Isosceles Triangle Theorem)**.** *If $\triangle ABC$ is an isosceles triangle in which $\overline{AB} \cong \overline{AC}$ then $\angle ABC \cong \angle ACB$.*

Corollary 3.13. *If $\triangle ABC$ is an equilateral triangle, then all three angles in $\triangle ABC$ are congruent.*

Note. Note that this is a conditional. We are not declaring that an equilateral triangle actually exists.

Proposition 3.14. *If A, B, and C are noncollinear points, then $\triangle ABC \cong \triangle ABC$.*

Proposition 3.15. *If $\triangle ABC$ and $\triangle A'B'C'$ are triangles with $\triangle ABC \cong \triangle A'B'C'$ then $\triangle A'B'C' \cong \triangle ABC$.*

Proposition 3.16. *If $\triangle ABC$, $\triangle A'B'C'$, and $\triangle A''B''C''$ are triangles with $\triangle ABC \cong \triangle A'B'C'$ and $\triangle A'B'C' \cong \triangle A''B''C''$, then $\triangle ABC \cong \triangle A''B''C''$.*

Proposition 3.17. *Suppose that angles $\angle BAD$ and $\angle DAC$ are a linear pair and that angles $\angle B'A'D'$ and $\angle D'A'C'$ are a linear pair. If $\angle BAD \cong \angle B'A'D'$ then $\angle DAC \cong \angle D'A'C'$.*

Definition. Suppose ℓ and m are two lines intersecting at a point A. Suppose that B and D are on m and C and D are on ℓ so that $B * A * D$ and $C * A * E$. Then $\angle BAC$ and $\angle DAE$ are *vertical angles.*

Theorem 3.18 (Vertical Angles Theorem)**.** *Vertical angles are congruent.*

Proposition 3.19 (Converse of Vertical Angles Theorem)**.** *Suppose that $A * B * C$ and that D and E are on opposite sides of $\overleftrightarrow{AB}$. If $\angle DBC \cong \angle EBA$ then D, B, and E are collinear.*

Proposition 3.20. *Suppose that $\overleftrightarrow{AD}$ is a line, and let B and C be on opposite sides of $\overleftrightarrow{AD}$. Suppose that α and β are are a linear pair so that $\alpha \cong \angle BAD$ and $\beta \cong \angle DAC$. Then B, A, and C are collinear and $\angle BAD$ and $\angle DAC$ form a linear pair.*

Definition. An angle α is a *right angle* if α forms a linear pair with an angle congruent to α.

Corollary 3.21. *Suppose that A, B, C, and D are points so that $\angle DAB$ and $\angle DAC$ are right angles. Then A, B, and C are collinear.*

Proposition 3.22 (Angle Addition)**.** *Suppose that D is an interior point of the angle $\angle BAC$ and that D' is an interior point of the angle $\angle B'A'C'$. If $\angle BAD \cong \angle B'A'D'$ and $\angle DAC \cong \angle D'A'C'$ then $\angle BAC \cong \angle B'A'C'$.*

Proposition 3.23. *Suppose $\angle BAC \cong \angle B'A'C'$ and that $\overrightarrow{AD}$ is a ray between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. There is a ray $\overrightarrow{A'D'}$ between $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ so that $\angle BAD \cong \angle B'A'D'$ and $\angle DAC \cong \angle D'A'C'$.*

Proposition 3.24 (Angle Subtraction)**.** *Suppose that D is an interior point of the angle $\angle BAC$ and that D' is an interior point of the angle $\angle B'A'C'$. If $\angle BAC \cong \angle B'A'C'$ and $\angle BAD \cong \angle B'A'D'$ then $\angle DAC \cong \angle D'A'C'$.*

Definition. If $\angle BAC$ and $\angle EDF$ are angles then we write $\angle BAC < \angle EDF$ or $\angle EDF > \angle BAC$ if there is a ray $\overrightarrow{DG}$ between $\overrightarrow{DE}$ and $\overrightarrow{DF}$ so that $\angle BAC \cong \angle EDG$.

Proposition 3.25. *Suppose that $\angle BAC < \angle EDF$ and $\angle EDF < \angle HGI$. Then $\angle BAC < \angle HGI$.*

Proposition 3.26 (Angle Substitution)**.** *Let α, α', β, and β' be angles. If $\alpha < \beta$ and $\alpha \cong \alpha'$ and $\beta \cong \beta'$ then $\alpha' < \beta'$.*

Proposition 3.27. *If $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ then $\angle BAD < \angle BAC$.*

Proposition 3.28. *Suppose that α and β are a linear pair and that α' and β' are a linear pair. If $\alpha > \alpha'$ then $\beta < \beta'$.*

Proposition 3.29. *If α is any angle, then it is not the case that $\alpha < \alpha$.*

Proposition 3.30 (Angle Trichotomy)**.** *If α and β are angles, then exactly one of these must hold: $\alpha < \beta$ or $\alpha \cong \beta$ or $\alpha > \beta$.*

3.3 Right Angles and Perpendicular Lines

Definition. An angle α is a *right angle* if α forms a linear pair with an angle congruent to α. If α is a right angle and β is an angle so that $\beta < \alpha$, then β is an *acute* angle. If $\alpha < \beta$ then β is an obtuse angle.

Proposition 3.31. *If α is a right angle and $\gamma \cong \alpha$, then γ is a right angle.*

Proposition 3.32. *If α is a right angle and γ forms a linear pair with α, then γ is a right angle.*

Proposition 3.33. *Any two right angles are congruent.*

Definition. Two lines ℓ and m are *perpendicular* if ℓ and m intersect at a point A and there exist $B \neq A$ on ℓ and $C \neq A$ on m so that $\angle BAC$ is a right angle. In this case, we write $\ell \perp m$.

Proposition 3.34. *Suppose that ℓ and m are lines that intersect at a point A and that there exist $B \neq A$ on ℓ and $C \neq A$ on m so that $\angle BAC$ is a right angle. Then for all $B' \neq A$ on ℓ and $C' \neq A$ on m $\angle B'AC'$ is a right angle.*

Proposition 3.35. *If ℓ is a line and P is a point not on ℓ then there is a line m passing through P which is perpendicular to ℓ.*

Corollary 3.36. *Right angles exist.*

Corollary 3.37. *Perpendicular lines exist.*

Proposition 3.38. *If ℓ is a line and P is a point on ℓ then there is a unique line m passing through P which is perpendicular to ℓ.*

3.4 Parallel Lines

Definition. Two lines ℓ and m are *parallel* if they do not intersect. In this case, we write $\ell \parallel m$.

Theorem 3.39 (Alternate Interior Angles Theorem)**.** *Suppose that C and D are on opposite sides of $\overleftrightarrow{AB}$. If $\angle CAB \cong \angle DBA$ then $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$.*

Definition. Suppose that ℓ and ℓ' are two distint lines. A *transversal* of ℓ and ℓ' is a third line which intersects ℓ and ℓ'. Suppose that t is a transversal of ℓ and ℓ'. Let B be the intersection of t with ℓ, and let B' be the intersection of t with ℓ'. Let A and C be points on ℓ on opposite sides of t, and let A' and C' be points on ℓ' on opposite sides of t so that A and A' are on the same side of t. The four angles $\angle ABB'$, $\angle CBB'$, $\angle A'B'B$, and $\angle C'B'B$ are *interior angles.* The angles $\angle ABB'$ and $\angle C'B'B$ are *alternate interior angles* as are $\angle CBB'$ and $\angle A'B'B$.

Note. An alternative statement of the Alternate Interior Angles Theorem is: If ℓ and ℓ' are two lines cut by a transversal t in such a way that a pair of alternate interior angles are congruent, then $\ell \parallel \ell'$.

Definition (Continuing Definition 3.4)**.** Let D and D' be points on t so that $D * B * B' * D'$. These pairs of angles are *corresponding angles*:

$$\angle ABD \text{ and } \angle A'B'B;\ \angle ABB' \text{ and } \angle A'B'D'$$

$$\angle CBD \text{ and } \angle C'B'B;\ \angle CBB' \text{ and } \angle C'B'D'.$$

Proposition 3.40 (Corresponding Angles Theorem)**.** *If ℓ and ℓ' are two lines cut by a transversal t in such a way that a pair of corresponding angles are congruent, then $\ell \parallel \ell'$.*

Proposition 3.41. *Suppose that ℓ and ℓ' are distinct lines. If there is one line m so that $\ell \perp m$ and $\ell' \perp m$ then $\ell \parallel \ell'$.*

Proposition 3.42. *If ℓ is a line and P is a point not on ℓ then there is a unique line m passing through P which is perpendicular to ℓ.*

Definition. If P is a point not on a line ℓ, and if m is the unique line passing through P perpendicular to ℓ, then the point of intersection of ℓ and m is the *foot* of the perpendicular.

Proposition 3.43 (Existence of Parallels)**.** *Suppose that P is a point not on a line ℓ. There is at least one line ℓ' through P parallel to ℓ.*

Note. The method of proof you (probably) used in 3.43 is called the *Double Perpendicular Construction.*

3.5 Inequalities for Triangles

Definition. Suppose that A, B, and C are noncollinear points. The angles $\angle ABC$, $\angle BCD$, $\angle CAB$ are called *interior* angles of $\triangle ABC$. Any angle forming a linear pair with an interior angle of a triangle is an *exterior* angle. If an exterior angle forms a linear pair with one of the interior angles, the other two interior angles are *remote* interior angles (remote to the given exterior angle). In other words if $B * C * D$, then the angle $\angle ACD$ is an exterior angle with remote interior angles $\angle CAB$ and $\angle CBA$.

Lemma 3.44. *Suppose that A, B, and C are noncollinear points. If $B * C * D$ then the exterior angle $\angle ACD$ is greater than the remote interior angle $\angle CAB$ (whose vertex is not on $\overleftrightarrow{CD}$).*

Theorem 3.45 (Exterior Angle Theorem)**.** *Suppose that A, B, and C are noncollinear points. If $B * C * D$ then the exterior angle $\angle ACD$ is greater than both remote interior angles $\angle CAB$ and $\angle CBA$.*

Definition. Suppose that $\triangle ABC$ is a triangle. The side $\overline{BC}$ is *opposite* to the angle $\angle CAB$. The side $\overline{AB}$ is opposite to the angle $\angle BCA$, and the side $\overline{CA}$ is opposite to the angle $\angle ABC$.

Definition. A *right triangle* is a triangle in which at least one angle is a right angle.

Proposition 3.46. *If $\triangle ABC$ is a triangle and $\angle ABC$ is a right angle, then $\angle CAB$ and $\angle BCA$ are acute angles.*

Corollary 3.47. *A right triangle has exactly one right angle.*

Definition. In any right triangle, the side opposite the right angle is the *hypotenuse*. The other two sides are *legs*.

Proposition 3.48. *If α and β are a linear pair, then α is acute if and only if β is obtuse.*

Proposition 3.49. *A triangle can have at most one obtuse angle.*

Proposition 3.50. *Suppose that $\triangle ABC$ is a triangle and let F be the foot of the perpendicular from C to $\overleftrightarrow{AB}$. If the angles $\angle CAB$ and $\angle CBA$ are acute, then $A * F * B$.*

Theorem 3.51 (Scalene Inequality)**.** *If A, B, and C are noncollinear points then $\overline{BC} > \overline{AC}$ if and only if $\angle BAC > \angle ABC$.*

This theorem is proven by the following claimable lemmas.

Lemma 3.52 (Part of the proof of 3.51)**.** *Suppose that A, B, and C are noncollinear points. If $\overline{BC} > \overline{AC}$ then $\angle BAC > \angle ABC$.*

Lemma 3.53 (Part of the proof of 3.51)**.** *Suppose that A, B, and C are noncollinear points. If $\angle BAC > \angle ABC$ then $\overline{BC} > \overline{AC}$.*

Proposition 3.54. *Suppose that P is a point not on a line ℓ. Let F be the foot of the perpendicular from P to ℓ. If Q is any point on ℓ other than F, then $\overline{PF} < \overline{PQ}$.*

Lemma 3.55. *Suppose that A, B, and C are noncollinear points and $B * D * C$. If $\overline{AC} > \overline{AB}$ or $\overline{AC} = \overline{AB}$ then $\overline{AC} > \overline{AD}$.*

Theorem 3.56 (Hinge Theorem)**.** *Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$. Then $\angle BAC > \angle B'A'C'$ if and only if $\overline{BC} > \overline{B'C'}$.*

This theorem is proven by the following claimable lemmas.

Lemma 3.57 (Part of the proof of 3.56)**.** *Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$. If $\angle BAC > \angle B'A'C'$ then $\overline{BC} > \overline{B'C'}$.*

Lemma 3.58 (Part of the proof of 3.56)**.** *Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$. If $\overline{BC} > \overline{B'C'}$ then $\angle BAC > \angle B'A'C'$.*

Theorem 3.59 (Triangle Inequality Form 1)**.** *Let $\triangle ABC$ be a triangle and let D be a point such that $A * B * D$ and $\overline{BD} \cong \overline{BC}$. Then $\overline{AC} < \overline{AD}$.*

3.6 Triangle Congruence

Theorem 3.60 (Angle-Angle-Side Congruence or AAS)**.** *Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\angle CAB \cong \angle C'A'B'$ and $\angle ABC \cong \angle A'B'C'$ and $\overline{BC} \cong \overline{B'C'}$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Theorem 3.61 (Angle-Side-Angle Congruence or ASA)**.** *Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\angle ABC \cong \angle A'B'C'$ and $\overline{BC} \cong \overline{B'C'}$ and $\angle BCA \cong \angle B'C'A'$ Then $\triangle ABC \cong \triangle A'B'C'$.*

Proposition 3.62 (Converse of Isosceles Triangle Theorem)**.** *If $\triangle ABC$ is a triangle in which $\angle ABC \cong \angle ACB$ then $\overline{AB} \cong \overline{AC}$.*

Corollary 3.63. *If all three angles in a triangle are congruent, then the triangle is equilateral.*

Theorem 3.64 (Side-Side-Side Congruence or SSS)**.** *Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$ and $\overline{BC} \cong \overline{B'C'}$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Exercise 3.65. Draw a picture indicating that Angle-Side-Side congruence is not a thing.

Theorem 3.66 (Hypotenuse-Leg Congruence or HL)**.** *Let $\triangle ABC$ and $\triangle A'B'C'$ be right triangles with $\angle CAB$ and $\angle C'A'B'$ right angles. If $\overline{BC} \cong \overline{B'C'}$ and $\overline{AB} \cong \overline{A'B'}$ then $\triangle ABC \cong \triangle A'B'C'$.*

3.7 Midpoints and Bisectors

Definition. A *midpoint* of a segment $\overline{AB}$ is a point M so that $A * M * B$ and $\overline{AM} \cong \overline{MB}$.

Lemma 3.67. *If a segment has a midpoint, then it has exactly one.*

Lemma 3.68. *Suppose that A and B are distinct points and that C and D are points on opposite sides of $\overleftrightarrow{AB}$ so that $\angle BAD$ and $\angle ABC$ are right angles. If M is the point of intersection of $\overline{CD}$ and $\overleftrightarrow{AB}$ then $M \neq A$ and $M \neq B$.*

Lemma 3.69. *Suppose that A and B are distinct points and that C and D are points on opposite sides of $\overleftrightarrow{AB}$ so that $\angle BAD$ and $\angle ABC$ are right angles. If M is the point of intersection of $\overline{CD}$ and $\overleftrightarrow{AB}$ then $A * M * B$.*

Note. The proof here that the midpoint of $\overline{AB}$ exists uses points off the line $\overleftrightarrow{AB}$. That may seem odd. If we had already assumed Dedekind's Axiom or if we had assumed a Ruler Postulate we could prove existence of the midpoint from within $\overleftrightarrow{AB}$.

Theorem 3.70 (Midpoint Existence)**.** *Every segment has a unique midpoint.*

Definition. A *perpendicular bisector* of a segment $\overline{AB}$ is a line ℓ perpendicular to $\overleftrightarrow{AB}$ which intersects $\overline{AB}$ at it midpoint.

Theorem 3.71 (Existence of Perpendicular Bisectors)**.** *Every segment has a unique perpendicular bisector.*

Theorem 3.72 (Pointwise Characterization of Perpendicular Bisectors)**.** *Suppose that A and B are distinct points and that P is any point. P lies on the perpendicular bisector of $\overline{AB}$ if and only if $\overline{PA} \cong \overline{PB}$.*

This theorem is proven by the following claimable lemmas.

Lemma 3.73 (Part of the proof of 3.72)**.** *Suppose that A and B are distinct points. If P lies on the perpendicular bisector of $\overline{AB}$ then $\overline{PA} \cong \overline{PB}$.*

Lemma 3.74 (Part of the proof of 3.72)**.** *Suppose that A and B are distinct points. If P is a point so that $\overline{PA} \cong \overline{PB}$ then P lies on the perpendicular bisector of $\overline{AB}$.*

Definition. Suppose that $\angle BAC$ is an angle. A ray $\overrightarrow{AD}$ is a *bisector* of $\angle BAC$ if $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and if $\angle DAB \cong \angle DAC$.

Lemma 3.75. *If an angle has a bisector, then it has exactly one.*

Theorem 3.76. *Every angle has a unique bisector.*

Theorem 3.77 (Pointwise Characterization of Angle Bisector)**.** *Suppose that P is a point in the interior of $\angle BAC$. Let F be the foot of the perpendicular from P to $\overline{AB}$, and let G be the foot of the perpendicular from P to $\overline{AC}$. Then $\overrightarrow{AP}$ is the bisector of $\angle BAC$ if and only if $\overline{PF} \cong \overline{PG}$.*

This theorem is proven by the following claimable lemmas.

Lemma 3.78 (Part of the proof of 3.77)**.** *Suppose that P is a point in the interior of $\angle BAC$. Let F be the foot of the perpendicular from P to $\overline{AB}$, and let G be the foot of the perpendicular from P to $\overline{AC}$. If $\overrightarrow{AP}$ is the bisector of $\angle BAC$ then $\overline{PF} \cong \overline{PG}$.*

Lemma 3.79 (Part of the proof of 3.77)**.** *Suppose that P is a point in the interior of $\angle BAC$. Let F be the foot of the perpendicular from P to $\overline{AB}$, and let G be the foot of the perpendicular from P to $\overline{AC}$. If $\overline{PF} \cong \overline{PG}$ then $\overrightarrow{AP}$ is the bisector of $\angle BAC$.*

Proposition 3.80. *If $\overrightarrow{AD}$ is a bisector of an angle $\angle BAC$ then the angle $\angle BAD$ is acute.*

3.8 Quadrilaterals

Definition. Suppose that A, B, C, and D are four distinct points. If $\overline{AB}$ and $\overline{CD}$ do not intersect and $\overline{AD}$ and $\overline{CB}$ do not intersection then

$$\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$$

is a *quadrilateral*. A, B, C, and D are the *vertices* of $\square ABCD$. The line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ are the *sides* of $\square ABCD$. $\overline{AB}$ and $\overline{CD}$ are *opposite sides*, as are $\overline{AD}$ and $\overline{CB}$.

Definition. Two quadrilaterals $\square ABCD$ and $\square XYZW$ are *congruent* if

$$\overline{AB} \cong \overline{XY} \text{ and } \overline{BC} \cong \overline{YZ} \text{ and } \overline{CD} \cong \overline{ZW} \text{ and } \overline{DA} \cong \overline{WX} \text{ and}$$

$$\angle ABC \cong \angle XYZ \text{ and } \angle BCD \cong \angle YZW \text{ and } \angle CDA \cong \angle ZWX \text{ and } \angle DAB \cong \angle WXY.$$

This is expressed as $\square ABCD \cong \square XYZW$.

Definition. A *convex quadrilateral* is a quadrilateral in which each vertex is in the interior of the angle formed by the other three vertices.

Note. This is standard but rather inconvenient terminology. A quadrilateral is not convex in the sense that we have been using that word. The interior of a convex quaderilateral is convex. However, defining the interior is awkward for general quadrilaterals.

Exercise 3.81. Draw a quadrilateral which is not a convex quadrilateral and explain.

Proposition 3.82. *If $\square ABCD$ is a convex quadrilateral, then C and D are on the same side of $\overline{AB}$. The same holds for all other pairs of points.*

Definition. The *interior* of a convex quadrilateral is the intersection of the interiors of the angles determined by its sides

Proposition 3.83. *The interior of a convex quadrilateral is convex.*

Definition. A quadrilateral $\square ABCD$ is a *parallelogram* if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{CB}$.

Proposition 3.84. *Parallelograms are convex quadrilaterals.*

Proposition 3.85. *Suppose that $\triangle ABC$ is a triangle. If $A * D * B$ and $A * E * C$ then $\square DBCE$ is a convex quadrilateral.*

Definition. The *diagonals* of $\square ABCD$ are the line segments $\overline{AC}$ and $\overline{BD}$

Proposition 3.86. *A quadrilateral is a convex quadrilateral if and only if its diagonals have an interior point in common.*

Corollary 3.87. *If $\square ABCD$ and $\square ACBD$ are both quadrilaterals then $\square ABCD$ is not a convex quadrilateral.*

Corollary 3.88. *If $\square ABCD$ is a quadrilateral but not a convex quadrilateral, then $\square ABCD$ and $\square ACBD$ are both quadrilaterals.*

3.9 Circles

Definition. Suppose that A and B are distinct points. The *circle* γ with center A and radius $\overline{AB}$ is the set of all points X so that $\overline{AX} \cong \overline{AB}$. We denote the circle as $\gamma = \mathcal{C}(A, B)$. The point A is the center of γ. If $\overline{AX} \cong \overline{AB}$, then $\overline{AX}$ is a *radius* of γ. We may also call $\mathcal{C}(A, B)$ *the circle centered at A passing through B* or *containing B*.

Definition. If γ is the circle with center A and radius $\overline{AB}$, then the *interior* of γ is the set of all points X so that $X = A$ or $\overline{AX} < \overline{AB}$. The *exterior* of γ is the set of all points X so that $\overline{AX} > \overline{AB}$.

Note. When $P \in \gamma = \mathcal{C}(A, B)$ we will usually say that P is *on* γ rather than P is *in* γ to avoid confusion with points inside the interior of γ, which we may say are *inside* γ.

Proposition 3.89 (Circle Trichotomy)**.** *Suppose that γ is any circle and X is any point. The exactly one of these holds: X is in the interior of γ or X is on γ or X is in the exterior of γ.*

Definition. If γ is a circle, then the union of γ with its interior is called a *closed disk*. The interior of γ is called an *open disk*.

Lemma 3.90. *Let X, Y, and C be noncollinear points so that $\overline{CX} < \overline{CY}$ or $\overline{CX} \cong \overline{CY}$. If $X * B * Y$ then $\overline{CB} < \overline{CY}$*

Proposition 3.91. *The interior of a circle is convex.*

Proposition 3.92. *Closed disks are convex.*

Proposition 3.93. *Suppose that γ is a circle, that A is in the interior of γ, and that B is in the exterior of γ. The intersection of $\overline{AB}$ with the interior of γ is convex.*

Proposition 3.94. *Suppose that γ is a circle, that A is in the interior of γ, and that B is in the exterior of γ. The intersection of $\overline{AB}$ with the exterior of γ is convex.*

Lemma 3.95. *Suppose that C is a point in the interior of a circle γ. There is a circle centered at C contained entirely in the interior of γ.*

Lemma 3.96. *Suppose that C is a point in the exterior of a circle γ. There is a circle centered at C contained entirely in the exterior of γ.*

Proposition 3.97. *Suppose that ℓ is a line and γ is a circle. The number of points in $\gamma \cap \ell$ is 0, 1, or 2.*

Definition. A *tangent* to a circle γ is a line that intersects γ exactly once. A *secant* to a circle γ is a line that intersects γ exactly twice. If C and D are points on a circle γ, then $\overline{CD}$ is a *chord* of γ. A chord that contains the center of γ is a *diameter* of γ. If $\overline{CD}$ is a diameter of γ, then C and D are *antipodal points* on γ.

Theorem 3.98 (Tangent Line Theorem)**.** *Let t be a line and let $\gamma = \mathcal{C}(C, B)$ be a circle. Let $P \in \gamma \cap \ell$. Then t is a tangent to γ at P if and only if $\overleftrightarrow{CP} \perp t$.*

This theorem is proven by the following claimable lemmas.

Lemma 3.99 (Part of the proof of 3.98)**.** *Let t be a line and let $\gamma = \mathcal{C}(C, B)$ be a circle. Let $P \in \gamma \cap \ell$. If t is a tangent to γ at P then $\overleftrightarrow{CP} \perp t$.*

Lemma 3.100 (Part of the proof of 3.98)**.** *Let t be a line and let $\gamma = \mathcal{C}(C, B)$ be a circle. Let $P \in \gamma \cap \ell$. If $\overleftrightarrow{CP} \perp t$ then t is a tangent to γ at P.*

Proposition 3.101. *If P is a point on a circle γ then there is a line t that is tangent to γ at P.*

Proposition 3.102. *If t is a line tangent to a circle γ at a point P, then every point on t except P is in the exterior of γ.*

Proposition 3.103 (Secant Line Theorem)**.** *If $\overline{AB}$ is a chord of the circle $\gamma = \mathcal{C}(C, R)$, then C lies on the perpendicular bisector of $\overline{AB}$.*

Proposition 3.104. *If $\overline{AB}$ is a chord of a circle γ and $A * B * D$, then D is in the exterior of γ.*

Definition. Two circles α and β are *tangent* to each other at a point P if $\alpha \cap \beta = \{P\}$.

Proposition 3.105 (Tangent Circles Theorem)**.** *If $\alpha = \mathcal{C}(A, R)$ and $\beta = \mathcal{C}(B, S)$ are tangent at a point P, then $A \neq B$, A, B, and P are collinear, and the circles have a common tangent at P.*

Definition. If γ is a circle that contains three noncollinear points A, B, and C, then γ is a *circumcircle* of $\triangle ABC$ and is said to *circumscribe* the triangle $\triangle ABC$. The center of γ is the *circumcenter* of $\triangle ABC$.

Theorem 3.106 (Circumscribed Triangle Theorem)**.** *A triangle can be circumscribed if and only if the perpendicular bisectors of the sides of the triangle are concurrent. In this case, the center of the circumscribing circle is the point of intersection of the perpendicular bisectors of the sides.*

This theorem is proven through the following claimable lemmas.

Lemma 3.107 (Part of the proof of Theorem 3.106)**.** *If the perpendicular bisectors of the sides of a triangle are concurrent, then the triangle can be circumscribed.*

Lemma 3.108 (Part of the proof of Theorem 3.106)**.** *If a triangle can be circumscribed, then the perpendicular bisectors of the sides of the triange intersect at the center of the circumscribing circle.*

Corollary 3.109. *Circumscribing circles are unique.*

Definition. A circle γ is *inscribed* in a triangle if the lines containing each side of the triangle are all tangent to the circle. In this case, the center of γ is an *incenter* of the triangle.

Proposition 3.110 (Inscribed Circle Theorem)**.** *Every triangle has an inscribed circle whose center is the intersection of the angle bisectors of the triangle.*

Corollary 3.111. *Inscribed circles are unique.*

4 Continuity and Measurement

In this section we add a new axiom, Dedekind's Axiom, which essentially declares that lines have no holes in them. The point of this section is to establish the following theorems. The first two theorems on intersections were assumed by Euclid without explanation. Their proofs require an intimate knowledge of topological notions related to continuity. The second two theorems on measurement are often treated as axioms. However, doing so hides the fact that we can accomplish much in geometry without the notion of measurement. Treating the ability to measure as an axiom also hides subtle geometric properties within (unstated) assumptions about the real numbers. Some approaches to geometry will prove the measurement theorems first and then use them to prove the intersection theorems. This approach hides the fact that the intersection theorems are a product of the topology of the plane and are not dependent on the measurement theorems.

Theorem. 4.24 (Circle-Segment Intersection or Elementary Circular Continuity.) *Suppose that γ is a circle, that A is in the interior of γ, and that B is in the exterior of γ. Then $\overline{AB}$ intersects the circle γ.*

Theorem. 4.28 (Circle-Circle Intersection or Circular Continuity.) *Suppose that γ and δ are two circles. If γ contains a point in the interior of δ and if γ contains a point in the exterior of δ, then γ and δ intersect in two points which are on opposite sides of the line connecting the centers of the circles.*

Theorem. 4.37 (Ruler Theorem.) *For any points A and B there is a unique real number called the* length *of $\overline{AB}$ denoted as $|\overline{AB}|$. Suppose that A, B, C, and D are points.*

1. $|\overline{AB}| > 0$.
2. $|AB| = 0$ *if and only if* $A = B$.
3. $\overline{AB} \cong \overline{CD}$ *if and only if* $|\overline{AB}| = |\overline{CD}|$.
4. $\overline{AB} < \overline{CD}$ *if and only if* $|\overline{AB}| < |\overline{CD}|$.
5. *If* $A * B * C$ *then* $|\overline{AC}| = |\overline{AB}| + |\overline{BC}|$.
6. *For every positive real number x, there is a segment $\overline{AB}$ so that $|\overline{AB}| = x$.*

Theorem. 4.45 (Protractor Theorem.) *For any angle α there is a unique real number called the* measure *of α denoted as $|\alpha|$. Suppose that $\angle CAB$ and $\angle ZXY$ are angles.*

1. $0 < |\angle CAB| < 180$.
2. *If $\angle CAB$ is a right angle, then $|\angle CAB| = 90°$.*
3. $\angle CAB \cong \angle ZXY$ *if and only if* $|\angle CAB| = |\angle ZXY|$.
4. $\angle CAB < \angle ZXY$ *if and only if* $|\angle CAB| < |\angle ZXY|$.
5. *If D is interior to $\angle CAB$ then $|\angle CAB| = |\angle CAD| + |\angle DAB|$.*
6. *For every real number x with $0 < x < 180$, there is an angle α so that $|\alpha| = x°$.*

4.1 Continuous Functions on the Plane

Definition. Suppose that S and T are subsets of $\mathbb{P}$, that $f : S \to T$ is a function, and that A is in S. The function f is *continuous* at A if for every circle ϵ centered at $f(A)$, there is a circle δ cenered at A so that if P is in the intersection S and the interior of ϵ then $f(P)$ is in the interior of δ. If f is continuous at every $P \in S$ then f is continuous on S.

Proposition 4.1. *Suppose that $\overline{AB}$ is a chord of the circle $\gamma = \mathcal{C}(C, R)$ not passing through C. Define a function $f : \overline{AB} \to \gamma$ in the following way. Suppose that $X \in \overline{AB}$. Since $C \notin \overline{AB}$, $C \neq X$, and we can consider the ray $\overrightarrow{CX}$. There is a unique point Y on $\overrightarrow{CX}$ so that $\overline{CY} \cong \overline{CR}$. Let $f(X) = Y$. The function f is continuous on $\overline{AB}$.*

Definition. We will call the function f in 4.1 the *projection* of $\overline{AB}$ to γ.

Proposition 4.2. *Suppose that A and R are distinct points. We define a function f mapping the entire plane $\mathbb{P}$ to $\overrightarrow{AR}$. First, let $f(A) = A$. Now, if $X \neq A$ there is a unique point Y on $\overrightarrow{AR}$ so that $\overline{AX} \cong \overline{AY}$. Let $f(X) = Y$. This function f is continuous on $\mathbb{P}$.*

Definition. We will call the function f in 4.2 the *distance function ruled by* $\overrightarrow{AR}$.

Proposition 4.3. *Suppose that $f : R \to S$ is continuous on R and that $g : S \to T$ is continuous on S. The function $g \circ f : R \to T$ is continuous on R.*

4.2 Dedekind Cuts

Definition. A *Dedekind cut* of a line ℓ is a partition $\{\Sigma_1, \Sigma_2\}$ of ℓ into two nonempty convex subsets. the sets Σ_1 and Σ_2 called the *slices* of the Dedekind cut.

Proposition 4.4. *If ℓ is any line, then there is at least one Dedekind cut of ℓ.*

Definition. A *cut point* of a Dedekind cut of a line ℓ is a point C on ℓ so that for all $X, Y \in \ell$, if $C * X * Y$ then X and Y are in the same slice.

Proposition 4.5. *Suppose that C is a cut point of a Dedekind cut of a line ℓ. If X and Y are points on ℓ so that $X * C * Y$ then X and Y are in difference slices.*

Proposition 4.6. *Suppose that $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of a line ℓ and that C is a cut point. If $X \in \Sigma_1$ and $Y \in \Sigma_2$, then $X * C * Y$.*

Proposition 4.7. *Suppose that $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of a line ℓ and that C is a cut point. If $X \in \Sigma_1$ and $X \neq C$, then there is some $Y \in \Sigma_1$ so that $X * Y * C$.*

Proposition 4.8. *If a Dedekind cut of a line has a cut point, then it has exactly one cut point.*

Completeness Axiom 1. *(Dedekind's Axiom) Every Dedekind cut of a line has a cut point.*

Exercise 4.9. Use Dedekind cuts to prove that midpoints of line segments exist.

Note. We can also define Dedekind cuts and cut points for segments and rays. In each case, a Dedekind cut is a a partition $\{\Sigma_1, \Sigma_2\}$ of the segment or ray into two nonempty convex subsets. If we do so, all of the theorems above hold for segments and rays also. The reason for this is the next proposition for line segments and its corresponding proposition for rays.

Proposition 4.10. *Every Dedekind cut of a line segment has a cut point.*

4.3 The Intermediate Value Theorem

Theorem 4.11 (Intermediate Value Theore)**.** *Suppose that $\overline{AC}$ is a segment and $\overleftrightarrow{XZ}$ is a line, and let $f : \overline{AC} \to \overleftrightarrow{XZ}$ be a function that is continuous on $\overline{AC}$. If $f(A) = X$ and $f(B) = Z$ and $X * Y * Z$, then there is a B so that $A * B * C$ and $f(B) = Y$.*

This theorem is proven through the following claimable lemmas.

Setup for the proof of Theorem 4.11: Let Σ_1 the set of all $P \in \overline{AC}$ for which there is a $Q \in \overline{AC}$ so that either $A * P * Q$ or $P * Q * C$ and $f(Q) * Y * Z$, and let Σ_2 be $\overline{AC} - \Sigma_1$.

Note. Informally, think of A as being to the left of C and X as being to the left of Z. Σ_1 is the set of all points which are to the left of a point Q so that $f(Q)$ is to the left of Y.

Lemma 4.12. *$A \in \Sigma_1$.*

Lemma 4.13. *$C \in \Sigma_2$.*

Lemma 4.14. *Σ_1 is convex.*

Lemma 4.15. *Σ_2 is convex.*

Lemma 4.16. *$\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut.*

More setup for the proof of 4.11: Let B be the cut point of $\{\Sigma_1, \Sigma_2\}$.

Lemma 4.17. *$B \neq A$*

Lemma 4.18. *$B \neq C$.*

Lemma 4.19. *If $C \neq U$ and $U \in \Sigma_2$ and $Y * f(U) * Z$ then there is some $V \in \Sigma_2$ so that $V * U * C$.*

More setup for the proof of 4.11: Either $f(B) = Y$ or $f(B) * Y * Z$ or $Y * f(B) * Z$.

Lemma 4.20. *It is not the case that $f(B) * Y * Z$.*

Lemma 4.21. *$B \in \Sigma_2$.*

Lemma 4.22. *It is not the case that $Y * f(B) * Z$.*

Lemma 4.23. *$f(B) = Y$.*

4.4 Intersections

Theorem 4.24. *(Circle-Segment Intersection or Elementary Circular Continuity) Suppose that γ is a circle, that A is in the interior of γ, and that B is in the exterior of γ. Then $\overline{AB}$ intersects the circle γ.*

Lemma 4.25. *Suppose that γ is a circle centered at C with radius $\overline{CR}$ and that D is any point other than C. Let A be the unique point on $\overrightarrow{CD}$ with $\overline{CA} \cong \overline{CR}$. If X is any point on γ other than A, then $\overline{DA} < \overline{DX}$.*

Lemma 4.26. *Suppose that γ is a circle centered at C with radius $\overline{CR}$ and that D is any point other than C. Let A be the unique point on the ray opposite $\overrightarrow{CD}$ with $\overline{CA} \cong \overline{CR}$. If X is any point on γ other than A, then $\overline{DA} > \overline{DX}$.*

Lemma 4.27. *Suppose that γ is the circle centered at C with radius $\overline{CR}$ and that δ is any circle centered at a point $D \neq C$. If γ contains a point in the interior of δ and a point in the exterior of δ, then there is a point A on $\overleftrightarrow{CD}$ and points B and B' on either side of $\overleftrightarrow{CD}$ so that*

- *A, B, and B' are on γ.*
- *Either A is in the interior of δ and both B and B' are in the exterior of δ or A is in the exterior of δ and both B and B' are in the interior.*
- *Angles $\angle ACB$ and $\angle ACB'$ are right angles.*

Theorem 4.28 (Circle-Circle Intersection or Circular Continuity)**.** *Suppose that γ and δ are two circles. If γ contains a point in the interior of δ and if γ contains a point in the exterior of δ, then γ and δ intersect in two points which are on opposite sides of the line connecting the centers of the circles.*

4.5 Segment Measurement

Definition. Let O and I be any two distinct points.

Definition. The *dyadic rationals* are all real numbers of the form $\frac{a}{2^b}$ where a is any integer and b is a nonnegative intger. Denote the set of dyadic rationals by $\mathbb{D}$.

Definition. For any nonnegative integer n and any point P on $\overleftrightarrow{OI}$, we define a point $n \cdot P$. First $n \cdot O = O$ for all nonnegative integers n. Let $P \neq O$ be a point on $\overleftrightarrow{OI}$. We use recursion to define $n\cdot$.

- $0 \cdot P = O$
- $1 \cdot P = P$.
- If $n \geq 1$ and $n \cdot P$ is defined, then $(n+1) \cdot P$ is the unique point on the ray opposite $\overrightarrow{(n \cdot P)O}$ so that $\overline{(n \cdot P)((n+1) \cdot P)} \cong \overline{OP}$.

Note. Think of $n \cdot P$ as the point you arrive at by starting off at O and marking off n segments congruent to P in the direction of $\overrightarrow{OP}$.

Proposition 4.29 (Archimedean Principle)**.** *Suppose that P is any point in $\overrightarrow{OI}$ and that Q is any point in $\overrightarrow{OI}$ different from O. There is a positive integer n so that $-I * P * n \cdot Q$.*

Note. We use $-I$ here only because we are allowing P to be O. In that case, we cannot write $O * O * n \cdot Q$.

Definition. For any nonnegative integer n and any point P on $\overleftrightarrow{OI}$ we define a point $\frac{1}{2^n} \cdot P$ on $\overleftrightarrow{OI}$. First $\frac{1}{2^n} \cdot O = O$ for all nonnegative integers n. Let $P \neq O$ be a point on $\overleftrightarrow{OI}$. We use recursion to define $\frac{1}{2^n} \cdot P$.

- $\frac{1}{2^0} \cdot P = P$
- If $n \geq 0$ and $\frac{1}{2^n} \cdot P$ is defined, then $\frac{1}{2^{(n+1)}} \cdot P$ is the midpoint of $\overline{O\left(\frac{1}{2^n}P\right)}$.

Note. To find $\frac{1}{2^n} \cdot P$ we first find the midpoint $\frac{1}{2} \cdot P$ of $\overline{OP}$. Then we find the midpoint $\frac{1}{2^2} \cdot P$ of $\overline{O(\frac{1}{2} \cdot P)}$. We continue until we have found n midpoints.

Proposition 4.30 (Archimedean Principle for Halving)**.** *Suppose that P and Q are points $\overrightarrow{OI}$ other than O. There is a nonnegative integer n so that $0 * \left[\left(\frac{1}{2^n}\right) \cdot Q\right] * P$.*

Definition. If a and b are nonnegative integers, then we use $\frac{a}{2^b} \cdot P$ to represent $a \cdot \left(\frac{1}{2^b} \cdot P\right)$.

Proposition 4.31. *Suppose that $P \neq Q$ are points on $\overrightarrow{OI}$. There are positive integers a an b so that $P * \left[\frac{a}{2^b} \cdot I\right] * Q$.*

Proposition 4.32. *Suppose that a, b, c, and d are nonnegative integers. Then $\frac{a}{2^b} < \frac{c}{2^d}$ if and only if $-I * \frac{a}{2^b} \cdot I * \frac{c}{2^d} \cdot I$.*

Definition. For each point P on $\overrightarrow{OI}$ and each nonnegative integer k, let $m_k(P)$ be the greatest nonnegative integer so that either $P = \left(\frac{m_k(P)}{2^k} \cdot I\right)$ or $-I * \left(\frac{m_k(P)}{2^k} \cdot I\right) * P$.

Lemma 4.33. *For any P, the sequence $\left\langle \frac{m_k(P)}{2^k} \right\rangle$ is bounded and increasing (nondecreasing) and therefore converges.*

Definition. Define a function $\# : \overrightarrow{OI} \to \mathbb{R}$ by $\#(P) = \lim_{k\to\infty} \frac{m_k(P)}{2^k}$.

Theorem 4.34. *The following hold for all $P, Q \in \overrightarrow{OI}$.*

- *$\#(P)$ is a nonnegative real number.*
- $\#\left(\frac{a}{2^b} \cdot I\right) = \frac{a}{2^b}$.
- $\#(O) = 0$
- $\#(I) = 1$
- *$\#(P) < \#(Q)$ if and only if $-I * P * Q$.*

Proposition 4.35. *For every positive real number x there is a point $P \in \overrightarrow{OI}$ so that $\#(P) = x$.*

Definition. If $A \neq B$ are points, let $P_{\overline{AB}}$ be the unique point on $\overline{OI}$ so that $\overline{OP_{\overline{AB}}} \cong \overline{AB}$. If $A = B$, let $P_{\overline{AB}} = O$. The distance between A and B is $d(A, B) = \#P_{\overline{AB}}$. For any line segment $\overline{AB}$, the *length* of $\overline{AB}$ is $|\overline{AB}| = d(A, B)$.

Proposition 4.36. *The following hold for any points P and Q*

$d(P, Q) \geq 0$.

$d(P, Q) = 0$ if and only if $P = Q$.

$d(P, Q) = d(Q, P)$.

Theorem 4.37 (Ruler Theorem)**.** *Suppose that A, B, C, and D are points.*

1. *$|\overline{AB}| > 0$.*
2. *$|AB| = 0$ if and only if $A = B$.*
3. *$\overline{AB} \cong \overline{CD}$ if and only if $|\overline{AB}| = |\overline{CD}|$.*
4. *$\overline{AB} < \overline{CD}$ if and only if $|\overline{AB}| < |\overline{CD}|$.*
5. *If $A * B * C$ then $|\overline{AC}| = |\overline{AB}| + |\overline{BC}|$.*

6. *For every positive real number x, there is a segment $\overline{AB}$ so that $|\overline{AB}| = x$.*

This theorem is proven by the following claimable lemmas.

Lemma 4.38. $|\overline{OI}| = 1$.

Lemma 4.39. *If $\overline{AB}$ and $\overline{CD}$ are segments, then $|\overline{AB}| = |\overline{CD}|$ if and only if $\overline{AB} \cong \overline{CD}$.*

Lemma 4.40. *If $\overline{AB}$ and $\overline{CD}$ are segments, then $|\overline{AB}| < |\overline{CD}|$ if and only if $\overline{AB} < \overline{CD}$.*

Lemma 4.41. *If $\overline{AB}$ is a segment, then $|\overline{AB}| = |\overline{BA}|$.*

Lemma 4.42. *If A, B, and C are three distinct points, then $A * B * C$ if and only if $|\overline{AC}| = |\overline{AB}| + |\overline{BC}|$.*

Lemma 4.43. *For every positive real number x, there is a segment $\overline{AB}$ so that $|\overline{AB}| = x$.*

4.6 Angle Measurement

For segment measurement, we basically chose arbitrary points O and I as a fundamental unit along with an (almost) arbitrary number 1, assigned the length of $\overline{OI}$ to be 1 and derived all other lengths from this assignment using dyadic rationals. We can do the same thing with angles. We can select an arbitrary right angle, assign it the arbitrary measure of 90° (read as *90 degrees*), and then use dyadic rationals to assign a measure between 0 and 180 to every other angle. The one thing we are missing is the notion of Dedekind cut for angles.

Definition. Suppose that $\overleftrightarrow{AB}$ is any line and D is any point not on $\overleftrightarrow{AB}$. Let $\mathbb{F}$ be the set of all angles $\angle CAB$ where C is a point on the same side of $\overleftrightarrow{AB}$ as D. (In case you are wondering, $\mathbb{F}$ is for *fan*.) A Dedekind cut of $\mathbb{F}$ is a partition $\{\Sigma_1, \Sigma_2\}$ of $\mathbb{F}$ into two nonempty disjoint sets called *slices* so that if α and γ are angles in one slice and β is an angle between α and γ, then β is in the same slice as α and γ. A cut point of a Dedekind cut is an angle α in $\mathbb{F}$ so that if β an γ are angles so that β is between α and γ, then β and γ are in the same slice.

Theorem 4.44 (Completeness for Angles)**.** *Every Dedekind cut of $\mathbb{F}$ has a cut point.*

Armed with this completeness theorem, we can define a measure of angles (denoted $|\angle ABC|$) so that the following theorem is true.

Theorem 4.45 (Protractor Theorem)**.** *Suppose that $\angle CAB$ and $\angle ZXY$ are angles.*

1. $0 < |\angle CAB| < 180$.
2. *If $\angle CAB$ is a right angle, then $|\angle CAB| = 90°$.*
3. *$\angle CAB \cong \angle ZXY$ if and only if $|\angle CAB| = |\angle ZXY|$.*
4. *$\angle CAB < \angle ZXY$ if and only if $|\angle CAB| < |\angle ZXY|$.*
5. *If D is interior to $\angle CAB$ then $|\angle CAB| = |\angle CAD| + |\angle DAB|$.*
6. *For every real number x with $0 < x < 180$, there is an angle α so that $|\alpha| = x°$.*

5 Neutral Geometry

5.1 Adding Measurements

Proposition 5.1 (Triangle Inequality)**.** *If A, B, and C are three noncollinear points, then $|\overline{AC}| < |\overline{AB}| + |\overline{BC}|$.*

Proposition 5.2. *If A and B are distinct points and x is a positive real number, then there exists a unique point $C \in \overrightarrow{AB}$ so that $|\overline{AC}| = x$.*

Proposition 5.3. *If A and B are distinct points and x is a real number between 0 and 180, then on either side of $\overleftrightarrow{AB}$ there exists a unique point C with $|\angle CAB| = x°$.*

Proposition 5.4. *Suppose that A, B, C, and D are four distinct points and that C and D are on the same side of $\overleftrightarrow{AB}$. Then $|\angle DAB| < |\angle CAB|$ if and only if D is in the interior of $\angle CAB$.*

Theorem 5.5 (Linear Pair Theorem)**.** *If $\angle DAC$ and $\angle DAB$ are a linear pair then $|\angle DAC| + |\angle DAB| = 180°$.*

5.2 The Saccheri-Legendre Theorem

Definition. The *angle sum* of a triangle $\triangle ABC$ is the sum of the measures of the three interior angles of $\triangle ABC$:

$$\sigma(\triangle ABC) = |\angle ABC| + |\angle BCA| + |\angle CAB|.$$

Proposition 5.6. *Suppose that $\triangle ABC$ is any triangle. Then $|\angle CAB| + |\angle ABC| < 180°$.*

Proposition 5.7. *If $\triangle ABC$ is any triangle and E is a point in the interior of $\overline{BC}$ then*

$$\sigma(\triangle AEB) + \sigma(\triangle AEC) = \sigma(\triangle ABC) + 180°.$$

Proposition 5.8. *If $\triangle ABC$ is a triangle, then there is a point D not on $\overleftrightarrow{AB}$ so that $\sigma(\triangle ABD) = \sigma(\triangle ABC)$ and so that the measure of one of the interior angles of $\triangle ABD$ is less than or equal to $\frac{1}{2}|\angle CAB|$.*

Theorem 5.9 (Saccheri-Legendre Theorem)**.** *If $\triangle ABC$ is any triangle then $\sigma(\triangle ABC) \leq 180°$.*

Corollary 5.10. *The sum of the measures of two interior angles of a triangle is less than or equal to the measure of their remote exterior angle.*

Corollary 5.11 (Converse to Euclid's Fifth Postulate)**.** *Let ℓ and ℓ' be two lines cut by a transversal t. If ℓ and ℓ' meet on one side of t, then the sum of the measures of the two interior angles on that side of t is strictly less than $180°$.*

5.3 More on Quadrilaterals

Definition. If $\square ABCD$ is a convex quadrilateral, then the *angle sum* of $\square ABCD$ is the sum of the measures of the four interior angles of the quadrilateral:

$$\sigma(\square ABCD) = |\angle ABC| + |\angle BCD| + |\angle CDA| + |\angle DAB|.$$

Proposition 5.12. *If $\square ABCD$ is a convex quadrilateral then $\sigma(\square ABCD) \leq 360°$.*

Definition. The *defect* of a triangle $\triangle ABC$ is

$$\delta(\triangle ABC) = 180 - \sigma(\triangle ABC).$$

The *defect* of a convex quadrialteral $\square ABCD$ is

$$\delta(\square ABCD) = 360 - \sigma(\square ABCD).$$

Proposition 5.13 (Additivity of Defect for Triangles)**.** *If $\triangle ABC$ is a triangle and $B * E * C$ then $\delta(\triangle ABC) = \delta(\triangle AEB) + \delta(\triangle AEC)$.*

Proposition 5.14 (Additivity of Defect for Quadrilaterals)**.** *If $\square ABCD$ is a convex quadrilateral, then $\delta(\square ABCD) = \delta(\triangle ACB) + \delta(\triangle ACD)$.*

Definition. A quadrilateral with four right angles is a *rectangle.*

Proposition 5.15. *Every rectangle is a parallelogram.*

Theorem 5.16. *The following are equivalent.*

1. *There is a triangle with defect $0°$.*
2. *There is a right triangle with defect $0°$.*
3. *There is a rectangle.*
4. *There exist rectangles with arbitrarily long sides.*
5. *The defect of every right triangle is $0°$.*
6. *The defect of every triangle is $0°$.*

This theorem is proven through the following sequence of claimable lemmas.

Lemma 5.17 (Part of proof of 5.16)**.** *If there is a triangle with defect $0°$ then there is a right triangle with defect $0°$.*

Lemma 5.18 (Part of proof of 5.16)**.** *If there is a right triangle with defect $0°$ then there is a rectangle.*

Lemma 5.19 (Part of proof of 5.16)**.** *If there is a rectangle then there exist rectangles with arbitrarily long sides.*

Lemma 5.20 (Part of proof of 5.16)**.** *If there exist rectangles with arbitrarily long sides then the defect of every right triangle is $0°$.*

Lemma 5.21 (Part of proof of 5.16)**.** *If the defect of every right triangle is $0°$ then the defect of every triangle is $0°$.*

Lemma 5.22 (Part of proof of 5.16)**.** *If the defect of every triangle is $0°$ thn There is a triangle with defect $0°$.*

Definition. A *Saccheri quadrilateral* is a quadrilateral $\square ABCD$ so that $\angle ABC$ and $\angle DAB$ are right angles and $\overline{AD} \cong \overline{CB}$. The segment $\overline{AB}$ is the *base* and the segment $\overline{CD}$ is the *summit.* The angles at A and B are the *base angles* while the angles at C and D are the *summit angles.*

Proposition 5.23. *There exists a Saccheri quadrilateral.*

Proposition 5.24 (Properties of Saccheri Quadrilaterals)**.** *If $\square ABCD$ is a Saccheri quadrilateral with base $\overline{AB}$ then*

1. *The diagonals $\overline{AC}$ and $\overline{BD}$ are congruent.*
2. *The summit angles $\angle ADC$ and $\angle BCD$ are congruent.*
3. *The segment joining the midpoint of the base $\overline{AB}$ and the midpoint of the summit $\overline{CD}$ is perpendicular to both the base and the summit.*
4. *$\square ABCD$ is a parallelogram.*
5. *$\square ABCD$ is a convex quadrilateral.*
6. *The summit angles $\angle ADC$ and $\angle BCD$ are either right or acute.*

This theorem is proven through the following sequence of claimable lemmas.

Lemma 5.25 (Part of proof of 5.24)**.** *The diagonals $\overline{AC}$ and $\overline{BD}$ are congruent.*

Lemma 5.26 (Part of proof of 5.24)**.** *The summit angles $\angle ADC$ and $\angle BCD$ are congruent.*

Lemma 5.27 (Part of proof of 5.24)**.** *The segment joining the midpoint of the base $\overline{AB}$ and the midpoint of the summit $\overline{CD}$ is perpendicular to both the base and the summit.*

Lemma 5.28 (Part of proof of 5.24)**.** *$\square ABCD$ is a parallelogram.*

Lemma 5.29 (Part of proof of 5.24)**.** *$\square ABCD$ is a convex quadrilateral.*

Lemma 5.30 (Part of proof of 5.24)**.** *The summit angles $\angle ADC$ and $\angle BCD$ are either right or acute.*

Definition. A *Lambert quadrilateral* is a quadrilateral with at least three right angles.

Proposition 5.31. *There exists a Lambert quadrilateral.*

Theorem 5.32 (Properties of Lambert Quadrilaterals)**.** *If $\square ABCD$ is a Lambert quadrilateral with right angles at vertices A, B, and C then*

1. *$\square ABCD$ is a parallelogram.*
2. *$\square ABCD$ is a convex quadrilateral.*
3. *$\angle ADC$ is either right or acute.*
4. *$\overline{BC} < \overline{AD}$ or $\overline{BC} \cong \overline{AD}$.*

This theorem is proven through the following sequence of claimable lemmas.

Lemma 5.33 (Part of proof of 5.32)**.** *$\square ABCD$ is a parallelogram.*

Lemma 5.34 (Part of proof of 5.32)**.** *$\square ABCD$ is a convex quadrilateral.*

Lemma 5.35 (Part of proof of 5.32)**.** *$\angle ADC$ is either right or acute.*

Lemma 5.36 (Part of proof of 5.32)**.** *$\overline{BC} < \overline{AD}$ or $\overline{BC} \cong \overline{AD}$.*

5.4 Neutral Area

Recall that the *interior* of $\triangle ABC$ is the intersection of the interiors of the angles of $\triangle ABC$.

Definition. Suppose that $\triangle ABC$ is a triangle. Denote the interior of $\triangle ABC$ as $\text{Int}\triangle ABC$. The *triangular region associated with* $\triangle ABC$ is the union of the triangle an its interior. We denote this by $\blacktriangle ABC$, so $\blacktriangle ABC = \triangle ABC \cup \text{Int}\triangle ABC$.

Definition. A set $\{T_1, T_2, \ldots, T_n\}$ of triangular regions are *nonoverlapping* if for all $i \neq j$ either $T_i \cap T_j = \emptyset$ or $T_i \cap T_j$ is a subset of and edge of T_i and an edge of T_j.

Proposition 5.37. *If* $\triangle ABC$ *is a triangle and* $E \in \overline{AB}$ *then* $\blacktriangle ABC = \blacktriangle AEC \cup \blacktriangle BEC$

Proposition 5.38. *If* $\square ABCD$ *is a convex quadrilateral, then* $\blacktriangle ACB \cup \blacktriangle ACD = \blacktriangle BDA \cup \blacktriangle BDC$

Proposition 5.39. *The* quadrilateral region *defined by a convex quadrilateral* $\square ABCD$ *is*

$$\blacksquare ABCD = \blacktriangle ACB \cup \blacktriangle ACD = \blacktriangle BDA \cup \blacktriangle BDC.$$

Definition. Suppose that R is a subset of the plane. A *triangulation* of R is a finite set $\{T_1, T_2, \ldots, T_n\}$ of nonoverlapping triangular regions so that $R = T_1 \cup T_2 \cup \cdots \cup T_n$.

Exercise 5.40. Draw two different triangulations of a triangular region.

Exercise 5.41. Draw two different triangulations of a quadrilateral region.

Exercise 5.42. Give an example of a subset of the plane which has no triangulation.

Definition. A *polygonal region* is a subset of the plane that has a triangulation.

Exercise 5.43. Draw a somewhat complex polygonal region and two triangulations of it.

Proposition 5.44. *The intersection of two polygonal regions is a polygonal region.*

Proposition 5.45. *The union of two polygonal regions is a polygonal region.*

Axiom (Neutral Area Postulate)**.** Every polygonal region R is associated with a nonnegative real number $\alpha(R)$ called the *area* of R so that:

- Suppose $\triangle ABC$ and $\triangle DEF$ are two triangles. If $\triangle ABC \cong \triangle DEF$ then $\alpha(\blacktriangle ABC) = \alpha(\blacktriangle DEF)$.
- If R_1 and R_2 are nonoverlapping polygonal regions, then $\alpha(R_1 \cup R_2) = \alpha(R_1) + \alpha(R_2)$.

Proposition 5.46. *Suppose that* k *is any nonnegative real number. For any polygonal region* R*, let* $\beta(R) = k \cdot \alpha(R)$*. Then* β *satisfies the two conditions in the Neutral Area Postulate for* α*.*

Note. This proposition says that if there is an area function α, then any nonnegative multiple of α is also a legitimate area function.

5.5 Statements Equivalent to the Euclidean Parallel Postulate

Here are three options for a parallel postulate:

- **Euclidean Parallel Posulate:** For every line ℓ and for every point P not on ℓ, there is exactly one line m so that P lies on m and $m \parallel \ell$.
- **Elliptic Parallel Posulate:** For every line ℓ and for every point P not on ℓ, there is no line m so that P lies on m and $m \parallel \ell$.
- **Hyperbolic Parallel Posulate:** For every line ℓ and for every point P not on ℓ, there are at least two lines m and n so that P lies on m and n and $m \parallel \ell$ and $n \parallel \ell$.

Note. *Elliptic* means *too few*, and *hyperbolic* means *too many.*

Proposition 5.47. *The Elliptic Parallel Postulate does not hold in Neutral Geometry.*

Note. Below is a list of statements which are equivalent to the Euclidean Parallel Postulate, assuming the axioms of Neutral Geometry. What this means is that if we assume all of the axioms of Neutral Geometry and we assume the Euclidean Parallel Postulate, then we can prove each of these, and if we assume all of the axioms of Neutral Geometry and we assume any one of these, then we can prove the Euclidean Parallel Postulate. It also means that if we have any model of Neutral Geometry, then that model satisfies the Euclidean Parallel Postulate if and only if it satisfies these statements.

Definition. The triangles $\triangle ABC$ and $\triangle DEF$ are *similar* if $\angle ABC \cong \angle DEF$ and $\angle BCA \cong \angle EFD$ and $\angle CAB \cong \angle FDE$. We express this as $\triangle ABC \sim \triangle DEF$.

- **Euclid's Fifth Postulate:** If ℓ and ℓ' are two line cut by a transveral t in such a way that the sum of the measures of the two interior angles one one side of t is less than $180°$, then ℓ and ℓ' intersect on that side of t.
- **Hilbert's Parallel Posulate:** For every line ℓ and for every point P not on ℓ, there is at most one line m so that P lies on m and $m \parallel \ell$.
- **Converse to the Alternating Interior Angles Theorem:** If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.
- **Angle Sum Posulate:** The sum of the angles in any triangle is $180°$.
- **Clairaut's Axiom:** There is a rectangle.
- **Transitivity of Parallelism:** If $\ell \parallel m$ and $m \parallel n$ then either $\ell = n$ or $\ell \parallel n$.
- **Proclus's Axiom:** Supose ℓ and ℓ' are parallel lines and that t is a line other than ℓ. If t intersects ℓ, then t intersects ℓ'.
- **Perpendicular Transversal Condition:** Suppose that ℓ and ℓ' are parallel lines cut by a transversal t. If $t \perp \ell$, then $t \perp \ell'$.
- $\perp \circ \parallel \circ \perp$ **Condition:** If ℓ, m, n, and k are lines so that $m \perp k$, $k \parallel \ell$, and $\ell \perp n$, then either $m = n$ or $m \parallel n$.
- **Wallis's Postulate:** If $\triangle ABC$ is a triangle and $\overline{DE}$ is a segment, then there exists a point F so that $\triangle ABC \sim \triangle DEF$.

- **The Pythagorean Theorem:** Suppose that $\triangle ABC$ is a right triangle with right angle at C. Then $|\overline{AB}|^2 = |\overline{AC}|^2 + |\overline{BC}|^2$.
- **Circumscribed Triangle Condition:** Every triangle can be circumscribed.

Theorem 5.48. *The Euclidean Parallel Postulate is equivalent to Euclid's Fifth Postulate.*

This is proven by the following claimable lemmas.

Lemma 5.49 (Part of the proof of 5.48)**.** *The Euclidean Parallel Postulate implies Euclid's Fifth Postulate.*

Lemma 5.50 (Part of the proof of 5.48)**.** *Euclid's Fifth Postulate implies the Euclidean Parallel Postulate.*

Theorem 5.51. *The following statements are equivalent to the Euclidean Parallel Postulate.*

- *The Converse to the Interior Angles Theorem*
- *The Angle Sum Postulate*
- *Clairaut's Axiom*

This is proven by the following claimable lemmas.

Lemma 5.52 (Part of the proof of 5.51)**.** *The Euclidean Parallel Postulate implies the Converse to the Interior Angles Theorem.*

Lemma 5.53 (Part of the proof of 5.51)**.** *The Converse to the Interiors Angles Theorem implies the Angle Sum Postulate.*

Lemma 5.54 (Part of the proof of 5.51)**.** *Suppose that $\overline{PQ}$ is a segment and that Q' is a point so that $\angle PQQ'$ is a right angle. For every $\epsilon > 0$ there is a point T on $\overrightarrow{QQ'}$ so that $|\angle PTQ| < \epsilon$.*

Lemma 5.55 (Part of the proof of 5.51)**.** *The Angle Sum Postulate implies the Euclidean Parallel Postulate.*

Lemma 5.56 (Part of the proof of 5.51)**.** *The Angle Sum Postulate is Equivalent to Clairaut's Axiom.*

Theorem 5.57. *The Euclidean Parallel Postulate is equivalent to Hilbert's Parallel Postulate.*

Theorem 5.58. *The Euclidean Parallel Postulate is equivalent to the Transitivity of Parallelism.*

This is proven by the following claimable lemmas.

Lemma 5.59 (Part of the proof of 5.58)**.** *The Euclidean Parallel Postulate implies the Transitivity of Parallelism.*

Lemma 5.60 (Part of the proof of 5.58)**.** *Transitivity of Parallelism implies the Euclidean Parallel Postulate.*

Theorem 5.61. *The Euclidean Parallel Postulate is equivalent to Proclus's Axiom.*

This is proven by the following claimable lemmas.

Lemma 5.62 (Part of the proof of 5.61)**.** *The Euclidean Parallel Postulate implies Proclus's Axiom.*

Lemma 5.63 (Part of the proof of 5.61)**.** *Proclus's Axiom implies the Euclidean Parallel Postulate.*

Theorem 5.64. *The Euclidean Parallel Postulate is equivalent to the Perpendicular Transversal Condition.*

This is proven by the following claimable lemmas.

Lemma 5.65 (Part of the proof of 5.64)**.** *The Converse to the Alternate Interior Angle Theorem implies the Perpendicular Transversal Condition.*

Lemma 5.66 (Part of the proof of 5.64)**.** *The Perpendicular Transversal Condition implies the Euclidean Parallel Postulate.*

Theorem 5.67. *The Euclidean Parallel Postulate is equivalent to the* $\perp \circ \parallel \circ \perp$ *Condition.*

This is proven by the following claimable lemmas.

Lemma 5.68 (Part of the proof of 5.67)**.** *Proclus's Axiom and the Perpendicular Transversal Condition together imply the* $\perp \circ \parallel \circ \perp$ *Condition.*

Lemma 5.69 (Part of the proof of 5.67)**.** *The* $\perp \circ \parallel \circ \perp$ *Condition implies the Euclidean Parallel Postulate.*

Theorem 5.70. *The Euclidean Parallel Postulate is equivalent to Wallis's Postulate.*

This is proven by the following claimable lemmas.

Lemma 5.71 (Part of the proof of 5.70)**.** *Euclid's Fifth Postulate and the Angle Sum Postulate together imply Wallis's Postulate.*

Lemma 5.72 (Part of the proof of 5.70)**.** *Wallis's Postulate implies the Euclidean Parallel Postulate.*

5.6 The Euclidean Parallel Postulate vs. The Hyperbolic Parallel Postulate

Proposition 5.73. *In any model of Neutral Geometry, if the Hyperbolic Parallel Postulate holds, then the Euclidean Parallel Postulate fails.*

Proposition 5.74. *In any model of Neutral Geometry, if the Euclidean Parallel Postulate fails, then the Hyperbolic Parallel Postulate holds.*

Note. At first glance 5.74 seems not to say much - if there are more than one parallel, then there are at least two. However, what the results says is that if the Euclidean Parallel Postulate fails anywhere in the geometry, then it fails everywhere. There are not some exterior points with just one parallel and others with two or more.

Corollary 5.75. *In any model of Neutral Geometry, either the Euclidean Parallel Postulate holds or the Hyperbolic Parallel Postulate holds.*

6 A Little Euclidean Geometry

In this section, we assume the Euclidean Parallel Postulate:

Axiom (Euclidean Parallel Posulate)**.** For every line ℓ and for every point P not on ℓ, there is exactly one line m so that P lies on m and $m \parallel \ell$.

Note. Since we are assuming the Euclidean Parallel Postulate, we can make use of all of the statements proven to be equivalent the the Euclidean Parallel Postulate in the last section.

6.1 Basic Parallelograms and Triangles

Proposition 6.1. *If $\square ABCD$ is a parallelogram, then $\triangle ABC \cong \triangle CDA$ and $\triangle ABD \cong \triangle CDB$.*

Proposition 6.2. *If $\square ABCD$ is a parallelogram, then $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$.*

Proposition 6.3. *If $\square ABCD$ is a parallelogram, then $\angle ABC \cong \angle CDA$ and $\angle DAB \cong \angle BCD$.*

Proposition 6.4. *If $\square ABCD$ is a parallelogram, then $\overline{AC}$ and $\overline{BD}$ intersect at a point M which is the midpoint of both $\overline{AC}$ and $\overline{BD}$.*

Proposition 6.5. *The sum of the angles in any convex quadrilateral is $360°$.*

Proposition 6.6. *Every Saccherri quadrilateral is a rectangle.*

Proposition 6.7. *Every Lambert quadrilateral is a rectangle.*

Proposition 6.8. *In any equilateral triangle, all of the angles are $60°$.*

Proposition 6.9. *There is an equilateral triangle.*

Proposition 6.10. *There is a triangle whose angles measure $30°$, $60°$, and $90°$.*

Proposition 6.11. *A right triangle is isosceles if and only if the angles of the triangle measure $45°$, $45°$, and $90°$.*

Proposition 6.12. *Suppose that ℓ and m are parallel lines, that $P, Q \in \ell$, and that R and S are, respectively, the feet of the perpendiculars from O and Q to m. Then $\overline{PR} \cong \overline{QS}$.*

Proposition 6.13. *Suppose that ℓ and m are different lines, that $P, Q \in \ell$ are on the same side of m, and that R and S are, respectively, the feet of the perpendiculars from O and Q to m. If $\overline{PR} \cong \overline{QS}$ then $\ell \parallel m$.*

Proposition 6.14. *If there are three positive numbers that add to 180, then there is a triangle in which the measures of the angles are those three positive numbers.*

6.2 The Parallel Projection Theorem

Proposition 6.15 (Equal Projections)**.** *Let ℓ, m, and n be distinct parallel lines. Let t be a transversal that cuts these lines at points A, B, and C, respectively, and let t' be a transversal that cuts these lines at points A', B', and C', respectively. Assume that $A * B * C$. If $\overline{AB} \cong \overline{BC}$ then $\overline{A'B'} \cong \overline{B'C'}$.*

Proposition 6.16 (Parallel Projection for Integer Ratios)**.** *Let ℓ, m, and n be distinct parallel lines. Let t be a transversal that cuts these lines at points A, B, and C, respectively, and let t' be a transversal that cuts these lines at points A', B', and C', respectively. Assume that $A * B * C$. If $\frac{|\overline{AC}|}{|\overline{AB}|} = n$ for some positive integer n then $\frac{|\overline{A'C'}|}{|\overline{A'B'}|} = n$.*

Note. Note that 6.15 is the special case of 6.16 when $n = 2$.

Proposition 6.17 (Parallel Projection for Rational Ratios)**.** *Let ℓ, m, and n be distinct parallel lines. Let t be a transversal that cuts these lines at points A, B, and C, respectively, and let t' be a transversal that cuts these lines at points A', B', and C', respectively. Assume that $A * B * C$. If $\frac{|\overline{AC}|}{|\overline{AB}|} = \frac{n}{m}$ for some positive integers n and m then $\frac{|\overline{A'C'}|}{|\overline{A'B'}|} = \frac{n}{m}$.*

Theorem 6.18 (Parallel Projection Theorem)**.** *Let ℓ, m, and n be distinct parallel lines. Let t be a transversal that cuts these lines at points A, B, and C, respectively, and let t' be a transversal that cuts these lines at points A', B', and C', respectively. Assume that $A * B * C$. Then $\frac{|\overline{AC}|}{|\overline{AB}|} = \frac{|\overline{A'C'}|}{|\overline{A'B'}|}$.*

Corollary 6.19. *Let ℓ, m, and n be distinct parallel lines. Let t be a transversal that cuts these lines at points A, B, and C, respectively, and let t' be a transversal that cuts these lines at points A', B', and C', respectively. Assume that $A * B * C$. Then $\frac{|\overline{AC}|}{|\overline{A'C'}|} = \frac{|\overline{AB}|}{|\overline{A'B'}|}$.*

6.3 Similar Triangles

Theorem 6.20 (Similar Triangles Theorem)**.** *Suppose that $\triangle ABC \sim \triangle DEF$. Then $\frac{|\overline{AB}|}{|\overline{DE}|} = \frac{|\overline{AC}|}{|\overline{DF}|} = \frac{|\overline{BC}|}{|\overline{EF}|}$.*

Corollary 6.21. *Suppose that $\triangle ABC \sim \triangle DEF$. There is a real number r so that $|\overline{DE}| = r \cdot |\overline{AB}|$, $|\overline{EF}| = r \cdot |\overline{BC}|$, and $|\overline{FD}| = r \cdot |\overline{CA}|$.*

Proposition 6.22 (SAS Similarity)**.** *If $\triangle ABC$ and $\triangle DEF$ are triangles so that $\angle CAB \cong \angle FDE$ and $\frac{|\overline{AC}|}{|\overline{AB}|} = \frac{|\overline{DF}|}{|\overline{DE}|}$ then $\triangle ABC \sim \triangle DEF$*

Proposition 6.23 (Converse to Similar Triangles Theorem)**.** *Suppose that $\triangle ABC$ and $\triangle DEF$ are triangles. If $\frac{|\overline{AB}|}{|\overline{DE}|} = \frac{|\overline{AC}|}{|\overline{DF}|} = \frac{|\overline{BC}|}{|\overline{EF}|}$ then $\triangle ABC \sim \triangle DEF$.*

6.4 Euclidean Area

Note. It is cumbersome always to refer to the area of a triangular region. Therefore, when we are working with a triangular region or a quadrilateral region, we will refer to the area of the triangle or the area of the quadrilateral when we really mean the area of the associated region.

Proposition 6.24. *Suppose that $\triangle ABC$ is a triangle. Let D be the foot of the perpendicular from C to $\overleftrightarrow{AB}$, and let E be the foot of the perpendicular from B to $\overleftrightarrow{AC}$. Then $|\overline{AB}| \cdot |\overline{CD}| = |\overline{AC}| \cdot |\overline{EB}|$.*

Definition. Let $\triangle ABC$ be an oriented triangle, and let D be the foot of the perpendicular from C to A. The *base* of $\triangle ABC$ is $\overline{AB}$ and the *height* is $|\overline{CD}|$.

Note. Proposition 6.24 tells us that the product (*length of base*) $\cdot$ *height* is the same for every orientation of $\triangle ABC$.

Definition. A *square* is a rectangle all of whose sides have the same length.

Definition. Suppose that $\square ABCD$ is a square with side length 1. For the rest of this section, let K be $\alpha(\square ABCD)$, the area of a square with side length 1.

Lemma 6.25. *Suppose that d is a positive integer. The area of a square with side length $\frac{1}{d}$ is $K \cdot \frac{1}{d^2}$.*

Lemma 6.26. *Suppose that a, c, and d are positive integers. A rectangle with a side of length $\frac{a}{d}$ and a side of length $\frac{c}{d}$ has area $K \cdot \frac{a}{d} \cdot cd$.*

Lemma 6.27. *Suppose that $\square ABCD$ is a rectangle so that $|\overline{AB}|$ and $\overline{CD}$ are rational. The area of $\square ABCD$ is $K \cdot |\overline{AB}| \cdot |\overline{CD}|$.*

Theorem 6.28. *The area of a rectangle $\square ABCD$ is $K \cdot |\overline{AB}| \cdot |\overline{CD}|$.*

Note. This theorem guarantees that the area of a rectangle $\square ABCD$ is a constant times $|\overline{AB}| \cdot |\overline{CD}|$. The constant in question is the area of a square with edge length 1. We know from 5.46 that any multiple of an area function is a legitimate area function, so we can basically choose the area of a square of edge length 1 to be anything we want. We choose 1.

Axiom (Euclidean Area Postulate)**.** The area of a square with edge length 1 is 1.

Theorem 6.29 (Euclidean Area)**.** *The area of a rectangle $\square ABCD$ is $\alpha(\square BCD) = |\overline{AB}| \cdot |\overline{CD}|$.*

Proposition 6.30. *The area of a right triangle is one half the product of the lengths of its legs.*

Proposition 6.31. *The area of a triangle is one half the product of the length of its base and its height.*

Proposition 6.32. *Suppose that $\triangle ABC$ and $\triangle DEF$ are similar triangles and let $r = \frac{|\overline{DE}|}{|\overline{AB}|}$. Then $\alpha(\triangle DEF) = r^2 \alpha(\triangle ABC)$.*

6.5 The Pythagorean Theorem

Theorem 6.33 (The Pythagorean Theorem)**.** *Suppose that $\triangle ABC$ is a right triangle with right angle at C. Then $|\overline{AB}|^2 = |\overline{AC}|^2 + |\overline{BC}|^2$.*

Note. It appears that the original proofs of the Pythagorean Theorem used similarity. However, the Pythagoreans discovered that not all real numbers are rational, so there was some skepticism attached to similarity proofs. By the time of Euclid, proofs of the Pythagorean Theorem relied on area arguments.

Proposition 6.34. *In Neutral Geometry (not assuming the Euclidean Parallel Postulate) the Pythagorean Theorem implies the Euclidean Parallel Postulate.*

Proposition 6.35 (Converse to the Pythagorean Theorem)**.** *Suppose that $\triangle ABC$ is a triangle in which $|\overline{AB}|^2 = |\overline{AC}|^2 + |\overline{BC}|^2$. The $\angle BCA$ is a right angle.*

6.6 Circles

Proposition 6.36. *In Euclidean Geometry, every triangle can be circumscribed.*

Proposition 6.37. *If the Euclidean Parallel Postulate fails, then there is a triangle that cannot be circumscribed.*

Theorem 6.38. *The Euclidean Parallel Postulate is equivalent to the statement that every triangle can be circumscribed.*

Proposition 6.39 (Thales's Theorem)**.**

Suppose that M is the midpoint of $\overline{AB}$ in a triangle $\triangle ABC$. If $\overline{AM} \cong \overline{MC}$ then the angle $\angle ACB$ is a right angle.

Proposition 6.40. *Suppose that M is the midpoint of $\overline{AB}$ in a triangle $\triangle ABC$. If $\angle ACB$ is a right triangle, then $\overline{AM} \cong \overline{MC}$.*

Proposition 6.41. *If the angles in a triangle are $30°$, $60°$, and $90°$. Then the length of the side opposite the $30°$ angle is half the length of the side opposite the $90°$ angle.*

Proposition 6.42. *If a right triangle is such that the length of one leg is half the length of the hypotenuse, then the angles in the triangle are $30°$, $60°$, and $90°$.*

Definition. Suppose γ is a circle with center C. An *inscribed angle* of γ is an angle $\angle PQR$ where P, Q, and R are on γ. The *arc intercepted* by an inscribed angle $\angle PQR$ is the set of all points on γ which are in the interior of $\angle PQR$. A *central angle* of γ is an angle of the form $\angle PCR$ where P and R are on γ.

Definition. Suppose $\angle PQR$ is an inscribed angle of a circle γ with center C. If P and Q are on opposite sides of $\overleftrightarrow{RC}$ or if R and Q are on opposite sides of $\overleftrightarrow{PC}$ then $\angle PCR$ is the *corresponding angle* of $\angle PQR$.

Exercise 6.43. Draw an inscribed angle which does not have a corresponding angle.

Theorem 6.44 (Central Angle Theorem)**.** *The measure of an angle corresponding to an inscribed angle is twice the measure of the inscribed angle.*

6.7 Dissection Theory

Definition. Two polygonal regions R and R' are *equivalent by dissection* if there are triangulations $R = T_1 \cup T_2 \cup \cdots \cup T_n$ and $R' = T_1' \cup T_2' \cup \cdots \cup T_n'$ containing the same number of triangles so that for each i, $T_i \cong T_i'$.

Lemma 6.45. *If R is any polygonal region, then $R \equiv R$.*

Lemma 6.46. *If R_1 and R_2 are polygonal regions and $R_1 \equiv R_2$, then $R_2 \equiv R_1$.*

Lemma 6.47. *If R_1, R_2, and R_3 are polygonal regions so that $R_1 \equiv R_2$ and $R_2 \equiv R_3$ then $R_1 \equiv R_3$.*

Lemma 6.48. *Every triangle is equivalent by dissection to a rectangle.*

Lemma 6.49. *If $\square ABCD$ is a rectangle and $1 \leq |\overline{BC}| < 2$ then $\square ABCD$ is equivalent by dissection to a rectangle with an edge of length 1.*

Lemma 6.50. *Any rectangle is equivalent by dissection to a rectangle with a side of length greater than 2.*

Lemma 6.51. *Any rectangle with a side of length greater than two is equivalent by dissection to the union of a rectangle with a side of length 1 and a rectangle with a side of length less than 2 but greater than or equal to 1.*

Lemma 6.52. *Any rectangle R is equivalent by dissection to a rectangle with edge lengths 1 and $\alpha(R)$.*

Lemma 6.53. *Every polygonal region R is equivalent by dissection to a rectangle with edge lengths 1 and $\alpha(R)$.*

Theorem 6.54 (Fundamental Theorem of Dissection Theory)**.** *Two polygonal regions are equivalent by dissection if and only if they have the same area.*

7 Even Less Hyperbolic Geometry

In this section, we assume the Hyperbolic Parallel Postulate:

Axiom (Hyperbolic Parallel Posulate)**.** For every line ℓ and for every point P not on ℓ, there are at least two lines m and n so that P lies on m and n and $m \parallel \ell$ and $n \parallel \ell$.

7.1 Triangles and Quadrilaterals

Proposition 7.1. *If $\triangle ABC$ is a triangle, then $\sigma(\triangle ABC) < 180°$ and $0° < \delta(\triangle ABC) < 180°$.*

Proposition 7.2. *For every convex quadrilateral $\square ABCD$, $\sigma(\square ABCD) < 360°$.*

Proposition 7.3. *There is no rectangle.*

Proposition 7.4. *The fourth angle in a Lambert quadrilateral is acute.*

Proposition 7.5. *The summit angles in a Saccheri quadrilateral are acute.*

Proposition 7.6. *In a Lambert quadrilateral, the length of a side between two right angles is less than the length of the opposite side.*

Definition. The segment joining the midpoints of the base and summit of a Saccheri quadrilateral is called the *altitude* of the quadrilateral. Its length is called the *height* of the quadrilateral.

Proposition 7.7. *In a Saccheri quadrilateral, the altitude is shorter than either side.*

Proposition 7.8. *In any Saccheri quadrilateral, the summit is longer than the base.*

Proposition 7.9 (AAA Congruence)**.** *Suppose $\triangle ABC$ and $\triangle DEF$ are triangles. If $\triangle ABC \sim \triangle DEF$ then $\triangle ABC \cong \triangle DEF$.*

Proposition 7.10. *Any two Saccheri quadrilaterals with congruent summits and equal defects are congruent.*

7.2 Common Perpendiculars

Definition. Suppose that P is a point not on a line ℓ. The *distance* from P to ℓ is $|\overline{PF}|$ where F is the foot of the perpendicular from P to ℓ. We denote this as $d(P, \ell)$.

Proposition 7.11. *Suppose that ℓ and m are distinct lines. There are at most two points P and Q on m with $d(P, \ell) = d(Q, \ell)$.*

Definition. Two lines ℓ and m *admit a common perpendicular* if there is a line n so that $n \perp \ell$ and $n \perp m$.

Proposition 7.12. *If $\ell \parallel m$ and if there are two points on m that are the same distance from ℓ, then ℓ and m admit a common perpedicular.*

Proposition 7.13. *If lines ℓ and m admit a common perpendicular, then they admit exactly one common perpendicular.*

Proposition 7.14. *Let ℓ and m be parallel lines cut by a transversal t. Alternate interior angles are congruent if and only if ℓ and m admit a common perpendicular and t intersects the perpendicular segment at its midpoint.*

This proposition is proven by the following claimable lemmas.

Lemma 7.15 (Part of the proof of 7.14)**.** *Let ℓ and m be parallel lines cut by a transversal t. If both pairs of alternate interior angles formed by t are congruent then ℓ and m admit a common perpendicular, and t intersects the perpendicular segment at its midpoint.*

Lemma 7.16 (Part of the proof of 7.14)**.** *Let ℓ and m be parallel lines cut by a transversal t. If ℓ and m admit a common perpendicular and t intersects the perpendicular segment at its midpoint then the alternate interior angles formed by t are congruent.*