An Introduction to Real Analysis

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Contents

Chapter 1

The Real Numbers

1.1 Preliminaries

The purpose of these notes is to begin with minimal assumptions and to build enough of the machinery for calculus to prove the Fundamental Theorem of Calculus. We assume that the reader is familiar with basic logic, set theory, and proof techniques including:

- logical operators
- logical equivalences
- rules of inference
- quantifiers
- set operations
- set builder notation
- functions
- inverse functions
- injectivity, surjectivity, bijectivity
- relations
- reflexivity, symmetry, transitivity
- equivalence relations
- cardinality
- proof
- direct proof
- proof of biconditionals
- cases
- proof of disjunctions
- proof by contrapositive
- proof by contradiction
- proofs about subsets
- proofs about set equality
- proofs about cardinality
- Mathematical Induction

We use this notation for number systems:

- Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
- Integers: $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$
- Rational Numbers: $\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$
- Real Numbers: $\mathbb{R}$
- Irrational Numbers: $\mathbb{R} - \mathbb{Q}$

This first chapter lays the foundation on which we will build throughout the rest of the notes. We treat the real numbers as a set with two operations (addition and multiplication) and an order relation satisfying a handful of nice properties. These nice properties declare that $\mathbb{R}$ is an **ordered field**. Most of the results in this chapter should be well-known to anyone studying this material. Theorems and proofs in this chapter are included both as a warm-up and to demonstrate that we can accomplish our goals beginning from a small set of assumptions. (The motivated student might want to begin with the Peano Axioms for $\mathbb{N}$, define $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ from $\mathbb{N}$ and derive these properties.)

The bare minimum which is essential to take away from this chapter aside from the usual arithmetic in $\mathbb{R}$ is: the Triangle Inequality 1.3.3 and Corollary 1.3.4; Theorem 1.3.5; the concepts of bounds, infima, and suprema; The Completeness Axiom 1.4.27; and the density of $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ in $\mathbb{R}$.

1.2 Ordered Field Axioms

We assume that the real numbers are a set $\mathbb{R}$ along with two binary operations, addition $+$ and multiplication $\cdot$, and a binary relation, less than $<$, which satisfy these properties.

Axiom 1.2.1. (Ordered Field Axioms)

A1. For all $x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.

A2. For all $x, y \in \mathbb{R}$, $x + y = y + x$.

A3. There is a unique element $0 \in \mathbb{R}$ *so that* $0 + x = x$ *for all* $x \in \mathbb{R}$.

A4. For each $x \in \mathbb{R}$ *there is an element* $-x \in \mathbb{R}$ *so that* $x + (-x) = 0$.

M1. For all $x, y, z \in \mathbb{R}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

M2. For all $x, y \in \mathbb{R}$, $x \cdot y = y \cdot x$.

M3. There is a unique element $1 \in \mathbb{R}$ *so that* $1 \cdot x = x$ *for all* $x \in \mathbb{R}$.

M4. For each $x \neq 0$ *in* $\mathbb{R}$ *there is an* $x^{-1} \in \mathbb{R}$ *so that* $x \cdot x^{-1} = 1$.

DL. For all $x, y, z \in \mathbb{R}$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

O1. For all $x, y \in \mathbb{R}$, *exactly one of these three relations holds:*

$$x < y \text{ or } y < x \text{ or } x = y.$$

O2. For all $x, y, z \in \mathbb{R}$, *if* $x < y$ *and* $y < z$ *then* $x < z$.

O3. For all $x, y, z \in \mathbb{R}$, *if* $x < y$ *then* $x + z < y + z$.

O4. For all $x, y, z \in \mathbb{R}$, *if* $x < y$ *and* $0 < z$ *then* $x \cdot z < y \cdot z$.

We will call $-x$ the *additive inverse* or the *opposite* of x. We call x^{-1} the *multiplicative inverse* or *reciprocal* or x. Before defining these notions, we should really have this lemma.

Lemma 1.2.2. (Uniqueness of Inverses) *Additive and multiplicative inverses are unique.*

Proof. We prove additive inverses are unique. Suppose that $x, y, z \in \mathbb{R}$ so that $x + y = 0$ and $x + z = 0$. By A2, we also know that $y + x = 0$. Then

$$\begin{aligned}
z &= 0 + z && A3 \\
&= (y + x) + z && \text{Assumption} \\
&= y + (x + z) && A1 \\
&= y + 0 && \text{Assumption} \\
&= 0 + y && A2 \\
&= y && A3
\end{aligned}$$

$\square$

Note that a consequence of this lemma is that the inverse of the inverse of x is x. That is, x is the unique element that can be added to $-x$ to get 0. This unique element is also called $-(-x)$, so $x = -(-x)$. A similar comment holds for multiplication.

The notations $x - y$ and $\dfrac{x}{y}$ are abbreviations for $x + (-y)$ and $x \cdot y^{-1}$. We typically use juxtaposition xy to indicate the multiplication $x \cdot y$, and we agree that multiplication takes precedence over addition to decrease the number of parentheses we must write. We use $x \leq y$ to abbreviate "$x < y$ or $x = y$." Note that $O2$, $O3$, and $O4$ also hold for $\leq$.

Lemma 1.2.3. (Cancellation) *For all* $x, y, z \in \mathbb{R}$

1. *If* $x + z = y + z$ *then* $x = y$.

2. *If* $xz = yz$ *and* $z \neq 0$ *then* $x = y$.

Proof. We prove (1). Suppose that $x + z = y + z$. Adding $-z$ gives $(x+z)-z = (y+z)-z$. $A1$ gives $x+(z-z) = y+(z-z)$, which implies by $A4$ that $x + 0 = y + 0$. By $A3$, this now implies that $x = y$. □

Lemma 1.2.4. (Multiplication by Zero) *For all* $x \in \mathbb{R}$, $x \cdot 0 = 0$.

Proof. Note that

$$x \cdot 0 + 0 = x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$$

by $A3$ (twice) and then DL. Now $x \cdot 0 + 0 = x \cdot 0 + x \cdot 0$. By cancellation, $x \cdot 0 = 0$. □

Lemma 1.2.5. (Additive Inverses and Multiplication) *For all* $x, y \in \mathbb{R}$

1. $(-x)y = -(xy) = x(-y)$.

2. $(-x)(-y) = xy$.

Proof. We will prove the second equality in (1). The first equality is proven similarly, and (2) follows from two applications of (1) along with the uniqueness of inverses. To prove that $x(-y)$ is $-(xy)$, we just need to show that $xy + x(-y) = 0$. Observe:

$$\begin{aligned} xy + x(-y) &= x(y + (-y)) && DL \\ &= x \cdot 0 && A4 \\ &= 0. && \textit{Lemma}\ 1.2.4 \end{aligned}$$

□

A consequence of this lemma is that $-x = -(1 \cdot x) = (-1) \cdot x$.

Lemma 1.2.6. (Divisors of Zero) *For all $x, y \in \mathbb{R}$, if $xy = 0$ then either $x = 0$ or $y = 0$*

Proof. Suppose that $xy = 0$ and $x \neq 0$. Since $x \neq 0$, x has a multiplicative inverse x^{-1}. Multiplying $xy = 0$ on both sides by x^{-1} and applying $M4$ and Lemma 1.2.4 gives $y = 0$. Thus, if $x \neq 0$, then $y = 0$. This is equivalent to either $x = 0$ or $y = 0$. □

Lemma 1.2.7. (Multiplication and Order) *For all $x, y, z \in \mathbb{R}$*

1. *If $x < y$ and $z < 0$ then $yz < xz$.*
2. *If $0 < x$ and $0 < y$ then $0 < xy$.*
3. *If $x < 0$ and $y < 0$ then $0 < xy$.*
4. *If $x < 0$ and $0 < y$ then $xy < 0$.*
5. *If $x \neq 0$, then $0 < x^2$.*
6. $0 < 1$.

Proof. We first prove (1). If $z < 0$, then by $O3$ we know that

$$z + (-z) < 0 + (-z).$$

Then $0 < -z$. Now by $O4$ we can multiply both sides of $x < y$ to get $x(-z) < y(-z)$. By Lemma 1.2.5, this is the same as $-xz < -yz$. Applying $O3$ and adding $xz + yz$ to both sides of this inequality gives $-xz + (xz + yz) < -yz + (xz + yz)$. This now reduces (via $A1 - 4$) to $yz < xz$. Assertion (2) follows directly from $O4$. For (3), suppose that $x < 0$ and $y < 0$. By adding $-x$ and $-y$ to these inequalities, we get $0 < -x$ and $0 < -y$. Now (2) and Lemma 1.2.5 give $0 < xy$. Statement (4) can be proven similarly.

If $x \neq 0$, then by $O1$ either $x < 0$ or $0 < x$. If $0 < x$ then (5) is a direct application of (2). If $x < 0$ then adding $-x$ by $O3$ gives $0 < -x$. Now applying (2) gives $0 < (-x)(-x)$, but $(-x)(-x) = x^2$ by Lemma 1.2.5. In either case, $0 < x^2$ when $x \neq 0$. Part (6) now follows immediately from (5). □

Lemma 1.2.8. (Inverses and Order) *For all $x, y, z \in \mathbb{R}$*

1. *If $x < y$ then $-y < -x$.*
2. *If $0 < x$ then $0 < x^{-1}$.*
3. *If $0 < x < y$ then $0 < y^{-1} < x^{-1}$.*

Proof. A proof of (1) is embedded in the proof of Lemm 1.2.7 (twice). For part (2), suppose by way of contradiction that $0 < x$ but $x^{-1} < 0$ (note that it cannot be that $x^{-1} = 0$). By $O4$ we have $x^{-1}x < 0 \cdot x$. But then $1 < 0$, which contradicts Lemma 1.2.7. Assertion (4) can now be proven by multiplying the two inequalities $0 < x < y$ by $x^{-1}y^{-1}$ (which we know by (2) must be greater than 0). □

1.3 Absolute Values

Definition 1.3.1. If $x \in \mathbb{R}$, then the *absolute value* of x is

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}.$$

If $x, y \in \mathbb{R}$ then the *distance between x and y* is $|x - y|$.

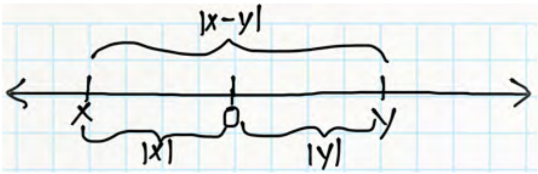

Figure 1.1: The absolute value of a number is the distance between that number and 0.

Lemma 1.3.2. *For all $x, y \in \mathbb{R}$*

1. *$0 \leq |x|$.*
2. *$-|x| \leq x \leq |x|$.*
3. *$|xy| = |x| \cdot |y|$.*
4. *If $0 < y$ then $|x| < y$ if and only if $-y < x$ and $x < y$.*

Proof. Statements (1) and (2) follow quickly from the definition of absolute values, Lemma 1.2.8, and $O1$. We prove (3) by cases. There are technically nine cases as to whether each of x and y is less than, greater than, or equal to zero. The cases are similar enough that we only have to address four. If either of x or y (or both) is 0, then

$$|xy| = 0 = |x| \cdot |y|.$$

If both x and y are greater than 0, then so is xy so

$$|x| \cdot |y| = xy = |xy|.$$

If both x and y are less than 0 then $0 < xy$ so

$$|x| \cdot |y| = (-x)(-y) = xy = |xy|.$$

We only have the cases left when one number is greater than 0 and one is less than 0. Without loss of generality, assume that $x < 0$ and $y > 0$. Then $xy < 0$ so

$$|x| \cdot |y| = (-x)y = -(xy) = |xy|.$$

In all cases, $|x| \cdot |y| = |xy|$.

For (4), suppose first that $-y < x$ and $x < y$. By Lemma 1.2.8, we know that $-x < y$. Since $|x|$ is either x or $-x$, and since $x < y$ and $-x < y$, then $|x| < y$. Suppose now that $|x| < y$. Then $-y < -|x|$, so

$$-y < -|x| \leq x \leq |x| < y.$$

Thus $-y < x$ and $x < y$. □

The following result is one of our most fundamental tools for working with inequalities. Its use will be pervasive throughout the rest of these notes.

Theorem 1.3.3. (Triangle Inequality) *For all* $x, y \in \mathbb{R}$

$$|x + y| \leq |x| + |y|.$$

Proof. By Lemma 1.3.2 we know that

$$-|x| \leq x \leq |x| \text{ and } -|y| \leq y \leq |y|.$$

Adding these inequalities (using $O3$ multiple times) gives

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

By Lemma 1.3.2, this implies that $|x + y| \leq |x| + |y|$. □

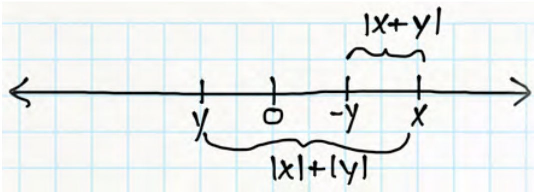

Figure 1.2: If x and y are opposite signs, then $|x + y| < |x| + |y|$. Otherwise, these two quantities are equal.

We note that this ineqality can also be applied when there is a subtraction within the abusolute values:

$$|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|.$$

The Triangle Inequality has the following corollary which will be useful later when we are proving limit theorems about the absolute value function.

Corollary 1.3.4. *For all $x, y \in \mathbb{R}$, $||x| - |y|| \leq |x - y|$.*

Proof. By the Triangle Inequality

$$|x| = |x - y + y| \leq |x - y| + |y|$$

so $|x| - |y| \leq |x - y|$. Exchanging x and y will give $|y| - |x| \leq |x - y|$, so $-|x - y| \leq |x| - |y|$. Now Lemma 1.3.2 gives $||x| - |y|| \leq |x - y|$. $\square$

We close this section by giving the standard method we will use to show that a real number is 0. We will most often use this result to prove that two real numbers are equal. To show that two real numbers a and b are equal, we will prove that $|a - b| < \epsilon$ for all $\epsilon > 0$. It will follow that $a - b = 0$ or $a = b$. This method of proof will become pervasive as we continue.

Theorem 1.3.5. *If $x \in \mathbb{R}$, then $x = 0$ if and only if $|x| < \epsilon$ for every real $\epsilon > 0$.*

Proof. If $x = 0$, then clearly $|x| < \epsilon$ for all $\epsilon > 0$. For the converse, we will use the contrapositive. Suppose that $x \neq 0$. This implies $0 < |x|$. Let $\epsilon = |x|$. Then $\epsilon > 0$ but $|x| \not< \epsilon$. Thus we have proven that if $x \neq 0$, then there is an $\epsilon > 0$ with $|x| \not< \epsilon$. The contrapositive of this is that if $|x| < \epsilon$ for all $\epsilon > 0$ then $|x| = 0$. $\square$

1.4 The Completeness Axiom

Definition 1.4.1. Suppose that $A \subseteq \mathbb{R}$. If there is an element $x \in A$ so that $a \le x$ for all $a \in A$, then x is the *maximum* of A. If $x \in A$ and $x \le a$ for all $a \in A$, then x is the *minimum* of A. The maximum of A is also called the greatest element of A, and the minimum is also called the least element.

Example 1.4.2. If $A = \{1, 2, 3\}$, then the maximum of A is 3, and the minimum is 1.

Example 1.4.3. The set $A = \{1/n : n \in \mathbb{N}\}$ has a greatest element 1 but no least element. That this set has no least element may appear obvious, but it follows from the Archimedean Property, Theorem 1.5.3.

Example 1.4.4. The open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ has no maximum and no minimum.

Example 1.4.5. The number 1 is the minimum of $\mathbb{N} \subseteq \mathbb{R}$, but $\mathbb{N}$ has no maximum.

Example 1.4.6. The set $\mathbb{Z} \subseteq \mathbb{R}$ has no maximum and no minimum.

In our definition of maximum and minimum we should really refer to "a maximum" rather than "the maximum" until after we have proven the following theorem.

Theorem 1.4.7. *If a set $A \subseteq \mathbb{R}$ has a maximum (or a minimum), then that maximum (minimum) is unique.*

Proof. Suppose that a and b are both maximum elements of $A \subseteq \mathbb{R}$. Then a and b are both elements of A. Since $a \in A$ and since b is a maximum element of A, then $a \le b$. On the other hand, since $b \in A$ and since a is a maximum element of A, then $b \le a$. Since $a \le b$ and $b \le a$, then $a = b$. $\square$

Notation 1.4.8. We denote the maximum element of a set A (when it exists) as $\max A$. The minimum is denoted $\min A$. If the elements of A can be listed such as $A = \{a_1, a_2, \ldots, a_n\}$, we may use $\max(a_1, a_2, \ldots, a_n)$ and $\min(a_1, a_2, \ldots, a_n)$ to denote the maximum and minimum. In particular, $\max(a, b)$ is the larger of two numbers a and b, and $\min(a, b)$ is the smaller.

Definition 1.4.9. Suppose that $A \subseteq \mathbb{R}$. A number x is an *upper bound* of A if $a \le x$ for all $a \in A$. On the other hand, x is a *lower bound* of A if $x \le a$ for all $a \in A$. If A has an upper bound, then A is *bounded above*. If A has a lower bound, then A is *bounded below*. If A is bounded above and below, then A is *bounded*.

Example 1.4.10. If $A = \{1, 2, 3\}$, then A is bounded above and below. Some upper bounds are 3, π, and 89. Some lower bounds are 1, 0, and $-\pi$.

Example 1.4.11. The set $A = \{1/n : n \in \mathbb{N}\}$ is also bounded above and below. Some upper bounds are 1, 2, 3, and 987. Some lower bounds are 0, -1, and -203.

Example 1.4.12. The open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is also bounded above and below. Some upper bounds are 1, 93, and 409. Some lower bounds are 0, -12, and -4π.

Example 1.4.13. The set $\mathbb{N} \subseteq \mathbb{R}$, is not bounded above. This is proven in Theorem 1.5.1. The set $\mathbb{N}$ is bounded below. Some lower bounds are 1, 0, and -1.

Example 1.4.14. The set $\mathbb{Z} \subseteq \mathbb{R}$ is not bounded above or below.

Example 1.4.15. The set $A = \{x \in \mathbb{R} : x^2 < 2\}$ is bounded above (by 2) and below (by -2).

Lemma 1.4.16. *A set $A \subseteq \mathbb{R}$ is bounded if and only if there is a number $M \in \mathbb{R}$ so that $|a| \le M$ for all $a \in A$.*

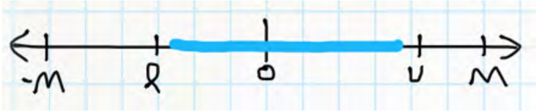

Figure 1.3: This set is bounded in absolute value by M. The numbers l and u from the proof of Lemma 1.4.16 are also shown.

Proof. Suppose first that A is bounded. In particular, suppose that l is a lower bound of $A \subseteq \mathbb{R}$ and that u is an upper bound of A. Let $M = max(|l|, |u|)$. Then $|l|, |u| \le M$, so $-M \le l \le M$ and $-M \le u \le M$. If $a \in A$ then $-M \le l \le a \le u \le M$ so $|a| \le M$. $\square$

Definition 1.4.17. If $A \subseteq \mathbb{R}$ is bounded above, and if A has a least (or minimum) upper bound, then we call it the *supremum* of A. If A is bounded below and has a greatest (or maximum) lower bound, then we call it the *infimum* of A.

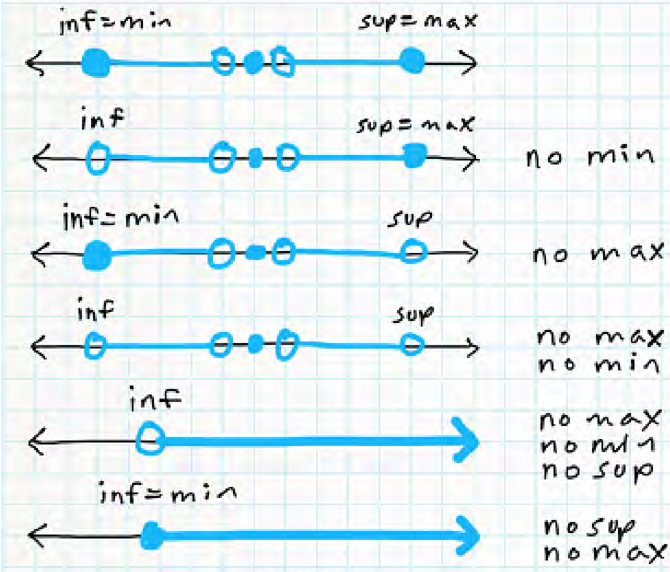

Figure 1.4: Some possible relationships between maxima and suprema and between minima and infima.

Example 1.4.18. If $A = \{1, 2, 3\}$, then A is bounded above and below. The infimum of A is 1 (which is also the minimum), and the supremum is 3 (which is also the maximum).

Example 1.4.19. The set $A = \{1/n : n \in \mathbb{N}\}$ is also bounded above and below. The infimum of A is 0, and the supremum is 1. Notice that the supremum is the same as the maximum, but the infimum is not the minimum of A – because A has no minimum.

Example 1.4.20. The infimum of the open interval

$$(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$$

is 0. The supremum is 1. Notice that these are not the minimum and maximum.

Example 1.4.21. The set $\mathbb{N} \subseteq \mathbb{R}$, is not bounded above, so $\mathbb{N}$ has no supremum. However, 1 is the infimum of $\mathbb{N}$.

Example 1.4.22. The set $\mathbb{Z} \subseteq \mathbb{R}$ is not bounded above or below and so has no infimum or supremum.

Example 1.4.23. The set $A = \{x \in \mathbb{R} : x^2 < 2\}$ is bounded above (by 2) and below (by -2). The supremum of A is $\sqrt{2}$, and the infimum is $-\sqrt{2}$.

Notation 1.4.24. When a set A has a supremum, we denote it as $\sup A$. When A has an infimum, we denote it as $\inf A$. Sometimes $\text{lub}A$ is used for the least upper bound of A and $\text{glb}A$ is used for the greatest lower bound.

As with the maximum and minimum of a set, we should really first define "a supremum" and "an infimum" and then prove the next theorem before we use the word "the."

Theorem 1.4.25. *If $A \subseteq \mathbb{R}$ and $\sup A$ (or $\inf A$) exists, then it is unique.* □

Theorem 1.4.26. *If $A \subseteq \mathbb{R}$ has a maximum, then $\sup A = \max A$. If $A \subseteq \mathbb{R}$ has a minimum, then $\inf A = \min A$.*

Proof. Suppose that m is the maximum of $A \subseteq \mathbb{R}$. We will prove that m is the supremum of A. To do so, we need to prove that m is an upper bound of A and that m is less than or equal to every upper bound of A. That m is an upper bound of A follows directly from the definition of maximum. Suppose now that u is any upper bound of A. Then u is greater than or equal to every element of A. In particular, since $m \in A$, $m \leq u$. Thus, m is an upper bound of A which is less than or equal to every upper bound of A. This makes m the least upper bound or supremum of A. □

We have seen examples of sets which are bounded and which are unbounded. We have also seen examples of sets which have suprema and which do not. One of the unique features of $\mathbb{R}$ is the following assertion as to which sets have suprema. We take this as an axiom. A proper derivation of $\mathbb{R}$ from the Peano Axioms for $\mathbb{N}$ can actually prove this result. In that case, we would call it the Completeness Property.

Axiom 1.4.27. (Completeness Axiom) *Every nonempty subset of $\mathbb{R}$ which is bounded above has a least upper bound.*

The Completeness Axiom is stated in terms of upper bounds, but it holds also for lower bounds.

Corollary 1.4.28. *Every nonempty subset of* $\mathbb{R}$ *which is bounded below has a greatest lower bound.*

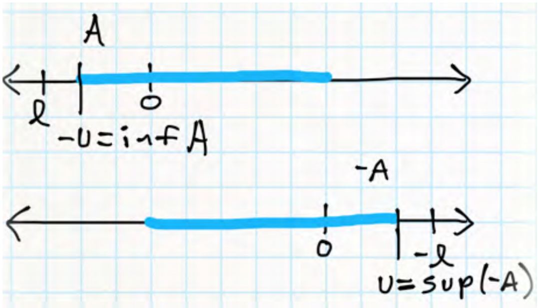

Figure 1.5: If l is a lower bound of A, then $-l$ is an upper bound of $-A$. If u is an upper bound of $-A$, then $-u$ is a lower bound of A.

Proof. Suppose that $A \subseteq \mathbb{R}$ is bounded below. Let $-A = \{-a : a \in A\}$. Let l be any lower bound of A. We will show that $-l$ is an upper bound of $-A$. Suppose that $x \in -A$. Then there is some $a \in A$ with $x = -a$. Since l is a lower bound of A, $l \leq a$. This implies that $x = -a \leq -l$. Thus $x \leq -l$ for all $x \in -A$, and $-l$ is an upper bound of $-A$.

Since $-A$ is bounded above, the Completeness Axiom tells us that $-A$ has a least upper bound. Let $u = \sup(-A)$. We will prove that $-u$ is the greatest lower bound of A. First, we show that $-u$ is a lower bound of A. Let $a \in A$. Then $-a \in -A$, so $-a \leq u$. But then $-u \leq a$. Hence, $-u$ is a lower bound of A. Now we must show that $-u$ is greater than or equal to every lower bound of A. Suppose that l is any lower bound of A. Then (as before) $-l$ is an upper bound of $-A$, so $u \leq -l$. This implies that $l \leq -u$ as desired. Thus $-u$ is a lower bound of A, and $-u$ is greater than or equal to every lower bound of A. It follows that $\inf A$ exists and is equal to $-u$. $\square$

1.5 Density of $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$

In this section we prove some results about the relationship between $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$. These results may appear to be obvious because they are so familiar to us. However, these are results that require the Completeness Axiom. There are environments (number systems) which contain $\mathbb{N}$ and $\mathbb{Q}$ which do not satisfy the Completeness Axiom in which these results do not hold. For example, there are number systems which contain $\mathbb{N}$ and which contain "numbers" that are strictly larger than every natural number and numbers which are strictly between 0 and $1/n$ for every $n \in \mathbb{N}$. The Completeness Axiom prevents such anomalies.

Theorem 1.5.1. *$\mathbb{N}$ is not bounded above in $\mathbb{R}$.*

Proof. Suppose by way of contradiction that $\mathbb{N}$ is bounded above in $\mathbb{R}$. By the Completeness Axiom $\sup \mathbb{N}$ exists (in $\mathbb{R}$). Let $z = \sup \mathbb{N}$. Then $z - 1$ is not an upper bound of $\mathbb{N}$ (since it is smaller than the least upper bound), so there is a natural number n with $z - 1 < n \leq z$. But then $z < n + 1$ and $n + 1 \in \mathbb{N}$. This contradicts the fact that z is an upper bound of $\mathbb{N}$. Hence, the assumption must be false. $\mathbb{N}$ cannot be bounded above in $\mathbb{R}$. $\square$

One consequence of this theorem sounds almost obvious: Every subset of $\mathbb{N}$ which is bounded above in $\mathbb{R}$ is finite. Suppose that $A \subseteq \mathbb{N}$ and that $x \in \mathbb{R}$ is an upper bound of A. Since x cannot be an upper bound of all of $\mathbb{N}$, then there is an $n \in \mathbb{N}$ with $x < n$. This implies that A has fewer than n elements. Thus:

Corollary 1.5.2. *Every subset of $\mathbb{N}$ which is bounded in $\mathbb{R}$ is finite.* $\square$

The next property of $\mathbb{R}$ is the mathematical equivalent of saying that you can empty the entire sea with the smallest spoon (though I am not sure where you would put all of the water).

Theorem 1.5.3. (Archimedean Property) *If $x, y \in \mathbb{R}$ are greater than 0, then there is some $n \in \mathbb{N}$ so that $nx > y$.*

Proof. Suppose that $x, y > 0$. By Theorem 1.5.1 there is a natural number n with $n > y/x$. Multiplying by x now gives $nx > y$. $\square$

Taking $y = 1$ in the Archimedean Property implies that for $x > 0$ there is an $n \in \mathbb{N}$ so that $x > 1/n$.

Lemma 1.5.4. *Suppose that $0 < x < y$ in $\mathbb{R}$ and that $y - x > 1$. There is an $m \in \mathbb{N}$ so that $x < m < y$.*

Proof. Suppose that $0 < x < y$ in $\mathbb{R}$ and that $y - x > 1$. Then $x + 1 < y$. Let $A = \{n \in \mathbb{N} : n \leq x\}$. If A is empty, then $x < 1 < x + 1 < y$ so $m = 1$ will satisfy the theorem. Suppose then that A is not empty. Then A is a bounded set of natural numbers, so A is finite. Let k be the greatest element of A, and let $m = k + 1$. Since m is greater than k, then $m \notin A$. Then we know that $x < m$. Now, since $k \leq x$, then $m = k + 1 \leq x + 1 < y$. We now have $x < m < y$ as desired. $\square$

Theorem 1.5.5. (Density of $\mathbb{Q}$) *If $x < y$ in $\mathbb{R}$, then there is some $r \in \mathbb{Q}$ so that $x < r < y$.*

Proof. We address the case where $0 < x < y$. The Archimedean Property guarantees some $n \in \mathbb{N}$ with $n(y - x) > 1$. So $ny - nx > 1$. By Lemma 1.5.4, there is a natural number m so that $nx < m < ny$. Dividing by n now gives

$$x < \frac{m}{n} < y.$$

Thus $r = \dfrac{m}{n}$ is a rational number strictly between x and y. $\square$

To prove that every interval contains an irrational number, we first need an irrational number to work with. This is given by the next theorem (which you should already be familiar with).

Theorem 1.5.6. *There are no integers m and n so that $\left(\dfrac{m}{n}\right)^2 = 2$.*

Proof. Suppose by way of contradiction that there are such integers m and n. We can suppose that m and n are positive and have no common factors (otherwise, we can reduce the fraction m/n). Multiplying in the equation $\left(\dfrac{m}{n}\right)^2 = 2$ gives us $m^2 = 2n^2$. This implies that m is even, so there is a positive integer k with $m = 2k$. The equation $m^2 = 2n^2$ can now be written $(2k)^2 = 2n^2$ or $4k^2 = 2n^2$. Canceling gives $2k^2 = n^2$, so n is also even, but then n and m are both even – contradicting the fact that m and n have no common factors. Thus our assumption must be false. There can be no integers m and n so that $\left(\dfrac{m}{n}\right)^2 = 2$. $\square$

Theorem 1.5.7. (Density of the Irrationals) *If $x < y$ in $\mathbb{R}$, then there is some irrational number s with $x < s < y$.*

Proof. By Theorem 1.5.5 there is a rational number r with

$$x\sqrt{2} < r < y\sqrt{2}.$$

We can assume that $r \neq 0$. (If $r = 0$, then there is another rational number between 0 and $y\sqrt{2}$ that we can use.) Then $s = \dfrac{r}{\sqrt{2}}$ is irrational (you should prove this) and $x < s < y$. □

Theorems 1.5.5 and 1.5.7 imply the next Theorem.

Theorem 1.5.8. *There are infinitely many irrational numbers and rational numbers in any interval of* $\mathbb{R}$. □

1.6 Chapter 1 Exercises

1.6.1 Which of the ordered field axioms fail in $\mathbb{N}$?
1.6.2 Which of the ordered field axioms fail in $\mathbb{Z}$?
1.6.3 If $x < y$, prove from the ordered field axioms that $x < \dfrac{x+y}{2} < y$.
1.6.4 If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \dfrac{x+y}{2}$. Hint: Use the fact that $(\sqrt{x} - \sqrt{y})^2 \geq 0$.
1.6.5 If $0 < x < y$, prove that $0 < x^2 < y^2$ and $0 < \sqrt{x} < \sqrt{y}$.
1.6.6 Prove that if $a, b \in \mathbb{R}$ then $\max(a, b) = \dfrac{a+b}{2} + \dfrac{|a-b|}{2}$.
1.6.7 Prove that if $a, b \in \mathbb{R}$ then $\min(a, b) = \dfrac{a+b}{2} - \dfrac{|a-b|}{2}$.
1.6.8 For each set below, list three upper bounds or declare that the set is not bounded.

a. $[2, 3]$

b. $(2, 3)$

c. $\{2, 3\}$

d. $\left\{n + \dfrac{1}{n} : n \in \mathbb{N}\right\}$

e. $\{r \in \mathbb{Q} : r^2 < 4\}$

f. $\left\{\dfrac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime}\right\}$

g. $\{x^2 : x \in \mathbb{Q}\}$

1.6.9 For each set in Exercise 1.6.8 either give the supremum or declare the set has no supremum.
1.6.10 Repeat Exercise 1.6.8 for lower bounds.
1.6.11 Repeat Exercise 1.6.9 for infima.
1.6.12 Prove that if $A \subseteq \mathbb{R}$ has a supremum and $\sup A \in A$, then $\sup A = \max A$.

1.6.13 Suppose that A and B are nonempty bounded subsets of $\mathbb{R}$ and that $A \subseteq B$. Prove that

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

1.6.14 Suppose that $A, B \subseteq \mathbb{R}$ are bounded and let

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

a. Prove that $\sup(A + B) = \sup A + \sup B$.

b. Prove that $\inf(A + B) = \inf A + \inf B$.

1.6.15 Suppose that $A, B \subseteq \mathbb{R}$ are bounded and that $x \geq 0$ for all $x \in A \cup B$. Let

$$AB = \{ab : a \in A \text{ and } b \in B\}.$$

a. Prove that $\sup(AB) = \sup A \cdot \sup B$.

b. Prove that $\inf(AB) = \inf A \cdot \inf B$.

1.6.16 Suppose that $A \subseteq \mathbb{R}$ is bounded and that $k \in \mathbb{R}$. Let

$$kA = \{kx : x \in A\}.$$

a. Prove that if $k \geq 0$ then $\sup(kA) = k \sup A$.

b. Prove that if $k \geq 0$ then $\inf(kA) = k \inf A$.

c. Prove that if $k < 0$ then $\sup(kA) = k \inf A$.

d. Prove that if $k < 0$ then $\inf(kA) = k \sup A$.

1.6.17 Suppose that $A \subseteq \mathbb{R}$ is bounded. Let $|A| = \{|x| : x \in A\}$. Prove that $\sup |A| - \inf |A| \leq \sup A - \inf A$.

1.6.18 Suppose that $A \subseteq \mathbb{R}$ is a bounded set of non-negative real numbers. Let $A^2 = \{a^2 : a \in A\}$. Prove that $\sup(A^2) = (\sup A)^2$. Give an example to show that we must assume that A does not contain negative numbers.

1.6.19 Suppose that $a, b \in \mathbb{R}$ and that $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$. Prove that $a \leq b$.

1.6.20 If $S \subseteq R$ and $x = \sup S$, show that for every $\epsilon > 0$ there is an $a \in S$ so that $x - \epsilon < a \leq x$.

1.6.21 If $S \subseteq R$ and $x = \inf S$, show that for every $\epsilon > 0$ there is an $a \in S$ so that $x \leq a < x + \epsilon$.
1.6.22 Prove part (2) of Lemma 1.2.3.
1.6.23 Prove part (2) of Lemma 1.2.5
1.6.24 Prove part (2) of Lemma 1.2.7.
1.6.25 Prove part (4) of Lemma 1.2.7.
1.6.26 Prove part (1) of Lemma 1.2.8.
1.6.27 Prove Theorem 1.4.25.
1.6.28 There are two cases not addressed in the proof of Theorem 1.5.5. State and prove these cases. You should be able to refer to what has already been proven.
1.6.29 Prove that if x is a positive real number then there is a positive real number y with $y^2 = x$. That is, the exercises is to prove that square roots of positive real numbers exist.
1.6.30 Suppose that r is rational. Prove that $\frac{r}{\sqrt{2}}$ is not rational. Hint: Use contradiction.
1.6.31 Prove Theorem 1.5.8.

Chapter 2

Sequences

Definition 2.0.1. A *sequence* is a function whose domain is of the form

$$\{n \in \mathbb{Z} : n \geq m\}$$

(where m is usually 0 or 1). If s is a sequence, we will usually write s_n for the value $s(n)$. The values s_n will be called *terms* of the sequence s.

Figure 2.1: The standard graph of a sequence $\langle s_n \rangle$ will look something like this. Notice that the function is only defined at integer values of n.

We may denote a sequence s with domain $\mathbb{N}$, in any of these ways:

$$\langle s_1, s_2, s_3, \ldots \rangle \text{ or } \langle s_n : n \in \mathbb{N} \rangle \text{ or } \langle s_n \rangle_{n=1}^{\infty} \text{ or simply } \langle s_n \rangle .$$

The manner in which we will usually depict the graph of a sequence is shown in Figure 2.1. We will most often be interested in sequences whose codomains are $\mathbb{R}$. These are called *sequences of real numbers* Henceforth, when we say sequence, we will mean sequence of real numbers.

2.1 Limits of Sequences

Definition 2.1.1. A sequence $\langle s_n \rangle$ of real numbers *converges* to a real number L if for every real number $\epsilon > 0$ there is a real number N so that for all integers n, if $n > N$ then $|s_n - L| < \epsilon$. In this case, we call L a *limit* of the sequence $\langle s_n \rangle$. A sequence which does not converge is said to *diverge*.

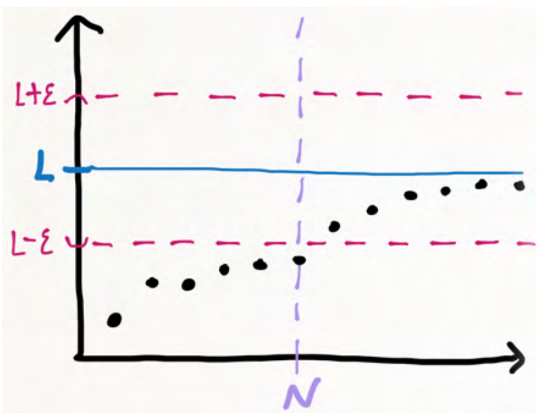

Figure 2.2: This figure depicts a "tube" of radius ϵ centered at L. If $n > N$, then s_n is in this tube. We might say that the sequence $\langle s_n \rangle$ is *eventually* in the ϵ-tube around L.

The situation described in the definition of a convergent sequence is depicted in Figure 2.2

Example 2.1.2. We prove that the sequence $\left\langle \dfrac{1}{n} \right\rangle$ converges to 0. Let $\epsilon > 0$. Let $N = 1/\epsilon$ and suppose that $n \in \mathbb{N}$ with $n > N$. Then

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \frac{1}{N} = \epsilon.$$

The value of N here comes from considering the inequality $\left|\frac{1}{n} - 0\right| < \epsilon$, solving for n, and then selecting a value of N which is "big enough." Any N which is at least $1/\epsilon$ would do.

Example 2.1.3. We prove that the sequence $\left\langle \frac{n}{2n+3} \right\rangle$ converges to $\frac{1}{2}$. Let $\epsilon > 0$. Let $N = \frac{3}{4\epsilon}$ and suppose that $n \in \mathbb{N}$ with $n > N$. Then

$$\begin{aligned} \left|\frac{n}{2n+3} - \frac{1}{2}\right| &= \left|\frac{2n-(2n+3)}{(2n+3)2}\right| \\ &= \frac{3}{4n+6} \\ &< \frac{3}{4N+6} \\ &< \frac{3}{4N} \\ &= \frac{3}{4\frac{3}{4\epsilon}} \\ &= \epsilon \end{aligned}$$

A nicer choice for N here would be $1/\epsilon$. Then this string of inequalities could finish this way:

$$\frac{3}{4N} < \frac{4}{4N} = \frac{1}{N} = \epsilon.$$

A much less nice but more obvious choice for N can be had by solving

$$\frac{3}{4N+6} = \epsilon$$

for N. This would give

$$N = \left(\frac{3}{\epsilon} - 6\right)\frac{1}{4}.$$

Here we would have to pay attention to the fact that $\frac{3}{\epsilon} - 6$ could be negative.

Example 2.1.4. We prove that the sequence

$$\langle s_n \rangle = \langle (-1)^n \rangle = \langle -1, 1, -1, 1, -1, 1, \ldots \rangle$$

does not converge. This proof is illustrated in Figure 2.3. We first consider the negation of the definition of convergence:

> For any L, there is an $\epsilon > 0$ so that for all $N \in \mathbb{R}$ there is an $n \in \mathbb{N}$ with $n > N$ and $|s_n - L| \geq \epsilon$.

Suppose that $L \in \mathbb{R}$. We prove that $\langle s_n \rangle$ does not converge to L. Let $\epsilon = 1$. Suppose that $N \in \mathbb{R}$. If $L \geq 0$, then let n be any odd natural number greater than N. If $L < 0$, let n be any even natural number greater than N. Then $|s_n - L| \geq |s_n - 0| = 1 \geq \epsilon$. Thus there is an ϵ so that for all N there is an $n > N$ with $|s_n - L| \geq \epsilon$. Hence, $\langle s_n \rangle$ cannot converge to L.

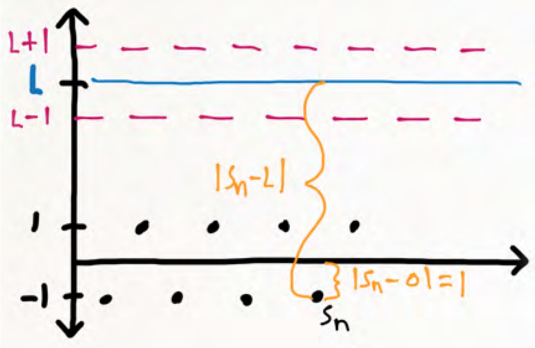

Figure 2.3: If $L > 0$ and n is odd, then s_n is closer to 0 than to L, but $|s_n - 0| = 1 = \epsilon$.

Example 2.1.5. We prove that the sequence $\langle s_n \rangle = \langle n \rangle$ does not converge. This proof is pictured in Figure 2.4. Let $L \in \mathbb{R}$. Let ϵ be 1. Let $N \in \mathbb{R}$. Let n be any natural number which is greater than $\max(L, N) + 1$. Then $|s_n - L| \geq 1 = \epsilon$. We have shown that there is an $\epsilon > 0$ so that for all $N \in \mathbb{R}$ there is an $n \in N$ with $n > N$ but $|s_n - L| \geq \epsilon$. Thus $\langle s_n \rangle$ cannot converge to L.

We would like to be able to speak about "the limit" of a sequence rather than "a limit." We would also like notation for "the limit" of a sequence. Before we can do so, we need to know that a sequence cannot have more than one limit. This may sound obvious, and the proof is not too difficult. However, the proof will demonstrate a couple of standard techniques that we will use frequently later. First, we illustrate a common method of showing that two numbers are equal. Second, we see the triangle inequality in action. Finally, this is an example of a *standard $\epsilon/2$ proof*.

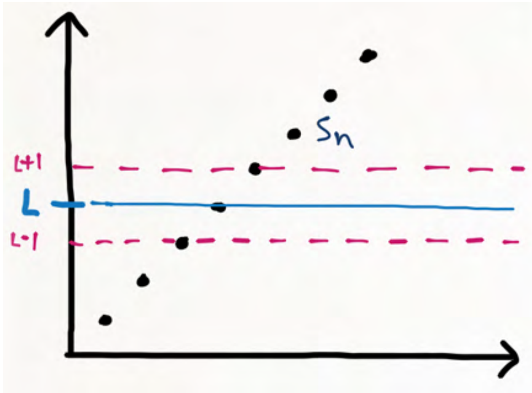

Figure 2.4: If $n > L+1$, then s_n is above the tube of radius 1 around L.

Theorem 2.1.6. *Limits of sequences are unique.*

Proof. Suppose that $\langle s_n \rangle$ is a sequence which converges to L and to M. We will prove that $L = M$. We do so by proving that the difference $|L - M|$ is less than every positive real number. Let $\epsilon > 0$. Since $\langle s_n \rangle$ converges to L, there is an $N_L \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N_L$, then $|s_n - L| < \epsilon/2$. Since $\langle s_n \rangle$ converges to M, there is an $N_M \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N_M$, then $|s_n - M| < \epsilon/2$. Let $N = \max(N_L, N_M)$. Suppose that $n \in \mathbb{N}$ and $n > N$. Then

$$\begin{aligned}
|L - M| &= |L - s_n + s_n - M| \\
&\leq |L - s_n| + |s_n - M| \\
&= |s_n - L| + |s_n - M| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus for all $\epsilon > 0$, $|L - M| < \epsilon$. It follows from Theorem 1.3.5 that $L = M$. $\square$

Since a sequence may have at most one limit, we are now allowed to give notation for this limit.

Notation 2.1.7. If a sequence $\langle s_n \rangle$ converges to number L, then we will write $\lim s_n = L$ or $s_n \to L$. If $\langle s_n \rangle$ converges, we say that the limit $\lim s_n$ *exists.* If $\langle s_n \rangle$ does not converge, then we will say that the limit $\lim s_n$ *does not exist.*

Frequently later we will be lazy and say something along the lines of, "Suppose $\lim s_n = L$" to mean, "Suppose that $\langle s_n \rangle$ is a convergent sequence and $\lim s_n = L$."

The notation $s_n \to L$ depicts convergence as a relation from sequences to real numbers. The theorems in the next section establish that we can do certain algebraic manipulations "to both sides of an $\to$" just like we might to both sides of an equality or inequality.

Exercises 2.1

2.1.1 Use the definition to prove that each of the following sequences converges:

a. $\left\langle 5 + \dfrac{1}{n} \right\rangle$

b. $\left\langle \dfrac{2-2n}{n} \right\rangle$

c. $\langle 2^{-n} \rangle$

d. $\left\langle \dfrac{3n}{2n+1} \right\rangle$

e. $\left\langle \dfrac{(-1)^n}{n} \right\rangle$

f. $\left\langle \dfrac{2n-1}{3n+2} \right\rangle$

g. $\left\langle \dfrac{4n+3}{7n-5} \right\rangle$

h. $\left\langle \sqrt{n^2+1} - n \right\rangle$

i. $\left\langle \sqrt{n^2+n} - n \right\rangle$

j. $\left\langle \sqrt{4n^2+n} - 2n \right\rangle$

2.1.2 Use the definition to prove that each of these sequences diverges:

a. $\langle (-1)^n n \rangle$

b. $\langle \sin(n\pi/3) \rangle$

2.1.3 Prove that the sequence $\langle a_n \rangle$ converges to L if and only if the sequence $\langle a_n - L \rangle$ converges to 0.

2.1.4 Prove or disprove that if $\langle |a_n| \rangle$ converges, then $\langle a_n \rangle$ converges.

2.1.5 Give an example of a sequence of rational numbers that converges to an irrational number.

2.1.6 Give an example of a sequence of irrational numbers that converges to a rational number.

2.1.7 Suppose that $\langle s_n \rangle$ is a sequence of positive terms that converges to 0. Prove that $\langle \sqrt{s_n} \rangle$ also converges to 0.
2.1.8 Suppose that $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences so that $|a_n| \leq b_n$ for all n and so that $\lim b_n = 0$. Prove that $\langle a_n \rangle$ converges to 0.

2.2 Algebraic Properties of Limits of Sequences

In this section, we prove several results related to algebraic operations and sequences. These results will allow us to calculate and manipulate limits more easily than we could with just the definition. We prove the results here as a list of lemmas and then gather all of the results together at the end of the section as one large theorem for easy reference.

The point of this first proof is that the differences $|c_n - k|$ are always 0 and, so, less than any positive ϵ. We give the proof as another example of a proof outline using the definition.

Lemma 2.2.1. (Limits of Constant Sequences) *If $k \in \mathbb{R}$ and $c_n = k$ is the sequence which is constantly k, then $\langle c_n \rangle$ converges to k (or $\lim k = k$).*

Proof. Let $\epsilon > 0$. Let $N = 0$. If $n \in \mathbb{N}$ and $n > N$ then

$$|c_n - k| = |k - k| = 0 < \epsilon.$$

Thus $\lim c_n = k$. □

This next proof illustrates both applying the definition of a convergent sequence and satisfying the definition.

Lemma 2.2.2. (Constant Multiples of Sequences) *Suppose that $\langle a_n \rangle$ is a sequence so that $\lim a_n = L$. If $k \in \mathbb{R}$, then $\langle ka_n \rangle$ converges to kL.*

Proof. If $k = 0$, then $ka_n = 0$ and the result follows from Lemma 2.2.1. Suppose then that $k \neq 0$. Let $\epsilon > 0$. There is an $N \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N$ then $|a_n - L| < \epsilon/|k|$. Suppose that $n \in \mathbb{N}$ and $n > N$. Then

$$|ka_n - kL| = |k||a_n - L| < |k|\epsilon/|k| = \epsilon.$$

Thus $\lim(ka_n) = kL$. □

We can express this last lemma as if $a_n \to L$ then $ka_n \to kL$. One might say we can multiply both sides of $\to$ by a constant. We can also add, subtract, multiply, and divide on both sides of an $\to$. This is the content of most of the next few lemmas. This first lemma is another example of an $\epsilon/2$ proof.

Lemma 2.2.3. (Sums of Sequences) *Suppose that $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences so that $\lim a_n = L$ and $\lim b_n = M$. Then $\langle a_n + b_n \rangle$ converges to $L + M$.*

Proof. Let $\epsilon > 0$. Since $\lim a_n = L$, there is an $N_a \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N_a$ then $|a_n - L| < \epsilon/2$. Since $\lim b_n = M$, there is an $N_b \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N_b$ then $|b_n - M| < \epsilon/2$. Let $N = \max(N_a, N_b)$. Suppose that $n \in \mathbb{N}$ and that $n > N$. Then

$$\begin{aligned} |(a_n + b_n) - (L + M)| &= |(a_n - L) + (b_n - M)| \\ &\le |(a_n - L)| + |(b_n - M)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Thus $\lim(a_n + b_n) = L + M$. □

If we change a few of the +s in this proof to −s, we get a proof of:

Lemma 2.2.4. (Differences of Sequences) *Suppose that $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences so that $\lim a_n = L$ and $\lim b_n = M$. Then $\langle a_n - b_n \rangle$ converges to $L - M$.* □

We have to be slightly more sly to make the ϵ argument work out for products. First, we need to know that convergent sequences are bounded. This is a fact which will be needed frequently later.

Definition 2.2.5. A sequence $\langle s_n : n \in \mathbb{N} \rangle$ is *bounded* if the set

$$\{s_n : n \in \mathbb{N}\}$$

is bounded.

Theorem 2.2.6. *All convergent sequences are bounded.*

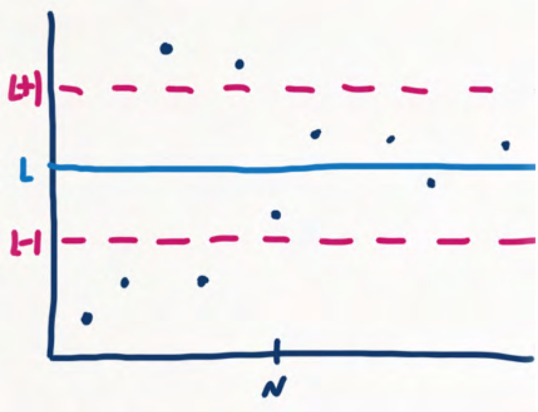

Figure 2.5: There are only finitely many terms of the sequence to the left of N. To the right of N, the sequence is bounded between $L+1$ and $L-1$.

Proof. Suppose that $\langle s_n \rangle$ is a sequence that converges to a number L. We apply the definition of convergence with $\epsilon = 1$. There is an $N_0 \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N_0$ then $|s_n - L| < 1$. Let N be the least natural number greater than N_0. Then, if $n > N$ we have $L - 1 < s_n < L + 1$. Now, let $A = \min(s_1, s_2, \ldots, s_N, L-1)$ and let $B = \max(s_1, s_2, \ldots, s_N, L+1)$. It follows that $A \leq s_n \leq B$ for all n so $\langle s_n \rangle$ is bounded. □

In the following proof, from $\lim a_n = L$, we will want to know that if n is large enough then $|M||a_n - L| < \epsilon/2$. We could arrive at this by selecting n large enough so that $|a_n - L| < \dfrac{\epsilon}{2|M|}$. This works if $M \neq 0$. If $M = 0$, then $|M||a_n - L| = 0 < \epsilon/2$ for all n. In either case, we can make n large enough to insure that $|M||a_n - L| < \epsilon/2$.

Lemma 2.2.7. (Products of Sequences) *Suppose that $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences so that $\lim a_n = L$ and $\lim b_n = M$. Then $\langle a_n b_n \rangle$ converges to LM.*

Proof. Since $\langle a_n \rangle$ is convergent, $\langle a_n \rangle$ is bounded. Let $B \in \mathbb{R}$ so that $|a_n| < B$ for all n. Let $\epsilon > 0$. Since $\lim a_n = L$, there is an $N_a \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N_a$ then $|M||a_n - L| < \dfrac{\epsilon}{2}$ (as in our discussion before the lemma). Since $\lim b_n = M$, there is an $N_b \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N_b$ then $|b_n - M| < \dfrac{\epsilon}{2B}$. Let $N = \max(N_a, N_b)$. Suppose that $n \in \mathbb{N}$ and $n > N$. Then

$$\begin{aligned}
|a_nb_n - LM| &= |a_nb_n - a_nM + a_nM - LM| \\
&\leq |a_nb_n - a_nM| + |a_nM - LM| \\
&= |a_n||b_n - M| + |M||a_n - L| \\
&< B\frac{\epsilon}{2B} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus $\lim(a_nb_n) = LM$. □

Example 2.2.8. Suppose that $\lim a_n = 4$. We will show using the lemmas above that $\lim(a_n^2 + 2a_n + 3)$ exists and is equal to 27. First, since $\lim a_n = 4$, Lemma 2.2.7 tells us that $\langle a_n^2 \rangle$ converges to 16. Lemma 2.2.2 tells us that $\langle 2a_n \rangle$ converges to 8, and Lemma 2.2.1 tells us that $\langle 3 \rangle$ converges to 3. Since $\lim(a_n^2) = 16$ and $\lim(2a_n) = 8$ and $\lim 3 = 3$, Lemma 2.2.3 tells us that $\langle a_n^2 + 2a_n + 3 \rangle$ converges to $16+8+3 = 27$. This chain of thought is expressed much more concisely in the following string of equalities.

$$\begin{aligned}
27 &= 16 + 8 + 3 \\
&= 4 \cdot 4 + 2 \cdot 4 + 3 \\
&= (\lim a_n) \cdot (\lim a_n) + 2 \cdot (\lim a_n) + \lim 3 && 2.2.1 \\
&= \lim(a_n \cdot a_n) + 2 \cdot (\lim a_n) + \lim 3 && 2.2.7 \\
&= \lim(a_n^2) + \lim(2a_n) + \lim 3 && 2.2.2 \\
&= \lim(a_n^2 + 2a_n + 3). && 2.2.3
\end{aligned}$$

Notice that in this string of equalities at each step we replace an expression with another expression known to be equal (and existent) by one of our lemmas. These equalities are frequently written backwards in a calculus class such as

$$\begin{aligned}
\lim(a_n^2 + 2a_n + 3) &= \lim(a_n^2) + \lim(2a_n) + \lim 3 \\
&= \lim(a_n \cdot a_n) + 2 \cdot (\lim a_n) + \lim 3 \\
&= (\lim a_n) \cdot (\lim a_n) + 2 \cdot (\lim a_n) + \lim 3 \\
&= 4 \cdot 4 + 2 \cdot 4 + 3 \\
&= 16 + 8 + 3 \\
&= 27.
\end{aligned}$$

This work is awkward at best since it is not until the end that we know that all of the limits exist and the equalities are valid. While this might be an abuse of notation, it is common practice. We will likely be guilty of this practice (frequently) later.

It should be clear that we could mimic the process in this example for any polynomial function. Thus:

Lemma 2.2.9. *Suppose that $\langle a_n \rangle$ is a convergent sequence with limit L. If $p(x)$ is any polynomial, then $\lim p(a_n) = p(L)$.* □

Now we move on to absolute values.

Lemma 2.2.10. *Suppose that $\langle a_n \rangle$ is a convergent sequence with limit L. Then $\langle |a_n| \rangle$ converges to $|L|$.*

Proof. Let $\epsilon > 0$. Since $\lim a_n = L$, there is an $N \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N$ then $|a_n - L| < \epsilon$. Suppose that $n \in \mathbb{N}$ and $n > N$. Then by 1.3.4

$$||a_n| - |L|| \leq |a_n - L| < \epsilon.$$

Thus $\lim |a_n| = |L|$. □

We will address order theorems about limits of sequences a bit more later; however, we need this lemma now in order to prove a result about square roots.

Lemma 2.2.11. *Suppose that $\langle a_n \rangle$ is a sequence converging to a number L. If $k \in \mathbb{R}$ so that $a_n \geq k$ for all n, then $L \geq k$. (A similar result holds for the case when $k \geq a_n$ for all n.)*

Proof. Suppose that $\langle a_n \rangle$ converges to L and that $L < k$. Let $\epsilon = (k - L)/2$ (which is positive). Note that $L + \epsilon < k$. By the definition of convergence, there is an $N \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N$ then $|a_n - L| < \epsilon$. This implies that $a_n < L + \epsilon < k$. Thus, if $L < k$ then there is an n so that $a_n < k$. This is the contrapositive of the lemma. □

By this lemma, if $\langle a_n \rangle$ is a convergent sequence of non-negative terms, then the limit of a_n is also non-negative. It makes sense then to consider the square root of the sequence and whether or not this converges to the square root of $\lim a_n$. Notice in this proof the use of the high-school algebra trick of multiplying by the conjugate.

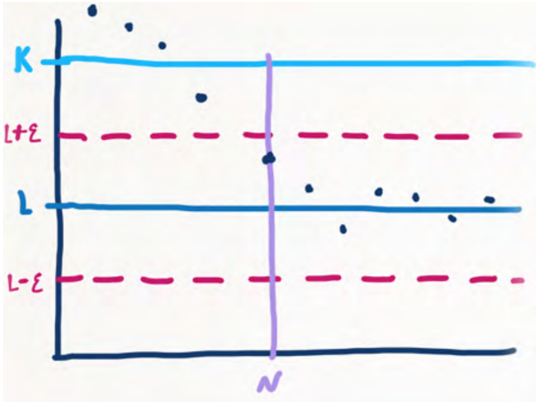

Figure 2.6: If $\langle a_n \rangle$ converges to L and if $L < k$, then $\langle a_n \rangle$ is eventually within $\dfrac{k-L}{2}$ of L. At this point, $\langle a_n \rangle$ is strictly below k.

Lemma 2.2.12. *Suppose that $\langle a_n \rangle$ is a convergent sequence of non-negative terms with limit L. Then* $\lim \sqrt{a_n} = \sqrt{L}$.

Proof. As noted before the lemma, by Lemma 2.2.11 we know that $L \geq 0$, so it is legal to discuss $\sqrt{L}$. We will consider two cases – $L \neq 0$ and $L = 0$. First, suppose that $L \neq 0$. Let $\epsilon > 0$. Since $\lim a_n = L$, there is an $N \in \mathbb{R}$ so that if $n \in N$ and $n > N$ then $|a_n - L| < \epsilon\sqrt{L}$. Suppose that $n \in \mathbb{N}$ and $n > N$. Then

$$\begin{aligned}
|\sqrt{a_n} - \sqrt{L}| &= |\sqrt{a_n} - \sqrt{L}| \frac{|\sqrt{a_n} + \sqrt{L}|}{|\sqrt{a_n} + \sqrt{L}|} \\
&= \frac{|a_n - L|}{|\sqrt{a_n} + \sqrt{L}|} \\
&\leq \frac{|a_n - L|}{\sqrt{L}} \\
&< \frac{\epsilon\sqrt{L}}{\sqrt{L}} \\
&= \epsilon.
\end{aligned}$$

Thus, if $L \neq 0$ then $\lim \sqrt{a_n} = \sqrt{L}$.

Suppose now that $L = 0$. Let $\epsilon > 0$. Since $\lim a_n = L = 0$, there is an $N \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N$ then $a_n = |a_n - 0| < \epsilon^2$. Suppose that $n \in \mathbb{N}$ and $n > N$. Since $a_n < \epsilon^2$, then $\sqrt{a_n} < \sqrt{\epsilon^2} = \epsilon$. Then $|\sqrt{a_n} - \sqrt{L}| = \sqrt{a_n} < \epsilon$. Thus $\lim \sqrt{a_n} = \sqrt{L}$ in this case also. □

At this point, we can perform most algebraic operations on sequences. We have only to deal with division. In addressing division, we will need to know that if the limit of a sequence is not 0, then if n is large enough s_n cannot be 0. This is the next lemma.

Lemma 2.2.13. *Suppose that $\langle b_n \rangle$ is a convergent sequence and that $\lim b_n \neq 0$. There is a positive real number B and a real number N so that if $n > N$ then $|b_n| > B$.*

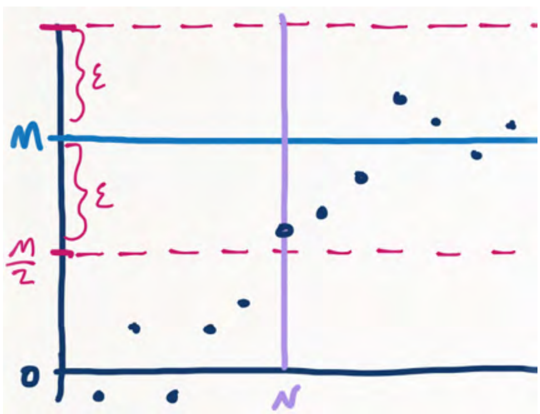

Figure 2.7: If $\langle b_n \rangle$ converges to $M > 0$, then $\langle b_n \rangle$ is eventually greater than $B = M/2$.

Proof. Suppose that $\langle b_n \rangle$ converges to a limit $M \neq 0$. By Lemma 2.2.10 we know that $\langle |b_n| \rangle$ converges to $|M|$. Let $B = |M|/2$. Applying the definition of convergence with $\epsilon = |M|/2$ gives us an $N \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N$ then $||b_n| - |M|| < |M|/2$. Then, if $n > N$ we have

$$-\frac{|M|}{2} < |b_n| - |M| < \frac{|M|}{2}.$$

Adding $|M|$ gives

$$\frac{|M|}{2} < |b_n| < |M| + \frac{|M|}{2}.$$

In particular, if $n > N$, then $|b_n| > |M|/2 = B$. □

We will approach quotients $\langle a_n/b_n\rangle$ as products of sequences $\langle a_n\rangle$ and $\langle 1/b_n\rangle$. Since we already know how to deal with products, we need only consider reciprocals at this point.

Lemma 2.2.14. *Suppose that $\langle b_n\rangle$ is a sequence of nonzero terms which converges to a number $M \neq 0$. The sequence $\langle 1/b_n\rangle$ converges to $1/M$.*

Proof. Since $\lim b_n = M \neq 0$, by Lemma 2.2.13 there is an $N_0 \in \mathbb{R}$ and a real number $B > 0$ so that if $n \in \mathbb{N}$ and $n > N$ then $|b_n| > B$. Then, if $n > N_0$, $\dfrac{1}{|b_n|} < \dfrac{1}{B}$. Let $\epsilon > 0$. Since $\lim b_n = M$, there is an $N_b \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N_b$ then $|b_n - M| < \epsilon B|M|$. Let $N = \max(N_0, N_b)$. Suppose that $n \in \mathbb{N}$ and $n > N$. Then

$$\begin{aligned}\left|\frac{1}{b_n} - \frac{1}{M}\right| &= \frac{|M - b_n|}{|b_n|\cdot|M|}\\ &= \frac{|b_n - M|}{|b_n|\cdot|M|}\\ &< \frac{\epsilon B|M|}{|b_n|\cdot|M|}\\ &< \frac{\epsilon B|M|}{B|M|}\\ &= \epsilon.\end{aligned}$$

Thus $\lim \dfrac{1}{b_n} = \dfrac{1}{m}$. □

We are finally ready to deal with quotients. We have done enough work now that the proof of the next lemma is quick. Under the hypothesis of the lemma, by Lemma 2.2.14 we know that since $\lim b_n = M$, then $\lim \dfrac{1}{b_n} = \dfrac{1}{M}$. By Lemma 2.2.3 since $\langle a_n\rangle$ converges to L and $\left\langle \dfrac{1}{b_n}\right\rangle$ converges to $\dfrac{1}{M}$, then $\left\langle \dfrac{a_n}{b_n}\right\rangle = \left\langle a_n\dfrac{1}{b_n}\right\rangle$ converges to $L\dfrac{1}{M}$ or $\dfrac{L}{M}$. Thus we have:

Lemma 2.2.15. (Quotients of Sequences) *Suppose that $\langle a_n\rangle$ and $\langle b_n\rangle$ are sequences so that $\lim a_n = L$ and $\lim b_n = M$. If $b_n \neq 0$ for all n and if $M \neq 0$, then $\lim(a_n/b_n) = L/M$.* □

We now summarize the results of this section into one big theorem about algebraic properties of limits of sequences.

Theorem 2.2.16. (Algebraic Properties of Limits of Sequences) *Suppose that $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences so that $\lim a_n = L$ and $\lim b_n = M$. Suppose that $k \in \mathbb{R}$ and that p is a polynomial.*

1. $\lim k = k$.
2. $\lim(ka_n) = kL$.
3. $\lim(a_n + b_n) = L + M$.
4. $\lim(a_n - b_n) = L - M$.
5. $\lim p(a_n) = p(L)$.
6. $\lim(a_n b_n) = LM$.
7. $\lim |a_n| = |L|$.
8. *If* $a_n \geq 0$ *for all* n*, then* $\lim \sqrt{a_n} = \sqrt{L}$.
9. *If* $b_n \neq 0$ *for all* n *and if* $M \neq 0$*, then* $\lim(a_n/b_n) = L/M$. □

We close this section with two order theorems about limits of sequences. The first is an extension of Lemma 2.2.11.

Theorem 2.2.17. (Order Theorem for Limits of Sequences) *Suppose that $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences so that $\lim a_n = L$ and $\lim b_n = M$. If $a_n \leq b_n$ for all n then $L \leq M$.*

Proof. Consider the sequence $\langle s_n \rangle$ given by $s_n = a_n - b_n$. By Theorem 2.2.16 we know that $\lim s_n = L - M$. For each n, since $a_n \leq b_n$, then $s_n \leq 0$. By Lemma 2.2.11

$$L - M = \lim s_n \leq 0.$$

Thus $L \leq M$. □

The next result, the Squeeze Theorem, will be essential frequently to argue that sequences we construct converge to the desired limits. A faulty approach to the proof of the Squeeze Theorem is to simply apply Theorem 2.2.17 twice – once to $a_n \leq b_n$ and once to $b_n \leq c_n$. The reason this approach does not work is that we do not know initially if $\langle b_n \rangle$ even converges.

Theorem 2.2.18. (Squeeze Theorem) *Suppose that $\langle a_n \rangle$, $\langle b_n \rangle$, and $\langle c_n \rangle$ are sequences so that*

$$a_n \leq b_n \leq c_n \text{ for all } n \text{ and } \lim a_n = \lim c_n = L.$$

Then $\langle b_n \rangle$ also converges to L.

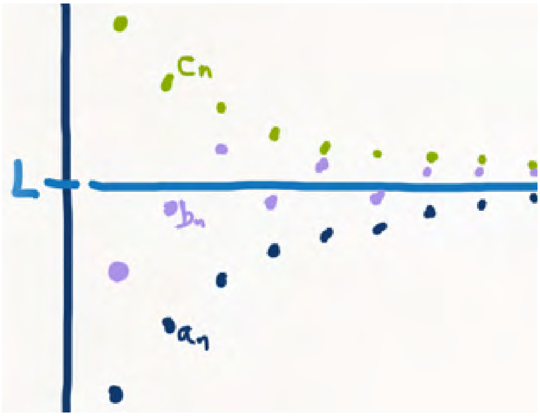

Figure 2.8: If $a_n \leq b_n \leq c_n$ and if $\langle a_n \rangle$ and $\langle c_n \rangle$ both converge to L, then $\langle b_n \rangle$ is also forced to converge to L.

Proof. Let $\epsilon > 0$. Since $\lim a_n = L$, there is an $N_a \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N_a$, then $|a_n - L| < \epsilon$. Since $\lim c_n = L$, there is an $N_c \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N_c$, then $|c_n - L| < \epsilon$. Let $N = \max(N_a, N_b)$. Suppose that $n \in \mathbb{N}$ and $n > N$. Then $|a_n - L| < \epsilon$, so $L - \epsilon < a_n$. Also, $|c_n - L| < \epsilon$, so $c_n < L + \epsilon$. It follows that

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$

so $|b_n - L| < \epsilon$. Thus $\langle b_n \rangle$ converges to L. $\square$

Example 2.2.19. Suppose that $A \subseteq \mathbb{R}$ is bounded. As an example application of the Squeeze Theorem, we prove that there is a a sequence $\langle a_n \rangle$ of elements of A so that $\lim a_n = \sup A$. Let $L = \sup A$. For each $n \in \mathbb{N}$, the number $L - \dfrac{1}{n}$ is not an upper bound of A, so there is some element $a_n \in A$ so that

$$L - \frac{1}{n} < a_n \leq L.$$

Now, $\lim \left(L - \dfrac{1}{n} \right) = L$ and $\lim L = L$, so by the Squeeze Theorem, $\langle a_n \rangle$ converges to L.

Exercises 2.2

2.2.1 Suppose that $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences so that $\langle a_n \rangle$ and $\langle a_n + b_n \rangle$ converge. Prove that $\langle b_n \rangle$ converges.

2.2.2 Give examples of divergent sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ for which $\langle a_n + b_n \rangle$ converges.

2.2.3 Find the limits of these sequences. Use theorems from this chapter to prove you have the correct limit.

a. $\left\langle \dfrac{n^2 + 4n}{n^2 - 5} \right\rangle$

b. $\left\langle \dfrac{\cos n}{n} \right\rangle$

c. $\left\langle \dfrac{\sin(n^2)}{n} \right\rangle$

d. $\left\langle \dfrac{n}{n^2 - 3} \right\rangle$

e. $\left\langle \left(\sqrt{4 - \dfrac{1}{n}} - 2 \right) n \right\rangle$

f. $\left\langle (-1)^n \dfrac{\sqrt{n}}{n + 7} \right\rangle$

2.2.4 Define a sequence $\langle s_n \rangle$ recursively by $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$ for $n \in \mathbb{N}$.

a. List the first several terms of $\langle s_n \rangle$.

b. The sequence $\langle s_n \rangle$ converges. Find the limit.

2.2.5 Define a sequence $\langle s_n \rangle$ recursively by $s_1 = 1$ and $s_{n+1} = \dfrac{s_n^2 + 2}{2s_n}$ for $n \in \mathbb{N}$.

a. List the first several terms of $\langle s_n \rangle$.

b. Assume that the sequence $\langle s_n \rangle$ converges. Find the limit.

2.2.6 Define a sequence $\langle s_n \rangle$ recursively by $s_1 = 1$ and $s_{n+1} = 3s_n^2$ for $n \in \mathbb{N}$.

a. List the first several terms of $\langle s_n \rangle$.

b. Assume that the sequence $\langle s_n \rangle$ converges. Find the limit.

c. Does $\lim \langle s_n \rangle$ actually exist?

d. What is the moral?

2.2.7 Suppose that $\langle s_n \rangle$ is a sequence so that $s_n \neq 0$ for all $n \in \mathbb{N}$. Suppose also that $\lim \left| \frac{s_{n+1}}{s_n} \right|$ exists. Call this limit L.

a. Prove that if $L < 1$ then $\langle s_n \rangle$ converges to 0.

b. Prove that if $L > 1$ then $\langle s_n \rangle$ is unbounded.

2.2.8 Prove that $\lim \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

2.2.9 Prove this generalization of the Squeeze Theorem 2.2.18: Suppose that $\langle a_n \rangle$, $\langle b_n \rangle$, and $\langle c_n \rangle$ are sequences so that b_n is between a_n and c_n for all n and so that $\lim a_n = \lim c_n = L$. Then $\langle b_n \rangle$ also converges to L.

2.3 Accumulation Points

Definition 2.3.1. A number $a \in \mathbb{R}$ is an *accumulation point* of $A \subseteq \mathbb{R}$ if every open interval containing a contains a point in A other than a.

Example 2.3.2. Perhaps the simplest examples of accumulation points are endpoints of intervals. The numbers a and b are accumulation points of the intervals $[a, b]$, (a, b), $(a, b]$, and $[a, b)$. Every number x with $a < x < b$ is also an accumulation point of these intervals. See Figure 2.9.

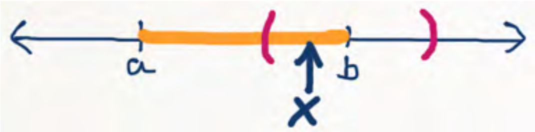

Figure 2.9: Any open interval around b must include points in the interval (a, b).

Example 2.3.3. Suppose that A is a bounded set, that $u = \sup A$, and that $u \notin A$. Then u is an accumulation point of A. To see this, suppose that (a, b) is any open interval containing u, Then $a < u$, so a is not an upper bound of A. This means that there is an element $x \in A$ so that $a < x$. Since $u = \sup A$, $x \le u$. Since $u \notin A$, $x < u$. Then x is an element of A distinct from u which is in the interval (a, b).

Example 2.3.4. If A is a bounded set and $\sup A \in A$, then $\sup A$ might not be an accumulation point of A. For example, if $A = \{1, 2\}$, then $\sup A = 2$, but 2 is not an accumulation point of A because the open interval $\left(\frac{3}{2}, \frac{5}{2}\right)$ contains 2 but does not contain any other element of A.

Example 2.3.5. If $A = \{1/n : n \in \mathbb{N}\}$, then 0 is an accumulation point of A. Suppose that (a, b) is any open interval containing 0. Then there is an $n \in \mathbb{N}$ so that $0 < 1/n < b$. Thus, $1/n$ is an element of A other than 0 which is in (a, b).

Example 2.3.6. It is not always the case that if a sequence $\langle a_n \rangle$ converges to L and if $A = \{a_n : n \in \mathbb{N}\}$ then L is an accumulation point of A. For example, if $\langle a_n \rangle$ is given by $a_n = 1$ then in this case A has no accumulation points.

Theorem 2.3.7. *The number $z \in \mathbb{R}$ is an accumulation point of $A \subseteq \mathbb{R}$ if and only if every open interval containing z contains infinitely many points of A.*

Proof. If every open interval around z contains infinitely many points of A, then every open interval around z contains a point of A other than z, and z is an accumulation point of A. Suppose now that z is an accumulation point of A, and let (a, b) be an open interval around z. We will recursively construct a sequence $\langle a_n \rangle$ of distinct elements of A in $(a, b) - \{z\}$. First, since z is an accumulation point of A, there is a point x_1 of A which is in $(a, b) - \{z\}$. Suppose that distinct points $x_1, \ldots, x_k$ in A have been chosen in $(a, b) - \{z\}$. Let $\epsilon = \min(|z - x_1|, \ldots, |z - x_k|)$. The open interval $(z - \epsilon, z + \epsilon)$ must contain an element x_{k+1} of A different from z. Since x_{k+1} is within ϵ of z, then x_{k+1} is also different from $x_1, \ldots, x_k$. (The point x_{k+1} is too close to z to be one of these. See Figure 2.10.) Thus, we have a sequence $\langle x_n \rangle$ of distinct elements of A which are all in $(a, b) - \{z\}$. This sequence gives infinitely many elements of A in (a, b). Hence, every open interval containing z contains infinitely many points in A. $\square$

Figure 2.10: If ϵ is chosen to be the minimum of the distances $|z-x_1|$, $|z-x_2|$, and $|z-x_3|$, then none of x_1, x_2, or x_3 is in the interval $(z-\epsilon, z+\epsilon)$.

Example 2.3.8. A quick consequence of Theorem 2.3.7 is that no finite set can have a limit point.

We can refine the proof technique of Theorem 2.3.7 to construct not only infinitely many points in A but actually a sequence in A converging to the accumulation point z.

Theorem 2.3.9. *A number $z \in \mathbb{R}$ is an accumulation point of the set $A \subseteq \mathbb{R}$ if and only if there is a sequence $\langle a_n \rangle$ in $A - \{z\}$ converging to z.*

Proof. Suppose first that z is an accumulation point of A. For each $n \in \mathbb{N}$ there is an element of A different from z in the open interval $(z - 1/n, z + 1/n)$. Call this element a_n. This gives a sequence $\langle a_n \rangle$ in $A - \{z\}$ so that $z - 1/n < a_n < z + 1/n$ for all n. Since $\langle z - 1/n \rangle$ and $\langle z + 1/n \rangle$ both converge to z, the sequence $\langle a_n \rangle$ converges to z by the Squeeze Theorem.

Suppose now that there is a sequence $\langle a_n \rangle$ in $A - \{z\}$ converging to z. We will prove that z is an accumulation point of A. Suppose that (a, b) is any open interval containing z. There is some $\epsilon > 0$ so that $(z - \epsilon, z + \epsilon) \subseteq (a, b)$. Since $\langle a_n \rangle$ converges to z, there is an $N \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N$ then $|a_n - z| < \epsilon$. If $n > N$, then $a_n \in (z - \epsilon, z + \epsilon) \subseteq (a, b)$. Since $a_n \in A - \{z\}$, then we see that (a, b) contains an element of A other than z. Since every open interval containing z contains an element of A other than z, z is an accumulation point of A. $\square$

Since every open interval contains infinitely many rational numbers and infinitely many irrational numbers, this theorem is not hard to prove:

Theorem 2.3.10. *Every rational number is an accumulation point of $\mathbb{R}$, of $\mathbb{Q}$, and $\mathbb{R} - \mathbb{Q}$. Every irrational number is an accumulation point of $\mathbb{R}$, of $\mathbb{Q}$, and $\mathbb{R} - \mathbb{Q}$.* □

Now, Theorem 2.3.9 immediately gives:

Theorem 2.3.11. *If x is any real number, then there is a sequence of irrational numbers which converges to x and there is a sequence of rational numbers which converges to x.* □

Exercises 2.3

2.3.1 Give an example of a set with exactly two accumulation points.
2.3.2 Give an example of a set with countably many accumulation points.
2.3.3 Give an example of a countable set with uncountably many accumulation points.
2.3.4 Give an example of a set that contains all of its accumulation points.
2.3.5 Give an example of a set that contains none of its accumulation points.
2.3.6 If $x \neq y$, prove that there are open intervals P and Q with $x \in P$, $y \in Q$, but $P \cap Q = \emptyset$. A hint for how to proceed is in Figure 2.11.

Figure 2.11: Suppose that $x < y$. There are open intervals containing x and y.

2.3.7 Suppose that I is an open interval and that $x \in I$. Prove that there is a $\delta > 0$ so that $(x - \delta, x + \delta) \subseteq I$. A hint for how to proceed is in Figure 2.12.
2.3.8 Let $S \subseteq \mathbb{R}$ be a bounded non-empty set and let $x = \sup S$. Prove that either $x \in S$ or x is an accumulation point of S. Hint: To prove a statement of the form $P \vee Q$, prove $\neg P \to Q$.
2.3.9 Suppose that $\langle a_n \rangle$ converges to L and that $\{a_n : n \in \mathbb{N}\}$ is infinite. Prove that L is an accumulation point of $\{a_n : n \in \mathbb{N}\}$. A hint for how to proceed is in Figure 2.13.
2.3.10 Give an example of a sequence $\langle a_n \rangle$ that converges to a number L so that L is **not** an accumulation point of $\{a_n : n \in \mathbb{N}\}$.

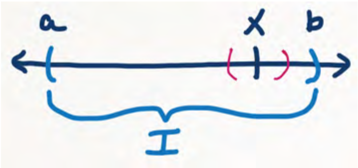

Figure 2.12: If x is in the open interval I, then there is an open interval around x which is contained entirely in I.

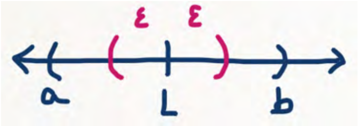

Figure 2.13: If $a < L < b$, then there is an $\epsilon > 0$ so that $a < L - \epsilon < L + \epsilon < b$. If $x_n \to L$, then $\langle x_n \rangle$ is eventually in the interval $(L - \epsilon, L + \epsilon)$. If $\langle x_n \rangle$ is infinite, then this interval must contain an x_n which is distinct from L.

2.4 Monotonic Sequences

In this section we introduce monotonic sequences. These are sequences which are always "going up" or "going down" as in Figure 2.14. These sequences are nice for us because we will always know exactly how a monotonic function behaves (as far as convergence). For example, if a sequence is increasing and bounded then the sequence must increase toward (and converge to) its least upper bound as in figure 2.15. Just knowing the existence of certain monotonic sequences will be good enough later to conclude some remarkable results such as the Bolzano Weierstrass Theorem.

Definition 2.4.1. A sequence $\langle s_n \rangle$ is *increasing* if $s_n \leq s_{n+1}$ for all n. If $s_n < s_{n+1}$ for all n, then $\langle s_n \rangle$ is *strictly increasing*. If $s_{n+1} \leq s_n$ for all n, then $\langle s_n \rangle$ is *decreasing*. If $s_{n+1} < s_n$ for all n, then $\langle s_n \rangle$ is *strictly decreasing*. A sequence which is either increasing or decreasing is *monotonic* or *monotone*.

Example 2.4.2. Define a sequence $\langle s_n \rangle$ recursively by $s_1 = 2$ and $s_{n+1} = \dfrac{s_n + 1}{2}$ for $n \in \mathbb{N}$. Then $\langle s_n \rangle$ is decreasing. To prove this, we use induction to prove that $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$. First, note

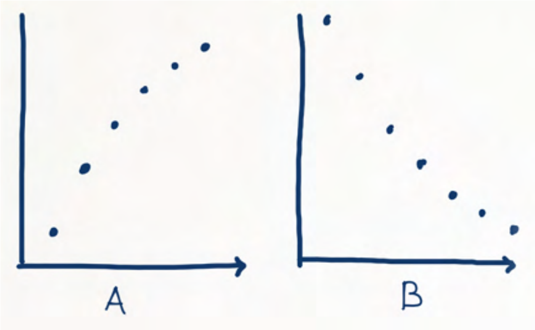

Figure 2.14: The sequence on the left is increasing. The sequence on the right is decreasing.

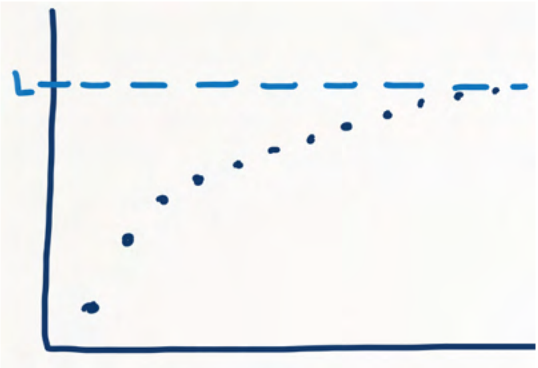

Figure 2.15: A sequence increasing toward its supremum.

that $s_1 = 2$ and $s_2 = 3/2$, so $s_1 \geq s_2$. Next, suppose that $k \in \mathbb{N}$ and that $s_k \geq s_{k+1}$. We will prove $s_{k+1} \geq s_{k+2}$. Since $s_k \geq s_{k+1}$, then $s_k + 1 \geq s_{k+1} + 1$. But then $\dfrac{s_k + 1}{2} \geq \dfrac{s_{k+1} + 1}{2}$. This last inequality is exactly $s_{k+1} \geq s_{k+2}$. By induction, $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$.

Theorem 2.4.3. *Suppose that $A \subseteq \mathbb{R}$ is bounded above. There is an increasing sequence of elements of A which converges to* $\sup A$.

Proof. This proof is pictured in Figure 2.16. Let $u = \sup A$. If $u \in A$, then the constant sequence $\langle u \rangle$ is a sequence of elements of A converging to u. Suppose then that $u \notin A$. We will recursively construct a strictly increasing sequence $\langle x_n \rangle$ of elements of A so that $u - \dfrac{1}{n} < x_n < u$ for all

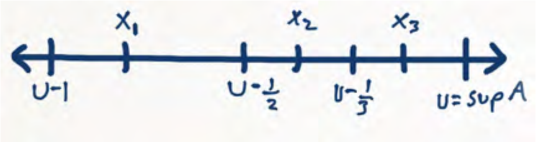

Figure 2.16: Once $x_1 < x_2 < x_3$ have been selected, x_4 must be selected to be greter than x_3 and greater than $u - 1/4$.

n. The Squeeze Theorem will then guarantee that $\langle x_n \rangle$ converges to u. First, note that $u-1$ is not an upper bound of A, so there is an element $x_1 \in A$ with $u - 1 < x_1 < u$ (the second less than follows from the fact that $u \notin A$). Suppose that strictly increasing elements $x_1, \ldots, x_k$ of A have been selected so that $u - \frac{1}{n} < x_n < u$ for $n = 1, \ldots, k$. Let $m = \max(u - \frac{1}{k+1}, x_k)$. Then $m < u$, so m is not an upper bound of A. This means that there is an element x_{k+1} of A with $m < x_{k+1} < u$. This forces $x_k < x_{k+1} < u$ and $u - \frac{1}{k+1} < x_{k+1} < u$ as desired. We now have a strictly increasing sequence $\langle x_n \rangle$ of elements of A so that $u - \frac{1}{n} < x_n < u$ for all n. By the Squeeze Theorem, $\lim x_n = u$. □

Of course, we could adjust this proof to show:

Theorem 2.4.4. *Suppose that $A \subseteq \mathbb{R}$ is bounded below. There is a decreasing sequence of elements of A which converges to* $\inf A$. □

Theorem 2.4.5. *Suppose that $\langle s_n \rangle$ is a monotonic sequence. Then $\langle s_n \rangle$ converges if and only if $\langle s_n \rangle$ is bounded.*

Proof. This proof is pictured in Figure 2.17. If $\langle s_n \rangle$ converges, then the sequence is bounded by Theorem 2.2.6. Suppose then that $\langle s_n \rangle$ is increasing and monotonic. Let $S = \{s_n : n \in \mathbb{N}\}$ and let $L = \sup S$. We will prove that $\langle s_n \rangle$ converges to L. Let $\epsilon > 0$. The number $L - \epsilon$ is not an upper bound of S. Therefore there is some $N \in \mathbb{R}$ so that $L - \epsilon < s_N \leq L$. Suppose now that $n \in \mathbb{N}$ and that $n > N$. Then $L - \epsilon < s_N \leq s_n \leq L$ so $|s_n - L| < \epsilon$. Thus $\langle s_n \rangle$ converges to L. A similar argument shows that if $\langle s_n \rangle$ is decreasing and bounded then $\langle s_n \rangle$ converges to $\inf\{s_n : n \in \mathbb{N}\}$. □

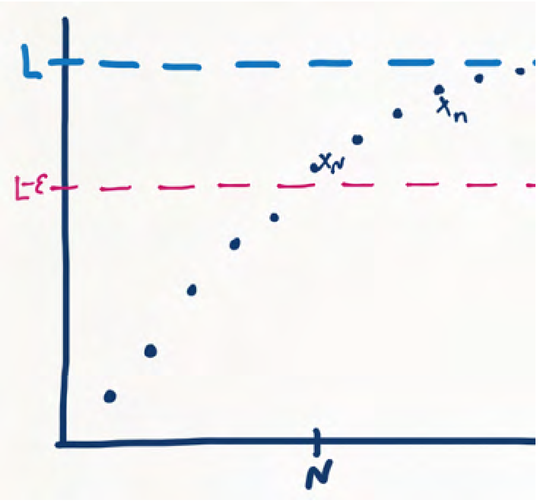

Figure 2.17: Once x_n is within ϵ of L, then x_n is between x_N and L for all $n > N$.

Example 2.4.6. Consider the sequence $\langle s_n \rangle$ recursively defined by $s_1 = 2$ and $s_{n+1} = \dfrac{s_n + 1}{2}$ for $n \in \mathbb{N}$. We proved in Example 2.4.2 that $\langle s_n \rangle$ is decreasing. Since $\langle s_n \rangle$ is decreasing, we know that $s_n \leq s_1 = 2$ for all n. Also, since we only ever add 1 or divide by 2 in computing the terms of $\langle s_n \rangle$, it follows that $s_n \geq 0$ for all n. Thus the sequence $\langle s_n \rangle$ is bounded and monotonic. By Theorem 2.4.5 $\langle s_n \rangle$ converges.

To find the limit of this sequence, we will use a trick which is sometimes useful for recursively defined sequences. Let $L = \lim s_n$. Consider the equality $s_{n+1} = \dfrac{s_n + 1}{2}$. Taking the limit of the left hand side of this equation gives L. Taking the limit of the right hand side of this equation gives $\dfrac{L+1}{2}$. Hence it must be that

$$L = \frac{L+1}{2}.$$

Solving this equality for L gives $L = 1$.

We can use the fact that bounded monotonic sequences converge to prove the following essential theorem. We will have two theorems that hold the name Bolzano-Weierstrass. We will use this first version to prove later that Cauchy Sequences are convergent (Theorem 2.2.6).

We can also use this first version to prove the second (Theorem 2.6.11); however, we will offer a more direct proof of that theorem (again employing Theorem 2.2.6). The proof of this first Bolzano-Weierstrass Theorem constructs two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$. The sequence $\langle a_n \rangle$ will be increasing, and the sequence $\langle b_n \rangle$ will be decreasing. The two sequences will converge toward each other, squeezing down on an accumulation point of A.

Theorem 2.4.7. (Bolzano-Weierstrass Theorem Version 1) *Any bounded infinite set has an accumulation point.*

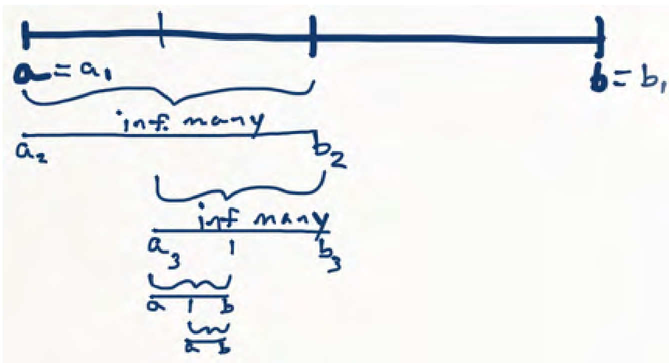

Figure 2.18: The process of bisecting intervals in the proof of Theorem 2.4.7.

Proof. This proof is pictured in Figure 2.18. Suppose that A is a bounded infinite set. This means that there is an interval $[a, b]$ so that $A \subseteq [a, b]$. We recursively define two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ so that $a_n < b_n$ for every n and so that there are infinitely many elements of A in $[a_n, b_n]$ for all n. First, let $a_1 = a$ and $b_1 = b$. Note that $a_1 < b_1$ and that there are infinitely many elements of A in $[a_1, b_1]$. Assuming that a_k and b_k have been defined so that $a_k < b_k$ and so that there are infinitely many elements of A in $[a_k, b_k]$, we show how to define a_{k+1} and b_{k+1}. Let $m = \dfrac{a_k + b_k}{2}$. Note that $a_k < m < b_k$. Since there are infinitely many elements of A in $[a_k, b_k]$, then there are infinitely many elements of A in $[a_k, m]$ or in $[m, b_k]$ (or both). If there are infinitely many elements of A in $[a_k, m]$, let $a_{k+1} = a_k$ and $b_{k+1} = m$. Otherwise, let $a_{k+1} = m$ and $b_{k+1} = b_k$. Then $a_{k+1} < b_{k+1}$ and there are infinitely many elements of A in $[a_{k+1}, b_{k+1}]$. This gives the desired two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$. We get a number of results as byproducts of how we defined our sequences.

1. At each step in the recursive construction, either $a_k = a_{k+1}$ or $a_k < a_{k+1}$. Thus, $\langle a_n \rangle$ is increasing.

2. Similarly, $\langle b_n \rangle$ is decreasing.

3. It was built into the design of our sequences that $a_n < b_n$ for every n.

4. It was also built into the design that the interval $[a_n, b_n]$ contains infinitely many elements of A for every n.

5. Notice from the definition of our sequences that $|b_{k+1} - a_{k+1}| = \frac{1}{2}|b_k - a_k|$ for each k. It follows that $|b_n - a_n| = \frac{1}{2^{n-1}}|b - a|$ for all n.

Notice that for all n we have $a \leq a_n < b_n \leq b$, so both $\langle a_n \rangle$ and $\langle b_n \rangle$ are bounded. Since both sequences are also monotonic (1 and 2 above), Theorem 2.4.5 tells us that both sequences converge. Let $L = \lim a_n$ and $R = \lim b_n$. It follows from 5 above that the sequence $\langle b_n - a_n \rangle$ converges to 0. Therefore

$$L - R = \lim b_n - \lim a_n = \lim(b_n - a_n) = 0$$

and $L = R$.

We now have only to argue that L is an accumulation point of A. Let (c, d) be any open interval containing L. Since $\lim(b_n - a_n) = 0$, there is an n so that $[a_n, b_n] \subseteq (c, d)$. Since $[a_n, b_n]$ contains infinitely many elements of A, so does (c, d). Hence, L is an accumulation point of A. $\square$

Exercises 2.4

2.4.1 Find upper and lower bounds of the sequence $\left\langle \dfrac{3n+7}{n} \right\rangle$.

2.4.2 Give an example of a sequence which is bounded but not convergent.

2.4.3 Which of the following sequences are increasing? decreasing? bounded?

a. $\left\langle \frac{1}{n} \right\rangle$

b. $\langle n^5 \rangle$

c. $\langle (-2)^n \rangle$

d. $\left\langle \frac{(-1)^n}{n^2} \right\rangle$

e. $\langle \sin(n\pi/7) \rangle$

f. $\left\langle \frac{n}{3^n} \right\rangle$

2.4.4 Prove that if $\langle a_n \rangle$ is decreasing and bounded, then $\langle a_n \rangle$ converges.

2.4.5 Prove that if $A \subseteq \mathbb{R}$ is a bounded set then there is a decreasing sequence of elements of A which converges to $\inf A$.

2.4.6 Simplify the proof from Exercise 2.4.5 to prove that if $A \subseteq \mathbb{R}$ is a bounded set then there is a sequence of elements of A which converges to $\inf A$. (We have left out "decreasing.")

2.4.7 Define a sequence $\langle s_n \rangle$ recursively by $s_1 = 1$ and $s_{n+1} = \frac{n}{n+1} s_n^2$ for $n \in \mathbb{N}$.

a. List the first several terms of $\langle s_n \rangle$.

b. Prove that $\langle s_n \rangle$ is monotonic.

c. Prove that $\langle s_n \rangle$ is bounded.

d. Prove that $\langle s_n \rangle$ converges and find the limit.

2.4.8 Define a sequence $\langle s_n \rangle$ recursively by $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \in \mathbb{N}$.

a. List the first several terms of $\langle s_n \rangle$.

b. Prove that $\langle s_n \rangle$ is monotonic.

c. Prove that $\langle s_n \rangle$ is bounded.

d. Prove that $\langle s_n \rangle$ converges and find the limit.

2.4.9 Define a sequence $\langle s_n \rangle$ recursively by $s_1 = 1$ and

$$s_{n+1} = \left(1 - \frac{1}{4n^2}\right) s_n$$

for $n \in \mathbb{N}$.

a. List the first several terms of $\langle s_n \rangle$.

b. Prove that $\langle s_n \rangle$ converges.

2.4.10 Define a sequence $\langle s_n \rangle$ recursively by $s_1 = 1$ and

$$s_{n+1} = \left(1 - \frac{1}{(n+1)^2}\right) s_n$$

for $n \in \mathbb{N}$.

a. List the first several terms of $\langle s_n \rangle$.

b. Prove that $\langle s_n \rangle$ converges.

c. Use induction to prove that $s_n = \dfrac{n+1}{2n}$ for all $n \in \mathbb{N}$.

2.5 Cauchy Sequences

To use the definition of convergence to prove that a sequence converges, we need a potential limit. It will occasionally be useful to be able to prove that a sequence converges without knowing the limit (since the limit might be difficult or impossible to find). A Cauchy sequence is a sequence which "looks like" it should converge because the terms of the sequence get close together. This definition focuses on the distance between terms of the sequence rather than the distance between terms and a limit.

Definition 2.5.1. A sequence $\langle s_n \rangle$ of real numbers is a *Cauchy sequence* if for every real number $\epsilon > 0$ there is a real number N so that for all integers m, n, if $m, n > N$ then $|s_m - s_n| < \epsilon$.

Example 2.5.2. The sequence $\langle s_n \rangle$ given by $s_n = \dfrac{1}{n}$ is a Cauchy sequence. Let $\epsilon > 0$. Let $N = 1/\epsilon$. Suppose that $m, n \in \mathbb{N}$ with

$m, n > N$. Without loss of generality, assume that $m < n$. Then

$$
\begin{aligned}
|s_m - s_n| &= \left| \frac{1}{m} - \frac{1}{n} \right| \\
&= \left| \frac{n-m}{mn} \right| \\
&= \frac{n-m}{mn} \\
&< \frac{n}{mn} \\
&= \frac{1}{m} \\
&< \frac{1}{N} \\
&= \epsilon.
\end{aligned}
$$

Thus $\langle s_n \rangle$ is Cauchy.

Theorem 2.5.3. *Every convergent sequence is a Cauchy sequence.*

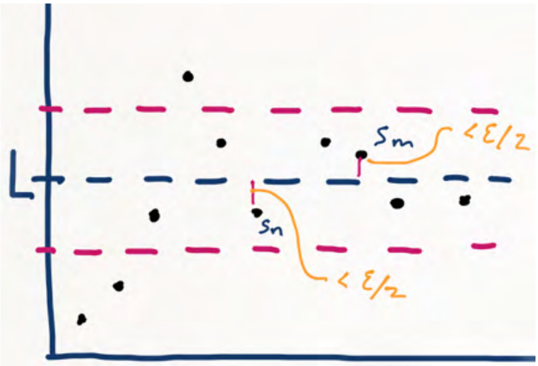

Figure 2.19: If the distance from s_n to L is less than $\epsilon/2$ and the distance from s_m to L is less than $\epsilon/2$, then the distance from s_n to s_m must be less than ϵ.

Proof. This proof is illustrated in Figure 2.19. Suppose that $\langle s_n \rangle$ is a sequence converging to a limit L. We will prove that $\langle s_n \rangle$ is Cauchy. Let $\epsilon > 0$. Since $\lim s_n = L$, there is an $N \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and

$n > N$ then $|s_n - L| < \epsilon/2$. Suppose that $m, n \in \mathbb{N}$ and $m, n > N$. Then

$$\begin{aligned} |s_m - s_n| &= |s_m - L + L - s_n| \\ &\leq |s_m - L| + |L - s_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus $\langle s_n \rangle$ is a Cauchy sequence. □

Every Cauchy sequence is also convergent. We will prove this with the help of our first Bolzano-Weierstrass Theorem 2.4.7. To do so, we need to know that Cauchy sequences are bounded.

Theorem 2.5.4. *Every Cauchy sequence is bounded.*

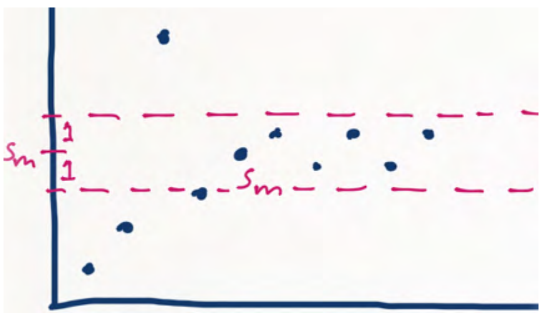

Figure 2.20: There are finitely many terms to the left of s_m. These are bounded. To the right of s_m, all terms are between $s_m - 1$ and $s_m + 1$.

Proof. This proof is illustrated in Figure 2.20. Suppose that $\langle s_n \rangle$ is a Cauchy sequence. We will prove that $\langle s_n \rangle$ is bounded. Applying the definition of Cauchy sequence with $\epsilon = 1$, We can find an $N \in \mathbb{R}$ so that if $m, n \in \mathbb{N}$ and $m, n > N$ then $|s_m - s_n| < 1$. Let m be the least natural number greater than N. Let $M = \max(|s_1|, |s_2|, \ldots, |s_m|, |s_m| + 1)$. We claim that M is a bound on $|s_n|$. If $n \leq m$, then $|s_n| \leq |M|$ simply by definition. Suppose that $n > m$. Then $|s_m - s_n| < 1$ so $s_m - 1 \leq s_n \leq s_m + 1$. But then

$$-|s_m| - 1 \leq s_m - 1 \leq s_n \leq s_m + 1 \leq |s_m| + 1.$$

This implies $|s_n| \leq |s_m| + 1 \leq M$. Thus $|s_n| \leq M$ for all n. □

We are now almost ready to prove that every Cauchy sequence is convergent. Our approach will be divided into two cases. Recall that a sequence $s = \langle s_n \rangle$ is really a function $s : \mathbb{N} \to \mathbb{R}$. As such, s has a range $\{s_n : n \in \mathbb{N}\}$. Our cases will be as to whether or not this range is infinite. If the range is infinite, then we can apply Theorem 2.4.7. If the range is finite, we see in the next lemma that s is eventually constant.

Lemma 2.5.5. *If $\langle s_n \rangle$ is a Cauchy sequence with a finite range, then there is an $N \in \mathbb{R}$ so that if $m, n \in \mathbb{N}$ and $m, n > N$ then $s_m = s_n$.*

Proof. Suppose that $\langle s_n \rangle$ is a Cauchy sequence with finite range. Let R be the range of $\langle s_n \rangle$. Let M be the minimum difference $|a - b|$ for $a \neq b$ in R (Note that such a minimum exists because R is finite). Let $\epsilon = M/2$. Now applying the definition of a Cauchy sequence gives an $N \in \mathbb{R}$ so that if $m, n \in \mathbb{N}$ and $m, n > N$ then $|s_m - s_n| < \epsilon$. Since ϵ is less than the minimum difference between any two terms, this implies that $s_m = s_n$ for $m, n > N$. □

Finally we can prove that every Cauchy sequence converges.

Theorem 2.5.6. *Every Cauchy sequence converges.*

Proof. Suppose that $\langle s_n \rangle$ is a Cauchy sequence. Let R be the range of $\langle s_n \rangle$. If R is finite, then the sequence is eventually constant by Lemma 2.5.5. In this case, the sequence converges. Assume then that R is infinite. By Theorem 2.5.4, R is bounded. Since R is bounded and infinite, by the Bolzano-Weierstrass Theorem 2.4.7, R has an accumulation point L. We prove that $\langle s_n \rangle$ converges to L. Let $\epsilon > 0$. Since $\langle s_n \rangle$ is Cauchy, there is an $N_0 \in \mathbb{R}$ so that if $m, n \in \mathbb{N}$ and $m, n > N_0$ then $|s_m - s_n| < \epsilon/2$. Since L is an accumulation point of R, there are infinitely many points in R which are in the interval $(L - \epsilon/2, L + \epsilon/2)$. Select $N \in \mathbb{N}$ so that $N > N_0$ and $s_N \in (L - \epsilon/2, L + \epsilon/2)$. Suppose that $n \in \mathbb{N}$ and $n > N$. Note that $|s_N - L| < \epsilon/2$ and $|s_n - s_N| < \epsilon/2$. Then

$$\begin{aligned} |s_n - L| &= |s_n - s_N + s_N - L| \\ &\leq |s_n - s_N| + |s_N - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus $\langle s_n \rangle$ converges to L. □

Exercises 2.5

2.5.1 Use the definition to prove that the sequence $\left\langle \dfrac{2n+1}{n} \right\rangle$ is Cauchy.

2.5.2 Suppose that $a_1 < a_2$ and define $\langle a_n \rangle$ recursively by $a_{n+2} = \dfrac{a_n + a_{n+1}}{2}$. Prove that $\langle a_n \rangle$ is Cauchy.

2.5.3 Find the limit of the sequence in Exercise 2.5.2 if $a_1 = 0$ and $a_2 = 3$.

2.6 Subsequences

Somewhat later (in the Extreme Value Theorem) we will be seeking the maximum value of a function f with domain D. That is, we will want a number u in D so that $f(x) \leq f(u)$ for all $x \in D$. At that point, we will know that the set $f(D) = \{f(x) : x \in D\}$ is bounded and has a supremum. By Theorem 2.4.3, we can find a sequence $\langle y_n \rangle$ in $f(D)$ converging to the supremum of $f(D)$. We can then find a sequence $\langle x_n \rangle$ in D so that $f(x_n) = y_n$ for all n. The natural desire at that point will be to let $u = \lim x_n$. However, we will have no guarantee that the sequence $\langle x_n \rangle$ will converge. This is but one example of an instance when we have a sequence which we would like to converge but which may not converge. The solution is to "throw out" some of the terms of the sequence in the hope that what remains will converge. The terms that remain will form a *subsequence* of the original sequence.

Definition 2.6.1. Suppose that $\langle s_n \rangle$ is any sequence and that

$$\langle n_k \rangle = \langle n_1, n_2, \ldots \rangle$$

is a strictly increasing sequence of natural numbers. The sequence $\langle s_{n_1}, s_{n_2}, s_{n_3}, \ldots \rangle$ is a subsequence of $\langle s_n \rangle$.

Remark 2.6.2. Note that if $\langle n_1, n_2, \ldots \rangle$ is a strictly increasing sequence then it has to be that $n_1 \geq 1$. Also, $n_2 > n_1 \geq 1$, so $n_2 \geq 2$. Similarly $n_3 > n_2 \geq 2$, so $n_3 \geq 3$. An induction argument can show that $n_k \geq k$ for all k.

Example 2.6.3. Consider the sequence

$$\left\langle (-1)^n + \frac{1}{n} \right\rangle = \left\langle 0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \frac{7}{6}, -\frac{6}{7}, \ldots \right\rangle.$$

Taking

$$\langle n_1, n_2, n_3, n_4, \ldots \rangle = \langle 1, 3, 5, 7, \ldots \rangle$$

gives the subsequence

$$\langle s_{n_k}\rangle = \langle s_1, s_3, s_5, s_7, \ldots\rangle = \left\langle 0, -\frac{2}{3}, -\frac{4}{5}, -\frac{6}{7}, \ldots\right\rangle.$$

Taking

$$\langle n_1, n_2, n_3, n_4, \ldots\rangle = \langle 2, 4, 6, 8 \ldots\rangle$$

gives the subsequence

$$\langle s_{n_k}\rangle = \langle s_2, s_4, s_6, s_8, \ldots\rangle = \left\langle \frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8} \cdots\right\rangle.$$

Notice that the original sequence $\langle s_n\rangle$ does not converge because its terms bounce back and forth between numbers close to -1 and numbers close to 1. However, these two subsequences both converge. The first converges to -1 and the second to 1.

Suppose that $\langle s_n\rangle$ is a sequence with a subsequence $\langle s_{n_k}\rangle$. If $\langle s_n\rangle$ converges to a number L, then *all* of the term of $\langle s_n\rangle$ are eventually close to L. It would make sense then that *some* of the terms (such as $\langle s_{n_k}\rangle$) eventually get close to L. Thus the subsequence should also converge to L.

Theorem 2.6.4. *If a sequence $\langle s_n\rangle$ converges to a number L, then every subsequence of $\langle s_n\rangle$ converges to L.*

Proof. Suppose that $\langle s_{n_k}\rangle$ is a subsequence of $\langle s_n\rangle$. We will prove that $\langle s_{n_k}\rangle$ converges to L. To do so, we must show that for all $\epsilon > 0$ there is an $N \in \mathbb{R}$ so that if $k \in \mathbb{N}$ and $k > N$ then $|s_{n_k} - L| < \epsilon$. Let $\epsilon > 0$. Since $\lim s_n = L$, there is an $N \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N$ then $|s_n - L| < \epsilon$. Suppose that $k \in \mathbb{N}$ and $k > N$. Then $n_k \geq k > N$ so $|s_{n_k} - L| < \epsilon$ as desired. Thus $\langle s_{n_k}\rangle$ converges to L. $\square$

Definition 2.6.5. A term s_N is a *dominant term* of the sequence $\langle s_n\rangle$ if $s_N \geq s_n$ for all $n > N$.

Example 2.6.6. If $\langle s_n\rangle$ is decreasing, then every term of $\langle s_n\rangle$ is dominant. If $\langle s_n\rangle$ is increasing, then no term is dominant.

Example 2.6.7. No terms of $\langle s_n\rangle = \langle(-1)^n n\rangle$ are dominant. See Figure 2.22.

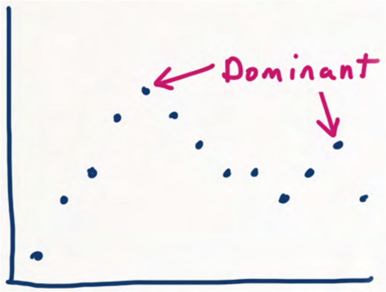

Figure 2.21: The indicated terms look as if they are higher than all the terms to the right of them. These are dominant.

Figure 2.22: For every term of this sequence, there is a higher term to the right. No terms are dominant.

Example 2.6.8. Suppose that

$$\langle s_n \rangle = \left\langle \frac{(-1)^n}{n} \right\rangle = \langle -1, 1/2, -1/3, 1/4, -1/5, 1/6, \ldots \rangle \,.$$

The dominant terms of $\langle s_n \rangle$ are $\langle 1/2, 1/4, 1/6, 1/8, \ldots \rangle$. Notice how the dominant terms form a decreasing sequence. See Figure 2.23.

Example 2.6.9. Suppose that $\langle s_n \rangle = \left\langle \dfrac{(n-3)(n-6)}{n^2} \right\rangle$. Then

$$s_1 = 10, s_2 = 1, s_3 = 0, \text{ and } s_4 = -1/8.$$

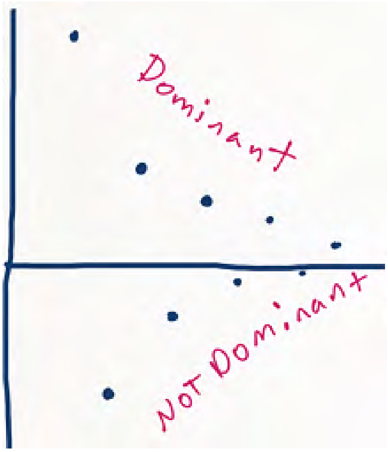

Figure 2.23: The terms above 0 are dominant. The terms below 0 cannot be dominant because of all of the positive terms. Note how the dominant terms form a decreasing subsequence.

After that point the terms increase toward 1. The only dominant terms of $\langle s_n \rangle$ are s_1 and s_2.

Lemma 2.6.10. *Every sequence has a monotonic subsequence.*

Proof. Suppose that $\langle s_n \rangle$ is any sequence. If $\langle s_n \rangle$ has infinitely many dominant terms, then the dominant terms of $\langle s_n \rangle$ form a decreasing subsequence and we are done. Assume then that $\langle s_n \rangle$ has only finitely many dominant terms. Select n_1 so that if $n \in \mathbb{N}$ and $n \geq n_1$, then s_n is not dominant. Then s_{n_1} is not dominant, so there is a natural number $n_2 > n_1$ so that $s_{n_1} < s_{n_2}$. Since $n_2 > n_1$, s_{n_2} is not dominant, so there is some $n_3 > n_2$ with $s_{n_2} < s_{n_3}$. Suppose that $n_1 < n_2 < \ldots < n_k$ have been chosen so that $s_{n_1} < s_{n_2} < \ldots < s_{n_k}$. Since $n_k > n_1$, then s_{n_k} is not dominant, so there is an $n_{k+1} > n_k$ with $s_{n_k} < s_{n_{k+1}}$. We have a recursively defined subsequence $\langle s_{n_k} \rangle$ of $\langle s_n \rangle$ which is strictly increasing. □

Suppose now that $\langle s_n \rangle$ is any bounded sequence. By Lemma 2.6.10, $\langle s_n \rangle$ has a monotonic subsequence. Since $\langle s_n \rangle$ is bounded, so is the subsequence. By Theorem 2.4.5, this subsequence must converge. Hence we have:

Theorem 2.6.11. (Bolzano-Weierstrass Theorem) *Every bounded sequence has a convergent subsequence.* □.

An alternative method of proving this Bolzano-Weierstrass Theorem is to use the first version of the Balzano-Weierstrass Theorem 2.4.7. If the range of a bounded sequence $\langle s_n \rangle$ is infinite then the range has a limit point L. Each interval of the form $(L - 1/n, L + 1/n)$ must contain infinitely many terms of $\langle s_n \rangle$. This fact can be used to recursively construct a subsequence of $\langle s_n \rangle$ converging to L. On the other hand, if the range of $\langle s_n \rangle$ is finite, then there have to be infinitely many terms of $\langle s_n \rangle$ which are equal. These equal terms form a convergent (constant) subsequence of $\langle s_n \rangle$.

We close this section with a result that is useful for proving that some bounded sequences converge. We will employ this trick in the proof of Theorem 3.7.3.

Theorem 2.6.12. *If every convergent subsequence of a bounded sequence $\langle s_n \rangle$ converges to the same number L, then $\langle s_n \rangle$ converges to L.*

Proof. Suppose that $\langle s_n \rangle$ does not converge to L. Then there is an $\epsilon > 0$ so that for all $N \in \mathbb{R}$ there is an $n \in \mathbb{N}$ with $n > N$ but $|s_n - L| \geq \epsilon$. We apply this fact repeatedly. First, if we take $N = 1$, then there is an $n_1 > 1$ so that $|s_{n_1} - L| \geq \epsilon$. Next, if we take $N = n_1$, then there is an $n_2 > n_1$ with $|s_{n_2} - L| \geq \epsilon$. Similarly, there is an $n_3 > n_2$ with $|s_{n_3} - L| \geq \epsilon$. Continuing in this way gives $n_1 < n_2 < n_3 < \ldots$ so that $|s_{n_k} - L| \geq \epsilon$ for all k. Thus we have a subsequence $\langle s_{n_k} \rangle$ whose terms are never closer than ϵ to L. This subsequence is bounded, so by the Bolzano-Weierstrass Theorem 2.6.11, $\langle s_{n_k} \rangle$ has a convergent subsequence. This subsequence is a convergent subsequence of $\langle s_n \rangle$ which cannot converge to L. We have proven that if $\langle s_n \rangle$ does not converge to L, then $\langle s_n \rangle$ has a convergent subsequence which does not converge to L. This is the contrapositive of the theorem. $\square$

Exercises 2.6

2.6.1 Let $s_n = 3 + 2(-1)^n$ for $n \in \mathbb{N}$.

a. List the first several terms of $\langle s_n \rangle$.

b. Find a subsequence of $\langle s_n \rangle$ which is convergent.

2.6.2 Find a convergent subsequence of the sequence $\left\langle (-1)^n \left(1 - \frac{1}{n}\right) \right\rangle$.

2.6.3 Suppose that $\langle a_n \rangle$ is a sequence and that x is an accumulation point of $\{a_n : n \in \mathbb{N}\}$. Prove that $\langle a_n \rangle$ has a subsequence converging to x.

2.6.4 Suppose that $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences which both converge to a number L. Define a new sequence $\langle c_n \rangle$ so that $c_{2n-1} = a_n$ and $c_{2n} = b_n$ for each $n = 1, 2, \ldots$. That is,

$$\langle c_n \rangle = \langle a_1, b_1, a_2, b_2, a_3, b_3, \ldots \rangle .$$

Prove that $\langle c_n \rangle$ converges to L.

2.6.5 Consider these sequences:

$$a_n = (-1)^n,\ b_n = \frac{1}{n},\ c_n = n^2,\ d_n = \frac{6n+4}{7n-3}.$$

a. For each sequence, give an example of a monotonic subsequence.

b. For each sequence find all limits of all subsequences.

c. Which of the sequences are bounded?

2.6.6 Suppose that $\langle s_n \rangle$ is a strictly increasing sequence of natural numbers. Prove that $n \leq s_n$ for all n.

Chapter 3

Limits and Continuity

3.1 Limits of Functions

Definition 3.1.1. Suppose that z is an accumulation point of $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is any function. The number L is a *limit* of f at z if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ so that for all $x \in D$, if $0 < |x - z| < \delta$ then $|f(x) - L| < \epsilon$.

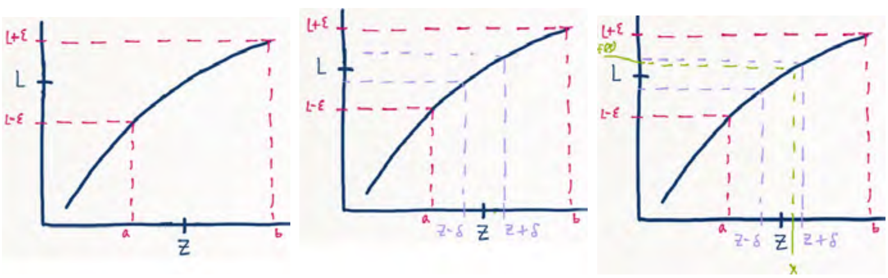

Figure 3.1: Consider the function $f(x)$ around z. In the first picture, we have marked off a distance of ϵ above and below L and have passed through the function back to the horizontal axis to an interval (a, b). (Note that this process can be extremely complicated for some functions. It is simple in this case.) In the second picture, we have selected a distance δ so that the interval $(z - \delta, z + \delta)$ is entirely between a and b. Finally, the third picture shows that if x is within δ of z, then $f(x)$ is within ϵ of L.

Example 3.1.2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2 - x$. Then 6 is a limit of f at 3. To prove this, we will shortly consider

$\epsilon > 0$ and have to specify a $\delta > 0$ so that if $0 < |x-3| < \delta$ then $|f(x)-6| < \epsilon$. To find our delta, we consider the difference $|f(x)-6|$ and try to manipulate this until we see $|x-3|$ like so:

$$|f(x)-6| = |x^2 - x - 6| = |x-3| \cdot |x+2|.$$

In this expression, we will assume that $|x-3| < \delta$. We will choose δ so that $\delta|x+2| < \epsilon$. There is a standard trick for doing so that works in many cases. We "fix" things so that $\delta \leq 1$. If $\delta \leq 1$, and if $|x-3| < \delta$, then x is between 2 and 4. This means that $|x+2| < 6$. If we want $\delta|x+2| < \epsilon$, it is good enough to have $\delta \cdot 6 < \epsilon$. We are now ready for the proof.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2 - x$. We prove that 6 is a limit of f at 3. Let $\epsilon > 0$. Let $\delta = \min(1, \epsilon/6)$. Suppose that $0 < |x-3| < \delta$. Then

$$\begin{aligned} |f(x)-6| &= |x^2 - x - 6| \\ &= |x-3| \cdot |x+2| \\ &< \delta \cdot 6 \\ &\leq \frac{\epsilon}{6} \cdot 6 \\ &= \epsilon. \end{aligned}$$

Thus 6 is a limit of f at 3.

Example 3.1.3. Suppose that $f : (0,1) \to \mathbb{R}$ is given by $f(x) = 1/x$. Then f has no limit at 0. To prove this, we must demonstrate that for any L there is an $\epsilon > 0$ so that for all $\delta > 0$ there is an $x \in (0,1)$ so that $0 < |x-0| < \delta$ but $|f(x)-L| \geq \epsilon$. Let $L \in \mathbb{R}$. Let $\epsilon = 1$. If $L \leq 0$, then $|f(x)-L| \geq \epsilon$ for all x in $(0,1)$, so L cannot be a limit of f at 0 (or anywhere else). Suppose then that $L > 0$. Let $\delta > 0$, and let x be any positive real number less than δ and less than $\dfrac{1}{1+L}$. Then $0 < |x-0| < \delta$, but since $x < \dfrac{1}{1+L}$, then $f(x) = 1/x > 1+L$ and

$$f(x) - L > 1 + L - L = 1 = \epsilon.$$

In this case, $f(x)-L$ is positive, so we have $|f(x)-L| > \epsilon$ even though $|x-0| < \delta$. Thus, L cannot be a limit of f at 0.

The terminology, "L is a limit of f at z" is a bit cumbersome. We would like to be able to refer to, "the limit" and to have notation for the limit of f at z. To do so, we need this result.

Theorem 3.1.4. *Limits of functions are unique.*

Proof. Suppose that z is an accumulation point of $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is any function. Suppose further that a and b are both limits of f at z. We will prove that $a = b$ by proving that $|a - b| < \epsilon$ for every $\epsilon > 0$. Suppose that $\epsilon > 0$. There is a δ_a so that if $x \in D$ and $0 < |x - z| < \delta_a$ then $|f(x) - a| < \epsilon/2$. Similarly, there is a δ_b so that if $x \in D$ and $0 < |x - z| < \delta_b$ then $|f(x) - b| < \epsilon/2$. Let $\delta = \min(\delta_a, \delta_b)$. Suppose that $x \in D$ and $0 < |x - z| < \delta$. (We know such an x exists because z is an accumulation point of D.) Then

$$\begin{aligned} |a - b| &= |a - f(x) + f(x) - b| \\ &\leq |a - f(x)| + |f(x) - b| \\ &= |f(x) - a| + |f(x) - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus $|a - b| < \epsilon$ for all positive ϵ. It follows that $a = b$. $\square$

Since limits of functions are unique, we can now introduce this notation.

Notation 3.1.5. We will use $\lim_{x \to z} f(x)$ to denote the limit of $f(x)$ at z when such a limit exists. If f has a limit at z, then we will say that $\lim_{x \to z} f(x)$ *exists*. Otherwise, we will say that $\lim_{x \to z} f(x)$ *does not exist.*

If we ever say something along the lines of, "Suppose $\lim_{x \to z} f(x) = L$," then implicitly we mean something like, "Suppose that f is a function, that z is an accumulation point of the domain of f, and that f has a limit of L at z."

Example 3.1.6. Suppose $f(x) : [0, \infty) \to \mathbb{R}$ is given by $f(x) = \sqrt{x}$. Then $\lim_{x \to 0} f(x) = 0$. Let $\epsilon > 0$ and let $\delta = \epsilon^2$. Suppose that $x \in [0, \infty)$ and $0 < |x - 0| < \delta$. Since $x \in [0, \infty)$, then $|x - 0| = |x| = x$ and $0 < x < \delta$. Since $0 < x < \delta$, then $0 < \sqrt{x} < \sqrt{\delta} = \epsilon$. Then

$$|f(x) - 0| = \sqrt{x} < \epsilon.$$

Thus $\lim_{x \to 0} f(x) = 0$.

Notice in this example that x must approach 0 along the domain of f. In a first semester calculus course, this might have been called a "limit from the right" at 0. Our notation here along with Theorem 3.2.1 below will relieve us of the necessity of one sided limits. Of course this will lead occasionally to apparent conflicts in terminology. For example, in many first semester calculus classes, it is said that $\lim_{x \to 0} \sqrt{x}$ does not exist.

Example 3.1.7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x$. If $z \in \mathbb{R}$, then $\lim_{x \to z} f(x) = z$. To see this, suppose that $\epsilon > 0$. Let $\delta = \epsilon$. Suppose that $x \in \mathbb{R}$ with $0 < |x - z| < \delta$. Then

$$|f(x) - z| = |x - z| < \delta = \epsilon.$$

Thus $\lim_{x \to z} f(x) = z$.

Example 3.1.8. Suppose that $m, b \in \mathbb{R}$ with $m \neq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = mx + b$. If $z \in \mathbb{R}$, then $\lim_{x \to z} f(x) = mz + b$. To see this, suppose that $\epsilon > 0$. Let $\delta = \epsilon/|m|$. Suppose that $x \in \mathbb{R}$ with $0 < |x - z| < \delta$. Then

$$|f(x) - (mz + b)| = |(mx + b) - (mz + b)| = |m||x - z| < |m|\delta = \epsilon.$$

Thus $\lim_{x \to z} f(x) = mz + b$.

Example 3.1.9. Suppose that $k \in \mathbb{R}$ and that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = k$. If $z \in \mathbb{R}$, then $\lim_{x \to z} f(x) = k$. Let $\epsilon > 0$. Let δ be any positive real number. Suppose that $x \in \mathbb{R}$ with $0 < |x - z| < \delta$. Then $|f(x) - k| = 0 < \epsilon$. Thus $\lim_{x \to z} f(x) = k$.

Example 3.1.10. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^3$. Then $\lim_{x \to 2} f(x) = 8$. Let $\epsilon > 0$. Let $\delta = \min(1, \epsilon/19)$. Suppose that $x \in \mathbb{R}$ with $0 < |x - 2| < \delta$. Note that since $|x - 2| < 1$, then $x < 3$. Consider

$$\begin{aligned}
|f(x) - 8| &= |x^3 - 8| \\
&= |x - 2| \cdot |x^2 + 2x + 4| \\
&\leq |x - 2|(x^2 + |2x| + 4) \\
&< |x - 2|(9 + 6 + 4) \\
&= |x - 2| \cdot 19 \\
&< \delta \cdot 19 \\
&< \epsilon.
\end{aligned}$$

Thus $\lim_{x\to 2} f(x) = 8$.

Example 3.1.11. Suppose $f : (0,1) \to \mathbb{R}$ is given by $f(x) = 1/x$. If $z \in (0,1)$, then $\lim_{x\to z} f(x) = 1/z$. Suppose that $\epsilon > 0$. Let $\delta = \min\left(\frac{z}{2}, \frac{\epsilon z^2}{2}\right)$. Suppose that $x \in (0,1)$ with $0 < |x - z| < \delta$. Since $|x - z| < z/2$, then $x > z/2$, so $0 < 1/x < 2/z$. Now consider

$$\begin{aligned}
\left|f(x) - \frac{1}{z}\right| &= \left|\frac{1}{x} - \frac{1}{z}\right| \\
&= \left|\frac{z - x}{xz}\right| \\
&< \frac{2|x - z|}{z^2} \\
&< \frac{2\delta}{z^2} \\
&\leq \epsilon.
\end{aligned}$$

Thus $\lim_{x\to z} f(x) = \frac{1}{z}$.

Example 3.1.12. Suppose $f : (1,\infty) \to \mathbb{R}$ is given by $f(x) = 1/x$. If $z \in (1,\infty)$, then $\lim_{x\to z} f(x) = 1/z$. Suppose that $\epsilon > 0$. Let $\delta = \epsilon$. Suppose that $x \in (1,\infty)$ with $0 < |x-z| < \delta$. Note that since $x, z \geq 1$, then $0 < \frac{1}{xz} \leq 1$. Now consider

$$\begin{aligned}
\left|f(x) - \frac{1}{z}\right| &= \left|\frac{1}{x} - \frac{1}{z}\right| \\
&= \left|\frac{z - x}{xz}\right| \\
&= \frac{1}{xz}|x - z| \\
&\leq |x - z| \\
&< \delta \\
&= \epsilon.
\end{aligned}$$

Thus $\lim_{x\to z} f(x) = \frac{1}{z}$.

There is a fundamental difference between Example 3.1.12 and Examples 3.1.10 and 3.1.11. Notice that in Examples 3.1.10 and 3.1.11

our δ depends on the value of z. In Example 3.1.12, the δ is independent of z. This will be an important difference later on.

One of the first tricks used in evaluating limits in a first semester calculus course is to algebraically manipulate a function to find a new function which agrees with the original almost everywhere. Usually this involves factoring and canceling. This method is made possible by the following theorem.

Theorem 3.1.13. *Suppose that z is an accumulation point of $D \subseteq \mathbb{R}$ and that $f, g : D \to \mathbb{R}$ are functions so that $f(x) = g(x)$ for all $x \in D - \{z\}$. If f has a limit at z then g has a limit at z and*

$$\lim_{x \to z} f(x) = \lim_{x \to z} g(x).$$

Proof. Let $L = \lim_{x \to z} f(x)$. We will prove that g has a limit at z and $\lim_{x \to z} g(x) = L$. Let $\epsilon > 0$. There is a $\delta > 0$ so that if $x \in D$ and $0 < |x - z| < \delta$ then $|f(x) - L| < \epsilon$. Suppose that $x \in D$ and $0 < |x - z| < \delta$. Then

$$|g(x) - L| = |f(x) - L| < \epsilon.$$

Thus the limit of g at z is L. $\square$

Example 3.1.14. Let $D = \mathbb{R} - \{0\}$. Suppose that $f, g : D \to \mathbb{R}$ are given by $f(x) = 2x + 1$ and $g(x) = \dfrac{2x^2 + x}{x}$. Then, for all $x \in D$, $f(x) = g(x)$. By Example 3.1.8 we know that $\lim_{x \to 0} f(x) = 1$. Therefore, by Theorem 3.1.13, we know that $\lim_{x \to 0} g(x)$ also is 1.

Example 3.1.15. In this example, we consider a limit of a function whose domain does not contain an interval. Let

$$D = \{1/n : n \in \mathbb{N}\} \cup \{0\}.$$

Define $f : D \to \mathbb{R}$ by $f(x) = x^2$. Note that 0 is the only accumulation point of D. We prove that $\lim_{x \to 0} f(x) = 0$. Let $\epsilon > 0$. Let $\delta = \sqrt{\epsilon}$. Suppose that $x \in D$ with $|x - 0| < \delta$. Then $|x| < \delta$, so $x^2 < \delta^2 = \epsilon$. It follows that

$$|f(x) - 0| = x^2 < \epsilon.$$

Thus, $\lim_{x \to z} f(x) = 0$.

Exercises 3.1

3.1.1 Define $f : (-2, 0) \to \mathbb{R}$ by $f(x) = \dfrac{x^2 - 4}{x + 2}$. Use the definition to prove that f has a limit at -2 and find it.

3.1.2 Define $f : (-2, 0) \to \mathbb{R}$ by $f(x) = \dfrac{2x^2 + 3x - 2}{x + 2}$. Use the definition to prove that f has a limit at -2 and find it.

3.1.3 Give an example of a function $f : (0, 1) \to \mathbb{R}$ that has a limit at every point of $(0, 1)$ except at $1/2$. Use the definition to prove your answer is correct.

3.1.4 Define $f : (0, 1) \to \mathbb{R}$ by $f(x) = \dfrac{x^3 - x^2 + x - 1}{x - 1}$. Prove that f has a limit at 1.

3.1.5 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}.$$

Use the definition to prove that f has no limit at 0.

3.1.6 Define $f : (-1, 1) \to \mathbb{R}$ by $f(x) = \dfrac{x + 1}{x^2 - 1}$. Either prove that f has a limit at 1 or prove that it does not.

3.2 Limits of Sequences and Functions

We would now like to prove several results for limits of functions similar to those for limits of sequences proven in Section 2.2. This next theorem is fundamental in simplifying those proofs (and many proofs later). In a first semester calculus class, when considering $\lim\limits_{x \to z} f(x)$, you may have considered a limit from the left and a limit from the right at z. The next theorem allows us to consider limits as we approach z along any sequence.

Theorem 3.2.1. *Suppose that z is an accumulation point of $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is any function. The number L is the limit of f at z if and only if for every sequence $\langle x_n \rangle$ in $D - \{z\}$ converging to z the sequence $\langle f(x_n) \rangle$ converges to L.*

Proof. The first half of this proof is depicted in Figure 3.2. Suppose first that L is a limit of f at z. Let $\langle x_n \rangle$ be any sequence in $D - \{z\}$ converging to z. We will use the definition of convergence of a sequence

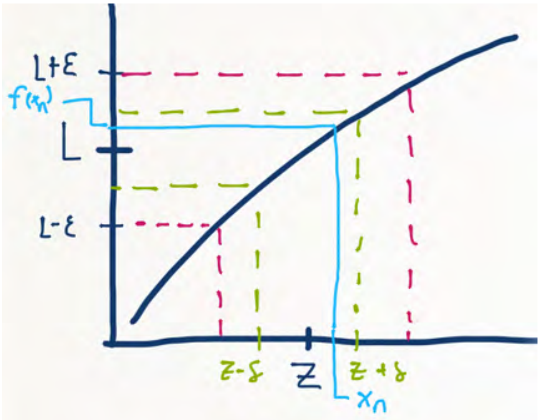

Figure 3.2: Since $\lim\limits_{x \to z} f(x) = L$, there is a δ so that if $x \neq z$ is within δ of z, then $f(x)$ is within ϵ of L. Since $\lim x_n = z$, $\langle x_n \rangle$ is eventually within δ of z. But then $\langle f(x_n) \rangle$ is eventually within ϵ of L.

to prove that $\langle f(x_n) \rangle$ converges to L. Let $\epsilon > 0$. Since $\lim\limits_{x \to z} f(x) = L$, there is a $\delta > 0$ so that if $x \in D$ and if $0 < |x - z| < \delta$ then $|f(x) - L| < \epsilon$. Since $\lim x_n = z$, there is an $N \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N$ then $|x_n - z| < \delta$. Suppose now that $n \in \mathbb{N}$ and $n > N$. Then $|x_n - z| < \delta$. Since $\langle x_n \rangle$ is a sequence in $D - \{z\}$ $x_n \neq z$ and $0 < |x_n - z| < \delta$. By our choice of δ, this implies that $|f(x_n) - L| < \epsilon$. Thus there is an $N \in \mathbb{R}$ so that if $n \in \mathbb{N}$ and $n > N$ then $|f(x_n) - L| < \epsilon$. Hence $\langle f(x_n) \rangle$ converges to L.

For the converse, we use the contrapositive. Suppose that the number L is not a limit of f at z. This means that there is an $\epsilon > 0$ so that for all $\delta > 0$ there is an $x \in D$ so that $0 < |x - z| < \delta$ but $|f(x) - L| \geq \epsilon$. For each $n \in \mathbb{N}$, we apply this fact with $\delta = 1/n$. Then for each $n \in \mathbb{N}$, there is an $x_n \in D$ with $0 < |x_n - z| < 1/n$ but $|f(x_n) - L| \geq \epsilon$. Since $0 < |x_n - z|$, then $x_n \in D - \{z\}$ for each n. We now have a sequence $\langle x_n \rangle$ in $D - \{z\}$. Since $|x_n - z| < 1/n$ for all n, we know that $\lim x_n = z$. However, since $|f(x_n) - L| \geq \epsilon$ for all n, the sequence $\langle f(x_n) \rangle$ cannot converge to L. Hence, we have proven that if L is not the limit of f at z then there is a sequence $\langle x_n \rangle$ in $D - \{z\}$ converging to z such that $\langle f(x_n) \rangle$ does not converge to L. This is the contrapositive of what we are trying to prove. $\square$

This theorem can be useful for showing that limits do not exist.

Example 3.2.2. Let $D = \mathbb{R} - \{0\}$ and let $f : D \to \mathbb{R}$ be given by $f(x) = \dfrac{|x|}{x}$. We show that $\lim\limits_{x\to 0} f(x)$ does not exist. Let $\langle l_n \rangle$ be any sequence of negative numbers approaching 0, and let $\langle r_n \rangle$ be any sequence of positive numbers approaching 0. (If you want actual sequences, you can use $l_n = -1/n$ and $r_n = 1/n$, but part of the point is that the actual sequences do not matter.) For any n, since $l_n < 0$, then $f(l_n) = \dfrac{|l_n|}{l_n} = -1$. For any n, since $r_n > 0$, then $f(r_n) = \dfrac{|r_n|}{r_n} = 1$. Then $-1 = \lim(-1) = \lim f(l_n)$ but $1 = \lim(1) = \lim f(r_n)$. Since $\lim l_n = \lim r_n = 0$ but $\lim f(l_n) \neq \lim f(r_n)$, then $\lim\limits_{x\to 0} f(x)$ cannot exist.

We can use Theorem 3.2.1 to prove a theorem for limits of functions parallel to Theorem 2.2.16 for sequences. Each part of the next theorem can be proven by invoking Theorems 3.2.1 and 2.2.16 on a sequence approaching z. Note that parts (1) and (2) also follow directly from Examples 3.1.8 and 3.1.9.

Theorem 3.2.3. (Algebraic Properties of Limits of Functions) *Suppose that z is an accumulation point of $D \subseteq \mathbb{R}$ and that $f, g : D \to \mathbb{R}$ are functions with $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$. Suppose also that $k \in \mathbb{R}$ and that p is a polynomial.*

1. $\lim\limits_{x\to z} k = k$
2. $\lim\limits_{x\to z} (kf(x)) = kL$.
3. $\lim\limits_{x\to z} (f+g)(x) = L + M$.
4. $\lim\limits_{x\to z} (f-g)(x) = L - M$.
5. $\lim\limits_{x\to z} p(x) = p(z)$.
6. $\lim\limits_{x\to z} (fg)(x) = LM$.
7. $\lim\limits_{x\to a} |f|(x) = |L|$.
8. *If* $f(x) \geq 0$ *for* $x \in D$, *then* $\lim\limits_{x\to a} \sqrt{f(x)} = \sqrt{L}$.
9. *If* $g(x) \neq 0$ *for* $x \in D$, *and if* $M \neq 0$, $\lim\limits_{x\to z} (f(x)/g(x)) = L/M$.

Proof. We prove part (3) and leave the other parts as exercises. Suppose that $\langle x_n \rangle$ is a sequence in $D - \{z\}$ converging to z. To prove (3), we need only prove that $\langle (f+g)(x_n) \rangle$ converges to $L + M$. Since $\lim\limits_{x\to z} f(x) = L$ and $\lim x_n = z$, by Theorem 3.2.1 $\lim f(x_n) = L$. Also, since $\lim\limits_{x\to z} g(x) = M$ and $\lim x_n = z$, we know by Theorem 3.2.1 that

$\lim g(x_n) = M$. Since $\lim f(x_n) = L$ and $\lim g(x_n) = M$, we know by Theorem 2.2.16 that $\lim(f(x_n) + g(x_n)) = L + M$. Hence, for any sequence $\langle x_n \rangle$ in $D - \{z\}$ converging to z, we have $\lim(f+g)(x_n) = L+M$. By Theorem 3.2.1, $\lim_{x \to z}(f + g)(x) = L + M$. □

We can also use Theorem 3.2.1 To extend the order theorems for limits (Theorems 2.2.17 and 2.2.18) to limits of functions.

Theorem 3.2.4. *Suppose z is an accumulation point of $D \subseteq \mathbb{R}$ and that $f, g : D \to \mathbb{R}$ are functions with $\lim_{x \to z} f(x) = L$ and $\lim_{x \to z} g(x) = M$. If $f(x) \leq g(x)$ for all $x \in D$, then $L \leq M$.*

Proof. Let $\langle x_n \rangle$ be any sequence in $D - \{z\}$ converging to z. By Theorem 3.2.1 we know that $\lim f(x_n) = L$ and $\lim g(x_n) = M$. Since $f(x_n) \leq g(x_n)$ for all n, Theorem 2.2.17 tells us that

$$L = \lim f(x_n) \leq \lim g(x_n) = M.$$

□

Theorem 3.2.5. (Squeeze Theorem) *Suppose that z is an accumulation point of $D \subseteq \mathbb{R}$ and that $f, g, h : D \to \mathbb{R}$ are functions with $f(x) \leq g(x) \leq h(x)$ for all $x \in D$. If $\lim_{x \to z} f(x) = \lim_{x \to z} h(x) = L$, then $\lim_{x \to z} g(x) = L$.*

Proof. Suppose that $\langle x_n \rangle$ is any sequence in $D - \{z\}$ converging to z. We need only prove that $\langle g(x_n) \rangle$ converges to L. Since $\lim_{x \to z} f(x) = L$ and $\lim_{x \to z} h(x) = L$, then $\lim f(x_n) = \lim g(x_n) = L$. Since

$$f(x_n) \leq g(x_n) \leq h(x_n)$$

for all n, by the Squeeze Theorem for sequences (Theorem 2.2.18) we have $\lim g(x_n) = L$. Thus $\lim g(x_n) = L$ for all sequences $\langle x_n \rangle$ converging to z in $D - \{z\}$. By Theorem 3.2.1, $\lim_{x \to z} g(x) = L$. □

Exercises 3.2

3.2.1 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}.$$

Use Theorem 3.2.1 to prove that f has no limit at 0.

3.2.2 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \notin \mathbb{Q} \end{cases}.$$

Find where f has a limit and where it does not. Support your answer with proofs. You may want to use Theorem 3.2.1

3.2.3 Consider the function $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = x^x$. Assume that f has a limit at 0 and find it.

3.2.4 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 8x & x \in \mathbb{Q} \\ 2x^2 + 8 & x \notin \mathbb{Q} \end{cases}.$$

Determine where f has a limit and where it does not. Support your answer with proofs.

3.2.5 Suppose that z is an accumulation point of $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is a function so that for all $\epsilon > 0$ there is a $\delta > 0$ so that for all $x, y \in D - \{z\}$, if $x, y \in (z - \delta, z + \delta)$ then $|f(x) - f(y)| < \epsilon$. Prove that f has a limit at z. Hints: Suppose that $\langle x_n \rangle$ is any sequence in $D - \{z\}$ converging to z. Prove that $\langle f(x_n) \rangle$ is Cauchy so that $\langle f(x_n) \rangle$ converges. Suppose now that $\langle y_n \rangle$ is any other such sequence. Then $\langle f(y_n) \rangle$ will also converge. Prove that $\lim f(x_n) = \lim f(y_n)$.

3.2.6 Define $f : (0, 1) \to \mathbb{R}$ by $f(x) = \dfrac{\sqrt{x+1} - 1}{x}$. Prove that f has a limit at 0 and find it.

3.2.7 Give examples of functions $f, g : \mathbb{R} \to \mathbb{R}$ which do not have limits at 0 so that $f + g$ does have a limit at 0.

3.2.8 Give examples of functions $f, g : \mathbb{R} \to \mathbb{R}$ which do not have limits at 0 so that fg does have a limit at 0.

3.2.9 Give examples of functions $f, g : \mathbb{R} \to \mathbb{R}$ which do not have limits at 0 so that f/g does have a limit at 0.

3.2.10 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function so that

$$f(x + y) = f(x)f(y)$$

for all $x, y \in \mathbb{R}$ and so that $\lim_{x \to 0} f(x)$ exists. Prove that either $f(x) = 0$ for all x or that $\lim_{x \to 0} f(x) = 1$.

3.3 The Definition of Continuity

Definition 3.3.1. Suppose that $a \in D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is *continuous* at a if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ so that for all $x \in D$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$. If f is continuous at all $x \in D$, then f is *continuous on* D.

At this point the reader should compare this definition with the definition of the limit 3.1.1.

Example 3.3.2. We use the definition to prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous at 3. Let $\epsilon > 0$. Let $\delta = \min(1, \epsilon/7)$. Suppose that $|x - 3| < \delta$. Since $|x - 3| < 1$ then $2 < x < 4$ so $|x + 3| < 7$. Now consider

$$\begin{aligned} |f(x) - f(3)| &= |x^2 - 3^2| \\ &= |x + 3| \cdot |x - 3| \\ &< 7\delta \\ &\leq \epsilon. \end{aligned}$$

Thus for every $\epsilon > 0$ there is a $\delta > 0$ so that if $|x - 3| < \delta$, then $|f(x) - f(3)| < \epsilon$. Therefore, f is continuous at 3.

Example 3.3.3. The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

is not continuous at 0. To prove this, we must look at the negation of the definition of continuity. For f to be not continuous at 0 there must exist $\epsilon > 0$ so that for all $\delta > 0$ there is an $x \in \mathbb{R}$ with $|x - 0| < \delta$ but $|f(x) - f(0)| \geq \epsilon$. Since $f(0) = 1$ but $f(x) = -1$ if x is negative (no matter how close x is to 0), it seems as if any $\epsilon \leq 2$ will work. Let $\epsilon = 2$. Suppose $\delta > 0$. Let $x = -\delta/2$. Then $|x - 0| < \delta$, but

$$|f(x) - f(0)| = |-1 - 1| = 2 \geq \epsilon.$$

Thus f is not continuous at 0.

Example 3.3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \notin \mathbb{Q} \end{cases}.$$

We prove that f is continuous at 0. Let $\epsilon > 0$. Since

$$|f(x) - f(0)| = |x| = |x - 0|$$

to make $|f(x) - f(0)|$ less than ϵ, it is adequate to make $|x - 0| < \epsilon$. Let $\delta = \epsilon$. Suppose that $|x - 0| < \delta$. Then

$$|f(x) - f(0)| = |x| = |x - 0| < \delta = \epsilon.$$

Thus f is continuous at 0. It just so happens that 0 is the only place where f is continuous. This will be easier to prove in the next section.

Exercises 3.3

3.3.1 Use the definition to prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is continuous at 2.
3.3.2 Use the definition to prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not continuous at 0.
3.3.3 Suppose that $z \in E \subseteq \mathbb{R}$ is not an accumulation point of E and that $f : E \to \mathbb{R}$. Use the definition to prove that f is continuous at z.
3.3.4 Suppose that $z \in E \subseteq \mathbb{R}$ is not an accumulation point of E and that $f : E \to \mathbb{R}$. Suppose also that $\langle x_n \rangle$ is a sequence in E converging to z. Prove that $\langle f(x_n) \rangle$ converges to $f(z)$.
3.3.5 A function $f : D \to \mathbb{R}$ satisfies the Lipschitz condition on D if there is an $M \in \mathbb{R}$ so that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in D$. Prove that if f satisfies the Lipschitz condition on D and if $z \in D$ then f is continuous at z.

3.4 Limits and Continuity

If you compare the definition of the limit of a function (Definition 3.1.1) with the definition of continuity (Definition 3.3.1) you should see that the two are quite similar. The differences between the definition of the limit of f at z being L and the definition of f being continuous at z are

- To discuss the limit of f at z, the point z must be an accumulation point of the domain of f. For continuity at z, the point z need not be an accumulation point.

- In the definition of the limit, the function value $f(z)$ is irrelevant because of the inequality "$0 < |x - z|$." In fact, z might not even be in the domain of f. In the definition of the continuity of f at z, the point z must be in the domain of f, and the value $f(z)$ is essential to the definition.

- The L in the definition of the limit at z is replaced by $f(z)$ in the definition of continuity at z.

With these differences in mind, the definitions should make this theorem obvious:

Theorem 3.4.1. *Suppose that $z \in D \subseteq \mathbb{R}$ is an accumulation point of D. A function $f : D \to \mathbb{R}$ is continuous at z if and only if*

$$\lim_{x \to z} f(x) = f(z).$$

$\square$

Example 3.4.2. Theorem 3.4.5 along with Example 3.1.8 reveal the unsurprising fact that linear functions are continuous.

So what happens at non-accumulation points of the domain? A function is continuous at all non-accumulation points in its domain.

Theorem 3.4.3. *If $z \in D \subseteq \mathbb{R}$ is not an accumulation point of D, then any function $f : D \to \mathbb{R}$ is continuous at z.*

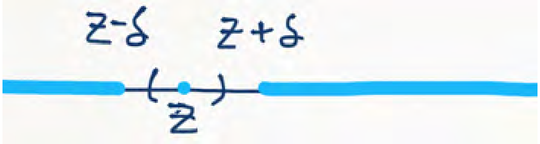

Figure 3.3: Since z is not an accumulation point of D, there is an interval $(z - \delta, z + \delta)$ which intersects D only at z. If $|x - z| < \delta$, then $x = z$ and $|f(x) - f(z)| = 0 < \epsilon$.

Proof. Suppose that $f : D \to \mathbb{R}$ is any function and that $z \in D$ is not an accumulation point of D. Since z is not an accumulation point of D, there is an interval (a, b) containing z which contains no points of D other than z. There is a $\delta > 0$ so that $(z - \delta, z + \delta) \subseteq (a, b)$. Note that if $x \in D$ and if $|x - z| < \delta$, then $x = z$. Now let $\epsilon > 0$. Suppose that $x \in D$ and that $|x - z| < \delta$. Then $x = z$ so

$$|f(x) - f(z)| = |f(z) - f(z)| = 0 < \epsilon.$$

Thus f is continuous at z. □

Example 3.4.4. Recall that in Example 3.1.15 we considered the function $f(x) = x^2$ on the domain $D = \{1/n : n \in \mathbb{N}\} \cup \{0\}$. We saw in that example that $\lim_{x \to z} f(x) = 0 = f(0)$. This implies by Theorem 3.4.1 that f is continuous at 0. Since D has no other accumulation points, f is continuous at every other point in its domain also.

Combining the previous two theorems gives:

Theorem 3.4.5. *Suppose that $z \in D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is continuous at z if and only if for every sequence $\langle s_n \rangle$ in D converging to z, $\lim f(s_n) = f(z)$.*

Proof. If z is an accumulation point of D, then this follows directly from Theorem 3.4.1. If z is not an accumulation point of D, then the only way a sequence $\langle s_n \rangle$ in D can converge to z is if that sequence is eventually equal to z. Then the sequence $\langle f(s_n) \rangle$ is eventually equal to $f(z)$, so $\lim f(s_n) = f(z)$. Thus the statements, "For every sequence $\langle s_n \rangle$ in D converging to z, $\lim f(s_n) = f(z)$" and, "f is continuous at z" are both true and the theorem holds when z is not an accumulation point of D. □

Example 3.4.6. Theorem 3.4.5 is useful in proving that certain functions are not continuous at given points. Recall that in Example 3.3.4 we considered the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \notin \mathbb{Q} \end{cases}.$$

We proved there that f is continuous at 0. We can now easily prove that f is not continuous at any other point in $\mathbb{R}$. Let $z \neq 0$ be a rational number. Let $\langle x_n \rangle$ be a sequence of irrational numbers which converges to z. Then $f(x_n) = -x_n$ for all n, so $\lim f(x_n) = -z \neq z = f(z)$. Since $\lim x_n = z$ but $\lim f(x_n) \neq f(z)$, f is not continuous at z. If $z \neq \mathbb{Q}$, we can argue similarly using a sequence of rational numbers converging to z.

We employ Theorem 3.4.5 in the proof of this next theorem that contains a fundamental property of continuous functions. If $f(z)$ is not 0 and if f is continuous at z, then f must be nonzero near z. An alternative approach to the proof of this theorem is pictured in Figure 3.4.

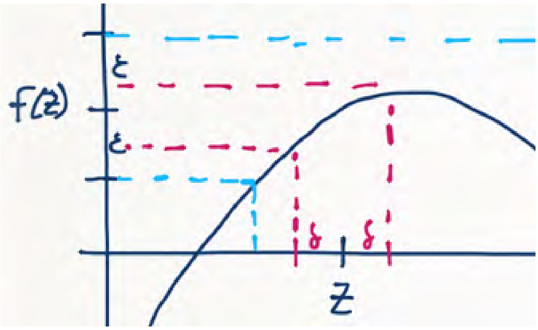

Figure 3.4: Suppose that $f(z) > 0$. Let $\epsilon = f(z)/2 > 0$. There is a $\delta > 0$ so that if x is within δ of z then $f(x)$ is within ϵ of $f(z)$. But this means that $f(x) > f(z) - \epsilon = f(z)/2 > 0$.

Theorem 3.4.7. *Suppose that $f : D \to \mathbb{R}$ is continuous at $z \in D$. If $f(z) > 0$, then there is an open interval (a, b) containing z so that $f(x) > 0$ for all $x \in (a, b) \cap D$.*

Proof. We prove the theorem in using the contrapositive along with Theorems 2.2.17 and 3.4.5. Suppose that for every open interval (a, b) containing z there is an $x \in (a, b) \cap D$ with $f(x) \leq 0$. We will prove that $f(z) \leq 0$. For each $n \in \mathbb{N}$, the interval $(z - 1/n, z + 1/n)$ contains z, so there is some $x_n \in (z - 1/n, z + 1/n) \cap D$ with $f(x_n) \leq 0$. The sequence $\langle x_n \rangle$ must converge to $f(z)$. By Theorem 3.4.5, the sequence $\langle f(x_n) \rangle$ converges to z. Since $f(x_n) \leq 0$ for all n, then by Theorem 2.2.17 $f(z) = \lim f(x_n) \leq 0$. We have established the contrapositive of the theorem. $\square$

Theorem 3.4.5 combined with Theorem 2.2.16 immediately give us these algebraic properties of continuity. We note that Theorem 3.4.7

is essential for part (9) as it implies that if $\langle x_n \rangle$ is any sequence that converges z in D then $\langle g(x_n) \rangle$ is eventually non-zero.

Theorem 3.4.8. (Algebraic Properties of Continuity) *Suppose that $z \in D \subseteq \mathbb{R}$ and that $f, g : D \to \mathbb{R}$ are functions which are continuous at z. Suppose also that $k \in \mathbb{R}$ and that p is a polynomial. These functions are continuous at z:*

1. *The constant function k*
2. *kf.*
3. *$f + g$.*
4. *$f - g$.*
5. *p.*
6. *fg.*
7. *$|f|$.*
8. *$\sqrt{f}$ (if $f(x) \geq 0$ for $x \in D$).*
9. *f/g (if $g(z) \neq 0$).* □

We close this section by addressing continuity and compositions.

Theorem 3.4.9. *Suppose that $z \in D \subseteq \mathbb{R}$, that $f : D \to \mathbb{R}$ is continuous at z, that $f(D) \subseteq E \subseteq \mathbb{R}$ and that $g : E \to \mathbb{R}$ is continuous at $f(z)$. Then $g \circ f$ is continuous at z.*

Proof. Suppose that $\langle x_n \rangle$ is any sequence in D converging to z. Since f is continuous at z, Theorem 3.4.5 implies $\langle f(x_n) \rangle$ is a sequence in E converging to $f(z)$. Since g is continuous at $f(z)$, the sequence $\langle g(f(x_n)) \rangle = \langle g \circ f(x_n) \rangle$ converges to $g \circ f(z)$ by Theorem 3.4.5. Again by Theorem 3.4.5, this implies that $g \circ f$ is continuous at z. □

Exercises 3.4

3.4.1 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere and that $f(r) = r^2$ for every rational number r. Find $f(\sqrt{2})$. Support your answer with a proof. Hint: There is a sequence $\langle r_m \rangle$ of rational numbers approaching $\sqrt{2}$.

3.4.2 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere and that $f(r) = 0$ for all rational numbers r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$. Hint: If x is irrational, then there is a sequence of rational numbers approaching x.

3.4.3 Suppose that $f : D \to \mathbb{R}$ is continuous at $z \in D$. Prove that there is an open interval I containing z so that f is bounded on $I \cap D$. That is, prove that there is a number M and an open interval I containing z so that if $x \in D \cap I$, then $|f(x)| < M$. Hint: There is a δ so that if x is within δ of z then $f(x)$ is within 1 of $f(z)$.

3.4.4 Suppose that $f, g : D \to \mathbb{R}$ are continuous. Prove that $\max(f, g)$ is continuous on D. Hint: Use Exercise 1.6.6.

3.4.5 Let $g(x) = x^2$ and $f(x) = \begin{cases} 4 & x \geq 0 \\ 0 & x < 0 \end{cases}$. Which of these functions are continuous at 0? f, g, $f + g$, fg, $f \circ g$, $g \circ f$?

3.4.6 Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Prove that $f = g$. Hint: If x is irrational, then there is a sequence of rational numbers approaching x.

3.4.7 Suppose that $S \subseteq \mathbb{R}$ has an accumulation point z which is not in S. Prove that there is a function $f : S \to \mathbb{R}$ which is continuous but unbounded. Hint: Consider as an example the function $1/x$.

3.5 Uniform Continuity

To say that a function $f : D \to \mathbb{R}$ is *continuous* at $x_0 \in D$ is essentially to say that if x is close to x_0, then $f(x)$ is close to $f(x_0)$. This is a local property that applies around the point x_0. It will be useful for us to have a more global idea of continuity that declares whenever x and y are close to each other in D, then $f(x)$ and $f(y)$ are close to each other. This is a property that functions such as $f(x) = 1/x$ fail. If x and y are both close to 0, then $1/x$ and $1/y$ might be any distance apart. This global notion of continuity is call *uniform continuity.*

Definition 3.5.1. Suppose that $E \subseteq D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is *uniformly continuous* on E if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ so that for all $x, y \in E$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Example 3.5.2. The function $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$. To demonstrate that f is not uniformly continuous, let us first call attention to the negation of the definition of uniform continuity. A function $f : D \to \mathbb{R}$ is not uniformly continuous on E if there is a real number $\epsilon > 0$ so that for all real numbers $\delta > 0$ there are $x, y \in E$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$. We will argue that $\epsilon = 1$ satisfies this, so let $\epsilon = 1$ and $\delta > 0$. We must find $x, y \in (0, 1)$ with $|x - y| < \delta$

but $|f(x)-f(y)| \geq \epsilon$. Let $x \in (0,1)$ with $x < \delta$, and let $y = x/2$. Then

$$|x-y| = x-y = x/2 < \delta/2 < \delta.$$

But

$$|f(x)-f(y)| = f(y)-f(x) = \frac{2}{x} - \frac{1}{x} = \frac{1}{x} > 1 = \epsilon.$$

Thus, no matter what δ may be given, we can always find x and y so that $|x-y| < \delta$ but $|f(x)-f(y)| \geq \epsilon$.

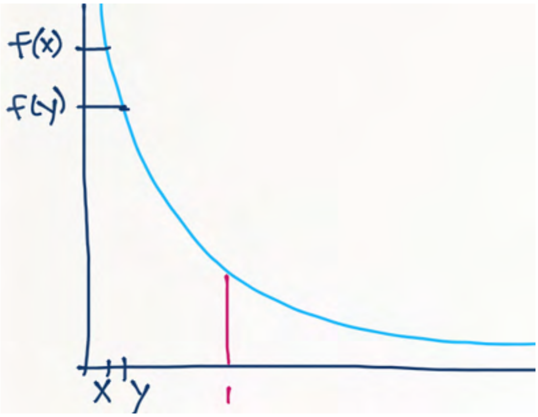

Figure 3.5: If x and y are near 0, then $1/x$ and $1/y$ might be some distance apart since $1/x$ is so steep near 0. On the other hand, if $x, y > 1$ then $1/x$ and $1/y$ should be somewhat close since $1/x$ is not steep here.

Example 3.5.3. The function $f(x) = 1/x$ is uniformly continuous on $(1,\infty)$. Let $\epsilon > 0$, and let $\delta = \epsilon$. Suppose that $x, y \in (1,\infty)$ with $|x-y| < \delta$. Then

$$|f(x)-f(y)| = |\frac{1}{x} - \frac{1}{y}| = |\frac{y-x}{xy}|.$$

Since $x, y \in (1,\infty)$, then $xy > 1$. This implies that

$$|\frac{y-x}{xy}| < |x-y| < \delta = \epsilon.$$

Thus f is uniformly continuous on $(1,\infty)$.

Example 3.5.4. The function $f(x) = x^2$ is not uniformly continuous on $[1, \infty)$. Let $\epsilon = 1$ (any positive number will work). Suppose that $\delta > 0$. Let x be any number in $[1, \infty)$ which is greater than $1/\delta$, and let $y = x + \delta/2$. Then

$$\begin{aligned}
|f(x) - f(y)| &= |x^2 - y^2| \\
&= |x + y||x - y| \\
&= (2x + \delta/2)\delta/2 \\
&= x\delta + \delta^2/4 \\
&\geq x\delta \\
&> (1/\delta)\delta \\
&= 1.
\end{aligned}$$

Example 3.5.5. The function $f(x) = x^2$ is uniformly continuous on $[1, 2]$. Let $\epsilon > 0$. Let $\delta = \epsilon/4$. Suppose that $x, y \in [1, 2]$ and that $|x - y| < \delta$. Note that if $x, y \in [1, 2]$, then $|x + y| \leq 4$.

$$\begin{aligned}
|f(x) - f(y)| &= |x^2 - y^2| \\
&= |x + y||x - y| \\
&< 4\delta \\
&= 4\epsilon/4 \\
&= \epsilon.
\end{aligned}$$

Theorem 3.5.6. *If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.*

Proof. Suppose by way of contradiction that f is continuous but not uniformly continuous on $[a, b]$. Since f is not uniformly continuous, then:

> There exists $\epsilon > 0$ so that for all $\delta > 0$ there are $x, y \in [a, b]$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$.

For each $n \in \mathbb{N}$, we apply this with $\delta = 1/n$. Then, for each $n \in \mathbb{N}$, there are $x_n, y_n \in [a, b]$ with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon$.

The sequence $\langle x_n \rangle$ is bounded and must have a subsequence $\langle x_{n_k} \rangle$ which converges to a number x_0. Note that since $a \leq x_n \leq b$ for all n, then $x_0 \in [a, b]$. The sequence $\langle y_{n_k} \rangle$ is bounded and must have a convergent subsequence $\left\langle y_{n_{k_l}} \right\rangle$. As a subsequence of $\langle x_{n_k} \rangle$, the sequence

$\left\langle x_{n_{k_l}} \right\rangle$ must converge to x_0. Since $\left\langle x_{n_{k_l}} \right\rangle$ and $\left\langle y_{n_{k_l}} \right\rangle$ each converge, so does their difference $\left\langle x_{n_{k_l}} - y_{n_{k_l}} \right\rangle$ Since $|x_{n_{k_l}} - y_{n_{k_l}}| < 1/(n_{k_l})$, this difference converges to 0. It follows that $\left\langle y_{n_{k_l}} \right\rangle$ also converges to x_0.

Since f is continuous at x_0, we know that $\left\langle f(x_{n_{k_l}}) \right\rangle$ and $\left\langle f(y_{n_{k_l}}) \right\rangle$ both converge to $f(x_0)$. This implies that $|f(x_{n_{k_l}}) - f(y_{n_{k_l}})|$ should converge to 0. However, we also know that $|f(x_n) - f(y_n)| \geq \epsilon$ for all n, so $|f(x_{n_{k_l}}) - f(y_{n_{k_l}})|$ cannot converge to 0. Based on this contradiction, we must conclude that the assumption that f is continuous but not uniformly continuous on $[a, b]$ is false. □

Lemma 3.5.7. *If $f : D \to \mathbb{R}$ is uniformly continuous and $\langle x_n \rangle$ is a convergent sequence in D then $\langle f(x_n) \rangle$ is a convergent sequence.*

Proof. We will prove that $\langle f(x_n) \rangle$ is Cauchy. Let $\epsilon > 0$. Since f is uniformly continuous on D, there is a $\delta > 0$ so that if $x, y \in D$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \delta$. Since $\langle x_n \rangle$ is convergent, $\langle x_n \rangle$ is Cauchy. Therefore, there is an N so that if $m, n \geq N$ then $|x_n - x_m| < \delta$. Now suppose that $m, n \geq N$. Then $|x_n - x_m| < \delta$, so $|f(x_n) - f(x_m)| < \epsilon$. This proves that $\langle f(x_n) \rangle$ is Cauchy and, hence, convergent. □

Theorem 3.5.8. *A continuous function $f : (a, b) \to \mathbb{R}$ is uniformly continuous if and only if f has limits at a and at b.*

Proof. Suppose first that f has limits at a and b. Let $L = \lim_{x \to a} f(x)$ and $R = \lim_{x \to b} f(x)$. Define a new function $\hat{f} : [a, b] \to \mathbb{R}$ by

$$\hat{f}(x) = \begin{cases} f(x) & x \in (a, b) \\ L & x = a \\ R & x = b \end{cases}.$$

Then $\hat{f}$ is continuous on $[a, b]$. That $\hat{f}$ is continuous at $x \in (a, b)$ follows because here $\hat{f} = f$, and we assumed that f is continuous on (a, b). At a, $\hat{f}$ is continuous because

$$\lim_{x \to a} \hat{f}(x) = \lim_{x \to a} f(x) = L = \hat{f}(a).$$

Note here that when evaluating $\lim_{x \to a} \hat{f}(x)$, the value at a is irrelavant so we can use f. That $\hat{f}$ is continuous at b is proven similarly. Since

$\hat{f}$ is continuous on $[a,b]$, $\hat{f}$ is uniformly continuous on $[a,b]$. Since $\hat{f}$ is uniformly continuous on $[a,b]$, $\hat{f}$ is uniformly continuous on (a,b). But since $\hat{f} = f$ on (a,b), this implies that f is uniformly continuous on (a,b).

Now assume that f is uniformly continuous on (a,b). We will prove that $\lim\limits_{x\to a} f(x)$ exists. The proof for b is similar. Suppose that $\langle x_n \rangle$ and $\langle y_n \rangle$ are any sequences in (a,b) which converge to a. By Lemma 3.5.7, we know that $\langle f(x_n) \rangle$ a $\langle f(y_n) \rangle$ both converge. We will prove that they converge to the same number L. This will imply that if $\langle z_n \rangle$ is any sequence in (a,b) converging to a, then $\langle f(z_n) \rangle$ converges to L. Therefore, we will know that $\lim\limits_{x\to a} f(x) = L$. Let s be the sequence

$$\langle x_1, y_1, x_2, y_2, x_3, y_3, \ldots \rangle .$$

That is, for each $n \in \mathbb{N}$, $s_{2n} = y_n$ and $s_{2n-1} = x_n$. Since

$$\lim x_n = \lim y_n = a$$

then also $\lim s_n = a$. By Lemma 3.5.7, we know that $\langle f(s_n) \rangle$ converges. Let $L = \lim f(s_n)$. Since $\langle f(x_n) \rangle$ and $\langle f(y_n) \rangle$ are subsequences of the convergent sequence $\langle f(s_n) \rangle$, it follows that $\langle f(x_n) \rangle$ and $\langle f(y_n) \rangle$ both converge to L. The theorem now follows as described above. □

Exercises 3.5

3.5.1 Use the definition to prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is uniformly continuous on $(0,1)$.

3.5.2 Use the definition to prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is not uniformly continuous on $(0,\infty)$.

3.5.3 Define $f : [3.4, 5] \to \mathbb{R}$ by $f(x) = \dfrac{2}{x-3}$. Use the definition to prove that f is uniformly continuous on $[3.4, 5]$.

3.5.4 Define $f : (2,7) \to \mathbb{R}$ by $f(x) = x^3 - x + 1$. Use the definition to prove that f is uniformly continuous on $(2,7)$.

3.5.5 Which of the following functions are uniformly continuous on the given interval? Justify your answer with any theorem you need.

a. $f(x) = \dfrac{x}{x+1}$ on $[2,\pi]$

b. $f(x) = x^4$ on $[0,1]$

c. $f(x) = x^4$ on $(0,1)$

d. $f(x) = x^4$ on $\mathbb{R}$

3.5.6 Prove that if $f : D \to \mathbb{R}$ satisfies the Lipschitz condition on D (see Exercise 3.3.5) then f is uniformly continuous on D.

3.6 The Extreme Value Theorem

One of the common applications of calculus is in optimization – finding the maximum or minimum values of a function. The first step in any such problem is knowing that a solution exists. We prove here that any continuous function on a closed interval achieves both a maximum value and a minimum value. This is the Extreme Value Theorem 3.6.2

Theorem 3.6.1. (Bounded Value Theorem) *If f is continuous on $[a, b]$, then f is bounded on $[a, b]$.*

We offer two proofs of this theorem (because it is so much fun). The first uses uniform continuity. The second simply establishes the contrapositive.

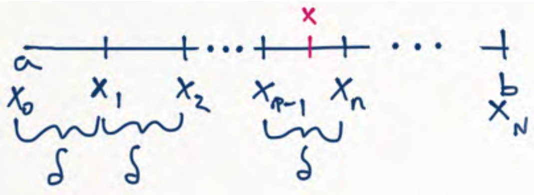

Figure 3.6: The numbers $x_0, x_1, \ldots, x_N$ are less than δ apart. If $x \in [a, b]$, then x is in one of the intervals $[x_{n-1}, x_n]$. This means that x is within δ of x_n, so $f(x)$ is within 1 of $f(x_n)$. Thus every value $f(x)$ is within 1 of one of $f(x_0), f(x_1), \ldots, f(x_N)$. It follows that $f(x)$ is bounded.

Proof. This proof is pictured in Figure 3.6. Since f is continuous on $[a, b]$, by 3.5.6 f is uniformly continuous on $[a, b]$. Select $\epsilon = 1$ in the definition of uniform continuity. There is a $\delta > 0$ so that if x and y are in $[a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < 1$. Select $N \in \mathbb{N}$ so that $(b - a)/N < \delta/2$. For $n = 0, 1, 2, \ldots, N$, let $x_n = a + n(b - a)/N$. Let

$$m = \min(f(x_0) - 1, f(x_2) - 1, \ldots, f(x_N) - 1)$$

and

$$M = \max(f(x_0) + 1, f(x_2) + 1, \ldots, f(x_N) + 1).$$

If $x \in [a, b]$, then there is an n with $|x - x_n| < \delta$. By uniform continuity and our choice of δ, this implies that $|f(x) - f(x_n)| < 1$, so we have

$$m \leq f(x_n) - 1 < f(x) < f(x_n) + 1 \leq M.$$

Thus, the set $\{f(x) : x \in [a, b]\}$ is bounded between m and M. □

And now we address the proof yet again. This time, we use the contrapositive.

Proof. Suppose that f is not bounded on $[a, b]$. Then f is either unbounded above or unbounded below. We will address the case when f is unbounded above. The case when f is unbounded below will work similarly. Suppose that f is unbounded above. For each positive integer n, there is an $x_n \in [a, b]$ so that $f(x_n) > n$. Since the sequence $\langle x_n \rangle$ is bounded within $[a, b]$, this sequence has a convergent subsequence $\langle x_{n_k} \rangle$. Let $z = \lim x_{n_k}$. Note that $z \in [a, b]$. Since $f(x_{n_k}) > n_k$ for all k, $f(x_{n_k})$ is unbounded and cannot converge. In particular, $f(x_{n_k})$ does not converge to $f(\lim x_{n_k})$, so f is not continuous at $z = \lim x_{n_k}$. □

Theorem 3.6.2. (Extreme Value Theorem) *If f is continuous on $[a, b]$, then there are $u, v \in [a, b]$ so that $f(u) \leq f(x) \leq f(v)$ for all $x \in [a, b]$.*

Proof. We will prove the existence of v. Let $S = \{f(x) : x \in [a, b]\}$. By 3.6.2, we know that S is bounded. Let $y_0 = \sup S$. We will prove that there is a $v \in [a, b]$ with $f(v) = y_0$. For each $n \in \mathbb{N}$, $y_0 - 1/n$ is not an upper bound of S, so there is some $y_n \in S$ so that $y_0 - 1/n < y_n \leq y_0$. By choice now, $\langle y_n \rangle$ is a sequence in S converging to y_0.

From the definition of S, we know that for each n there is some $x_n \in [a, b]$ so that $f(x_n) = y_n$. The sequence $\langle x_n \rangle$ is bounded (in $[a, b]$) and must therefore have a subsequence $\langle x_{n_k} \rangle$ which converges to a number v. Since $a \leq x_n \leq b$ for all n, it must be that $v \in [a, b]$, so f is continuous at v. Since f is continuous at v, $\langle f(x_{n_k}) \rangle$ must converge to $f(v)$. However, since $y_{n_k} = f(x_{n_k})$, this means that $\langle y_{n_k} \rangle$ converges to $f(v)$. Now, $\langle y_{n_k} \rangle$ is a subsequence of the sequence $\langle y_n \rangle$, which converges to y_0. Therefore, $\langle y_{n_k} \rangle$ must also converge to y_0. Since limits of sequences are unique, it must be that $f(v) = y_0 = \sup S$. □

3.7 The Intermediate Value Theorem

A grade school description of a continuous function is a function which has no holes, skips, or jumps. We know this to be somewhat naïve at this point. However, the fact that a continuous function has no breaks is an important property which requires proof. The first step is a special case that asserts that a continuous function cannot change signs without passing 0.

Theorem 3.7.1. (Bolzano's Theorem) *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $f(a)$ and $f(b)$ have opposite signs. There is a $z \in (a, b)$ such that $f(z) = 0$.*

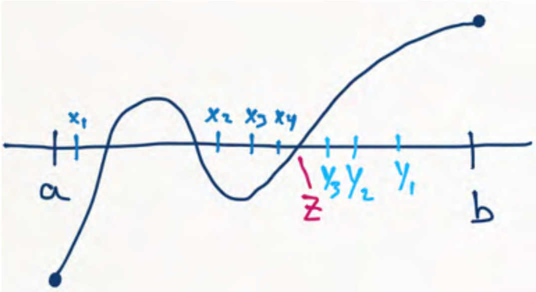

Figure 3.7: The sequence $\langle x_n \rangle$ approaches z from the left. By continuity, the sequence $\langle f(x_n) \rangle$ approaches $f(z)$ along negative values. This forces $f(x) \le 0$. The sequence $\langle y_n \rangle$ approaches z from the right. By continuity, the sequence $\langle f(y_n) \rangle$ approaches $f(z)$ along positive values. This forces $f(x) \ge 0$.

Proof. Suppose that $f(a) < 0 < f(b)$. Let

$$z = \sup\{x \in [a, b] : f(x) < 0\}.$$

(We know this set is not empty because $f(a) < 0$.) There is a sequence $\langle x_n \rangle$ in the set $\{x \in [a, b] : f(x) < 0\}$ which converges to z. Since $a \le x_n \le b$ for all n, it must be that $z \in [a, b]$. Since f is continuous at z, the sequence $\langle f(x_n) \rangle$ converges to $f(z)$. Since $f(x_n) < 0$ for all n, it follows that $f(z) \le 0$.

For each $n \in \mathbb{N}$, let $y_n = \min(z + 1/n, b)$. Then $\langle y_n \rangle$ is a sequence in $[a, b]$ converging to z. Since f is continuous at z, the sequence $\langle f(y_n) \rangle$ converges to $f(z)$. By our choice of z, it must be that $f(y_n) \ge 0$ for all n. Therefore, $f(z) \geq 0$. We now have $0 \leq f(z) \leq 0$, so $f(z) = 0$ as desired. $\square$

The following can now easily be proven by letting $f(x) = g(x) - y$ in Bolzano's Theorem:

Theorem 3.7.2. (Intermediate Value Theorem) *Suppose that the function $g : [a, b] \to \mathbb{R}$ is continuous and that y is between $g(a)$ and $g(b)$. There is a $z \in (a, b)$ such that $g(z) = y$.* $\square$

We close by proving the inverse of a continuous function is also continuous. This proof relies on the Extreme Value Theorem and the Intermediate Value Theorem to conclude that the image of a closed interval under a continuous function is a closed interval.

Theorem 3.7.3. *Suppose that f is an injective real valued function defined on an interval $[a, b]$. If f is continuous then f^{-1} is continuous.*

Proof. By the Extreme Value Theorem 3.6.2 and the Intermediate Value Theorem 3.7.2, we know that $f([a, b])$ is a closed interval $[c, d]$ and f^{-1} maps from $[c, d]$ to $[a, b]$. Let $y_0 \in [c, d]$ and let $\langle y_n \rangle$ be any sequence in $[c, d]$ converging to y_0. We must show that $\left\langle f^{-1}(y_n) \right\rangle$ converges to $f^{-1}(y)$. For each $n = 0, 1, 2, \ldots$ let $x_n = f^{-1}(y_n)$. Then $\langle x_n \rangle$ is a sequence in $[a, b]$. We will use Theorem 2.6.12 to show that $\langle x_n \rangle$ converges. Suppose that $\langle x_{n_k} \rangle$ is any convergent subsequence of $\langle x_n \rangle$. Let $z = \lim x_{n_k}$. Since f is continuous at z, we have

$$f(z) = \lim f(x_{n_k}) = \lim y_{n_k} = y_0 = f(x_0).$$

Since f is injective, it follows that $z = x_0$, so every convergent subsequence of $\langle x_n \rangle$ converges to x_0. By Theorem 2.6.12, $\langle x_n \rangle$ converges to x_0. But $x_n = f^{-1}(y_n)$ for all n, so this is the same as saying that $\left\langle f^{-1}(y_n) \right\rangle$ converges to $f^{-1}(y_0)$. Since this is true for all sequences converging to y_0, f^{-1} is continuous at y_0. Since this is true for all $y_0 \in [c, d]$, f^{-1} is continuous on $[c, d]$. $\square$

Exercises 3.7

3.7.1 Find an interval of length 1 that includes a solution to the equation $xe^x = 1$. Hint: By trial and error find a so that $f(a)$ and $f(a+1)$ are opposite signs where $f(x) = xe^x - 1$.

3.7.2 Find an interval of length 1 that includes a solution to the equation $x^3 - 6x^2 + 2.826 = 0$.

3.7.3 Prove that every odd degree polynomial has a real root. Hint: Consider the sign of $p(x)$ for very large positive x and and very large negative x.
3.7.4 Suppose that $f : [0, 1] \to [0, 1]$ is continuous. Prove that there is a $z \in [0, 1]$ so that $f(z) = z$. Such a z is called a fixed point of f. Hint: Consider the function $f(x) - x$ and use the Intermediate Value Theorem.
3.7.5 Prove that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ so that the equation $f(x) = c$ has exactly two solutions for every $c \in \mathbb{R}$. Hint: Suppose there is.
3.7.6 Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous, that $f(a) < g(a)$ and that $g(b) < f(b)$. Prove that there is some $z \in [a, b]$ where $f(z) = g(z)$. Hint: Consider the difference $f(x) - g(x)$.
3.7.7 Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Let $B = \{f(x) : x \in [a, b]\}$. Prove that B is a closed interval.
3.7.8 In the proof of Bolzano's Theorem 3.7.1, how do we know that $z \neq b$?

Chapter 4

Differentiation

In this chapter, we develop the theory of the derivative. In these discussions, it is best to think of slope, instantaneous velocity, or linear approximations. A linear function is a function $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = mx + b$ where $m, b \in \mathbb{R}$. The graph $y = f(x)$ of a linear function is a line. The graph of a linear function $f(x) = mx + b$ has the distinctive feature that for any two values $x_1 \neq x_2$ the difference quotient

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is equal to m. This number is the *slope* of the line and is sometimes called the *rate of change of f with respect to x*. If x changes by 1 unit, then $f(x)$ changes by m units. If f is interpreted as the position of an object moving along an axis and if x is interpreted as time, then the slope of $f(x) = mx + b$ is the rate of change in position with respect to time. This is velocity.

The slope m of a linear function $f(x) = mx + b$ gives a good deal of information about the function f. If $m = 0$, then f is constant. If $m > 0$, then f is *strictly increasing* – when $x_1 < x_2$, then $f(x_1) < f(x_2)$. If $m < 0$, then f is *strictly decreasing* – when $x_1 < x_2$, then $f(x_1) > f(x_2)$. The derivative is an attempt to extend this notion of slope to functions for which the difference quotient above may not be constant.

Exercises 4.0

4.0.1 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = mx + b$ and that $x_1, x_2 \in \mathbb{R}$. Prove that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = m.$$

4.0.2 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = mx + b$ and that $x_1 \in \mathbb{R}$ and $x_2 = x_1 + 1$. Prove that $f(x_2) - f(x_1) = m$.
4.0.3 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = mx + b$ where $m > 0$ and that $x_1 < x_2 \in \mathbb{R}$. Prove that $f(x_1) < f(x_2)$.

4.1 Tangent Lines

The notion of a line tangent to a circle is simple enough to describe. A tangent line is a line which intersects the circle at exactly one point, or it is a line which intersects but does not cross the circle. These descriptions are not adequate for describing lines tangent to graphs of functions as can be seen in Figure 4.1. For now, let us forget trying to

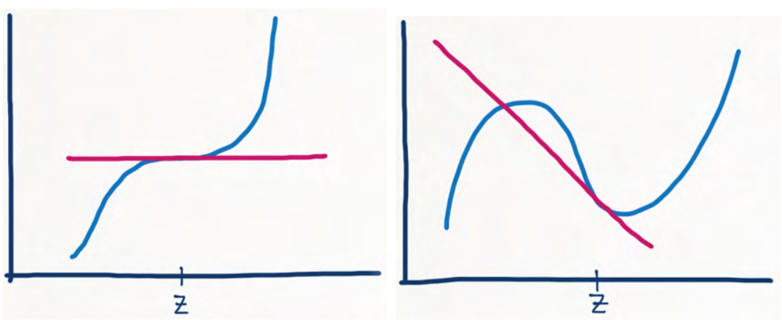

Figure 4.1: The graph on the left depicts a tangent line which crosses a function. The graph on the right depicts a tangent line that intersects a function at more then one point.

define a tangent line, and let us rather imagine how we might try to find the line tangent to the graph of a function $y = f(x)$ at a point z. To find this line, we need two things – a point on the line and the slope of the line. We have a point, $(z, f(z))$. We only need to find the slope. Suppose that x is close to z (and in the domain of f). Then $f(x)$ should be close to $f(z)$. (Well that really implies some sort of continuity, but this is an informal discussion.) The point $(x, f(x))$ should be close to the line tangent to $y = f(x)$ at z. This means that the slope of the

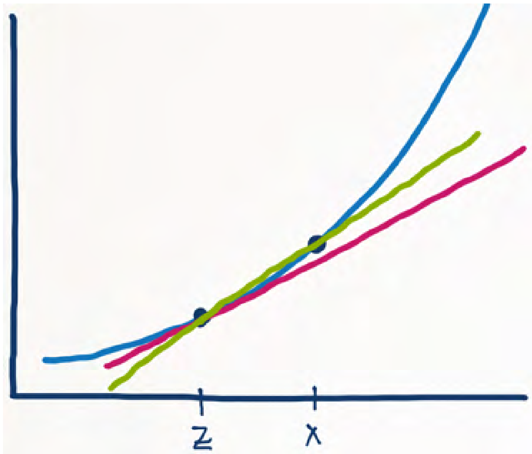

Figure 4.2: If x is close to z, then the slope of the secant line connecting $(z, f(z))$ and $(x, f(x))$ should be close to the slope of the line tangent to f at z.

line connecting $(x, f(x))$ and $(z, f(z))$ should be close to the slope of the tangent. That is

$$\text{slope of tangent at } z \approx \frac{f(x) - f(z)}{x - z}.$$

Presumably, this approximation should get better if x is allowed to move closer to z. Thus it appears that

$$\text{slope of tangent at } z = \lim_{x \to z} \frac{f(x) - f(z)}{x - z}.$$

Then we could *define* the tangent line to $y = f(x)$ at z to be the line through $(z, f(z))$ with this slope.

4.2 Linear Approximations

Another possible description of the line tangent to $y = f(x)$ at z is that the tangent line is the line which gives the best possible approximation to f near z. That is, this is a line $\ell(x) = mx + b$ so that $\ell(z) = f(z)$ and so that if x is close to z then $\ell(x)$ is close to $f(x)$. Moreover, this line should be "closer to f" than any other line. Now if f is somewhat nonlinear, then such an approximation might be good near z but much worse far from z. To find this line, we again only need the slope m. If x is any point, then the slope of the line is

$$\text{slope of linear approximation} = \frac{\ell(x) - \ell(z)}{x - z}.$$

If x is close to z, then $\ell(x)$ is close to $f(x)$ and $\ell(z) = f(z)$, so this slope is close to

$$\text{slope of linear approximation} \approx \frac{f(x) - f(z)}{x - z}.$$

Again, this approximation is better the closer x is to z, so perhaps

$$\text{slope of linear approximation} = \lim_{x \to z} \frac{f(x) - f(z)}{x - z}.$$

4.3 Instantaneous Velocity

Suppose that an object is moving along a number line and that we have a function f so that the position of the object at any time t is $f(t)$. We would like to find the velocity of the object at a particular time z. Suppose that t is a time shortly after z. (Actually, t may be before or after z.) We can easily find the average velocity from time z to time t. This is the change in position, $f(t) - f(z)$, divided by the change in time $t - z$:

$$\text{average velocity} = \frac{f(t) - f(z)}{t - z}.$$

Now the average velocity between z and t may not be that close to the velocity at z. It could be that the object "hits the breaks" at time z so that the velocity changes abruptly between z and t. On the other hand, velocity should be continuous, so if t is close enough to z, then this average velocity should be close to the velocity at z. It appears that

$$\text{velocity at } z = \lim_{t \to z} \frac{f(t) - f(z)}{t - z}.$$

4.4 The Definition of the Derivative

Based on the previous discussion, we make this definition.

Definition 4.4.1. Suppose that $z \in D \subseteq \mathbb{R}$ is an accumulation point of D and that $f : D \to \mathbb{R}$ is any function. If the limit

$$\lim_{x \to z} \frac{f(x) - f(z)}{x - z}$$

exists then f is *differentiable* at z and the *derivative* at z of f is

$$f'(z) = \lim_{x \to z} \frac{f(x) - f(z)}{x - z}.$$

If f is differentiable at z, then the *tangent line* to the curve $y = f(x)$ at the point $(z, f(z))$ is the line through $(z, f(z))$ with slope $f'(z)$. This is the line

$$y = f'(z)(x - z) + f(z).$$

Note that in this definition we allow z to be an accumulation point of the domain. Many texts restrict z to be an interior point of the domain. A statement of the form, "Suppose that f is differentiable at z" is meant to imply that f is a function with some domain $D \subseteq \mathbb{R}$, that z is an accumulation point of D, that $z \in D$, and that $f'(z)$ exists.

Example 4.4.2. (Derivative of a Constant) The derivative of any constant function is 0 everywhere. If $f : \mathbb{R} \to \mathbb{R}$ is constant, then f is differentiable everywhere and $f'(a) = 0$ for all $a \in \mathbb{R}$. Suppose that $c \in \mathbb{R}$ and $f(x) = c$ for all x. Suppose that $a \in \mathbb{R}$.

$$\begin{aligned}
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{c - c}{x - a} \\
&= \lim_{x \to a} \frac{0}{x - a} \\
&= \lim_{x \to a} 0 \\
&= 0
\end{aligned}$$

Example 4.4.3. (Derivative of a Linear Function) The derivative of any linear function is its slope. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = mx + b$. Then f is differentiable everywhere and $f'(a) = m$ for all $a \in \mathbb{R}$. Let $a \in \mathbb{R}$. Then

$$\begin{aligned}
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{(mx + b) - (ma + b)}{x - a} \\
&= \lim_{x \to a} \frac{m(x - a)}{x - a} \\
&= \lim_{x \to a} m \\
&= m
\end{aligned}$$

Thus, f' is constantly m.

Example 4.4.4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = |x|$. Then f is not differentiable at 0. To see this, suppose that $\langle x_n \rangle$ and $\langle y_n \rangle$ are the sequences given by $x_n = 1/n$ and $y_n = -1/n$. Note that

$$\lim \frac{|x_n| - |0|}{x_n - 0} = \lim \frac{1/n}{1/n} = \lim 1 = 1$$

but

$$\lim \frac{|y_n| - |0|}{y_n - 0} = \lim \frac{1/n}{-1/n} = \lim -1 = -1.$$

Since these two limits differ,

$$\lim_{x \to 0} \frac{|x| - |0|}{x - 0}$$

cannot exist. Thus f is not differentiable at 0.

Example 4.4.5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^3$. Then f is differentiable everywhere and $f'(x) = 3x^2$ for all $x \in \mathbb{R}$. Let $a \in \mathbb{R}$. Then

$$\begin{aligned}
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{x^3 - a^3}{x - a} \\
&= \lim_{x \to a} \frac{(x-a)(x^2 + xa + a^2)}{x - a} \\
&= \lim_{x \to a} (x^2 + xa + a^2) \\
&= a^2 + a^2 + a^2 \\
&= 3a^2
\end{aligned}$$

Thus, for any a, $f'(a) = 3a^2$. We would usually prefer to write this as $f'(x) = 3x^2$ for all $x \in \mathbb{R}$.

Recall that for any positive integer n

$$(x^n - a^n) = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}).$$

We can use this to extend the previous example.

Example 4.4.6. (Power Rule for Positive Integer Exponents) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^n$. Then f is differentiable everywhere and $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$. Let $a \in \mathbb{R}$. Then

$$\begin{aligned}
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{x^n - a^n}{x - a} \\
&= \lim_{x \to a} \frac{(x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\
&= \lim_{x \to a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}) \\
&= na^{n-1}
\end{aligned}$$

Thus, for any a, $f'(a) = na^{n-1}$. We would usually prefer to write this as $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$.

Example 4.4.7. Suppose that $f : [0, \infty) \to \mathbb{R}$ is given by $f(x) = \sqrt{x}$. Then f is differentiable at all $a \in (0, \infty)$ and $f'(a) = \dfrac{1}{2\sqrt{a}}$. Let $a > 0$. Then

$$\begin{aligned}
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \\
&= \lim_{x \to a} \left[\frac{\sqrt{x} - \sqrt{a}}{x - a} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right] \\
&= \lim_{x \to a} \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} \\
&= \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} \\
&= \frac{1}{2\sqrt{a}}.
\end{aligned}$$

This limiting process fails at 0. Suppose that $\langle x_n \rangle$ is given by $x_n = 1/n$. Then

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{\sqrt{1/n}}{1/n} = \sqrt{n}.$$

Since this sequence is unbounded, $\displaystyle\lim_{x \to 0} \frac{f(x_n) - f(0)}{x_n - 0}$ cannot exist.

Exercises 4.4

4.4.1 Use the definition of the derivative to find the derivative of the given function at the stated point.

a. $f(x) = x^4$ at $x = 5$

b. $f(x) = \dfrac{3x+4}{2x-1}$ at $x = 1$

4.4.2 Use the definition to find the derivative of $f(x) = x^2$.

4.4.3 Use the definition to find the derivative of $f(x) = \dfrac{1}{\sqrt{x}}$.

4.4.4 Prove that the function $f(x) = x^{1/3}$ is not differentiable at 0.

4.4.5 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere and that $z \in \mathbb{R}$. Prove that

$$\lim_{h \to 0} \frac{f(z+h) - f(z-h)}{2h}$$

exists and equals $f'(z)$.

4.4.6 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere and differentiable at $z \in \mathbb{R}$. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} \dfrac{f(x) - f(z)}{x - z} & x \neq z \\ f'(z) & x = z \end{cases}.$$

Prove that g is continuous everywhere.

4.4.7 Suppose that $f, h : \mathbb{R} \to \mathbb{R}$ are functions so that $f(x) \leq h(x)$ for all $x \in \mathbb{R}$. Suppose also that f and h are differentiable at $z \in \mathbb{R}$ and that $f(z) = h(z)$. Prove that $f'(z) = h'(z)$.

4.4.8 Let $f(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x \end{cases}$

a. Sketch the graph of f.

b. Prove that f is differentiable everywhere and find the derivative.

c. Sketch the graph of f'

4.4.9 Let $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ -x^2 & x \notin \mathbb{Q} \end{cases}.$

a. Prove that f is not continuous at any $z \neq 0$.

b. Prove that f is differentiable at 0.

4.5 Properties of the Derivative

Theorem 4.5.1. *If f is differentiable at a, then f is continuous at a.*

Proof. If f is differentiable at a, then

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

exists. Note that for $x \neq a$,

$$f(x) = \frac{f(x)-f(a)}{x-a}(x-a) + f(a).$$

Then

$$\begin{aligned} f(a) &= f'(a)\cdot 0 + f(a) \\ &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a) + \lim_{x\to a} f(a) \\ &= \lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a}(x-a) + f(a) \right) \\ &= \lim_{x\to a} f(x) \end{aligned}$$

Thus f is continuous at a. □

Theorem 4.5.2. *Suppose that f and g are differentiable at a and that $k \in \mathbb{R}$.*

1. **(Constant Multiple Rule)** *kf is differentiable at a and*

$$(kf)'(a) = kf'(a).$$

2. **(Sum Rule)** *$f+g$ is differentiable at a and*

$$(f+g)'(a) = f'(a) + g'(a).$$

3. **(Product Rule)** *fg is differentiable at a and*

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

4. **(Quotient Rule)** *If $g(a) \neq 0$, f/g is differentiable at a and*

$$(f/g)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Proof. 1. **(Constant Multiple Rule)** Note that

$$\begin{aligned} kf'(a) &= k \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{kf(x) - kf(a)}{x - a} \\ &= \lim_{x \to a} \frac{(kf)(x) - (kf)(a)}{x - a}. \end{aligned}$$

Since this last limit exists and is equal to $kf'(a)$, then kf is differentiable at a and $(kf)'(a) = kf'(a)$.

2. **(Sum Rule)** Note that

$$\begin{aligned} f'(a) + g'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \\ &= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} + \frac{g(x) - g(a)}{x - a} \right) \\ &= \lim_{x \to a} \frac{f(x) - f(a) + g(x) - g(a)}{x - a} \\ &= \lim_{x \to a} \frac{f(x) + g(x) - f(a) - g(a)}{x - a} \\ &= \lim_{x \to a} \frac{(f+g)(x) - (f+g)(a)}{x - a}, \end{aligned}$$

Since this last limit exists, $f + g$ is differentiable at a and

$$(f+g)'(a) = f'(a) + g'(a).$$

3. **(Product Rule)** This one may be slightly more exciting. Notice that

$$\begin{aligned} &f'(a)g(a) + f(a)g'(a) \\ &= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) g(a) + \left(\lim_{x \to a} f(x) \right) \left(\lim_{x \to a} \frac{g(x) - g(a)}{x - a} \right) \end{aligned}$$

$$\begin{aligned} &= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} g(a) + f(x) \frac{g(x) - g(a)}{x - a} \right) \\ &= \lim_{x \to a} \frac{f(x)g(a) - f(a)g(a) + f(x)g(x) - f(x)g(a)}{x - a} \\ &= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} \\ &= \lim_{x \to a} \frac{(fg)(x) - (fg)(a)}{x - a}. \end{aligned}$$

Since this last limit exists, fg is differentiable at a and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

4. **(Quotient Rule)** Suppose that $g(a) \neq 0$. Since g is continuous at a, this implies that $g(x) \neq 0$ for x near a. Thus we can safely divide by $g(x)$ near a. Then

$$\begin{aligned}
&\frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2} \\
&= \frac{\left(\lim_{x\to a} \frac{f(x) - f(a)}{x - a}\right) g(a) - f(a) \left(\lim_{x\to a} \frac{g(x) - g(a)}{x - a}\right)}{g(a) \lim_{x\to a} g(x)} \\
&= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a} g(a) - f(a) \frac{g(x)-g(a)}{x-a}}{g(a)g(x)} \\
&= \lim_{x\to a} \frac{f(x)g(a) - f(a)g(a) - f(a)g(x) + f(a)g(a)}{(x - a)g(a)g(x)} \\
&= \lim_{x\to a} \frac{f(x)g(a) - f(a)g(x)}{(x - a)g(a)g(x)} \\
&= \lim_{x\to a} \frac{\frac{f(x)g(a)}{g(a)g(x)} - \frac{f(a)g(x)}{g(a)g(x)}}{x - a} \\
&= \lim_{x\to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x - a} \\
&= \lim_{x\to a} \frac{(f/g)(x) - (f/g)(a)}{x - a}
\end{aligned}$$

Since this last limit exists, f/g is differentiable at a and

$$(f/g)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

□

Example 4.5.3. If $f(x) = 3x^2 + 4x + 5$ then

$$\begin{aligned}
f'(x) &= (3x^2)' + (4x)' + (5)' && \text{Sum Rule (twice)} \\
&= (3x^2)' + (4x)' + 0 && \text{Derivative of a Constant} \\
&= (3x^2)' + 4 && \text{Derivative of a Line} \\
&= 3(x^2)' + 4 && \text{Constant Multiple Rule} \\
&= 3(2x) + 4 && \text{Power Rule} \\
&= 6x + 4.
\end{aligned}$$

It should be clear that this process can be repeated for any polynomial. Therefore we have:

Theorem 4.5.4. *Every polynomial is differentiable everywhere.* □

Example 4.5.5. Suppose that $f(x) = \dfrac{2x+1}{3x+1}$. Then

$$\begin{aligned} f'(x) &= \frac{(2x+1)'(3x+1) - (2x+1)(3x+1)'}{(3x+1)^2} && \text{Quotient Rule} \\ &= \frac{2(3x+1) - (2x+1)3}{(3x+1)^2} && \text{Derivative of a Line} \\ &= \frac{-1}{(3x+1)^2}. \end{aligned}$$

Example 4.5.6. Suppose f is differentiable at 2 and that $f(2) = -2$ and $f'(2) = 3$. Suppose also that $g(x) = 4x^3 + 5x$ so that $g(2) = 42$. Note that $g'(x) = 12x^2 + 5$ so that $g'(2) = 53$. Let $h = fg$. Then

$$h'(2) = (fg)'(2) = f'(2)g(2) + f(2)g'(2) = 3 \cdot 42 + (-2) \cdot 53 = 20.$$

We now turn our attention to compositions. The idea behind the next proof is to consider

$$\begin{aligned} \frac{g \circ f(x) - g \circ f(a)}{x-a} &= \frac{g(f(x)) - g(f(a))}{x-a} \\ &= \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \frac{f(x) - f(a)}{x-a} \end{aligned}$$

and to take a limit as $x \to a$. The first fraction should approach $g'(f(a))$ and the second should approach $f'(a)$. The problem with this approach is that it may be that $f(x) = f(a)$ even when $x \neq a$, so the first fraction may not be defined. In the proof below, we replace this fraction with a function which is defined everywhere g is.

Theorem 4.5.7. (Chain Rule) *Suppose that f is differentiable at a and that g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a and $(g \circ f)'(a) = g'(f(a))f'(a)$.*

Proof. Define a function h to have the same domain as g so that

$$h(x) = \begin{cases} \dfrac{g(x) - g(f(a))}{x - f(a)} & x \neq f(a) \\ \\ g'(f(a)) & x = f(a). \end{cases}$$

Notice that since

$$\lim_{x\to f(a)} h(x) = \lim_{x\to f(a)} \frac{g(x)-g(f(a))}{x-f(a)} = g'(f(a)) = h(f(a))$$

h is continuous at $f(a)$. Also, since f is differentiable at a, f is continuous at a, so $h \circ f$ is continuous at a and

$$\lim_{x\to a} h(f(x)) = h(f(a)) = g'(f(a)).$$

If $x \neq f(a)$, then $g(x) - g(f(a)) = h(x)(x - f(a))$. Also,

$$g(f(a)) - g(f(a)) = 0 = h(f(a))(f(a) - f(a)),$$

so $g(x) - g(f(a)) = h(x)(x - f(a))$ for all x in the domain of g. In particular, $g(f(x)) - g(f(a)) = h(f(x))(f(x) - f(a))$. We can now consider the derivative of $g \circ f$.

$$\begin{aligned}
\lim_{x\to a} \frac{g\circ f(x) - g\circ f(a)}{x-a} &= \lim_{x\to a} \frac{g(f(x)) - g(f(a))}{x-a} \\
&= \lim_{x\to a} \frac{h(f(x))(f(x)-f(a))}{x-a} \\
&= \lim_{x\to a} h(f(x))\frac{f(x)-f(a)}{x-a} \\
&= g'(f(a))f'(a).
\end{aligned}$$

□

Example 4.5.8. Suppose that $f(x) = x^2 + x + 1$, $g(x) = x^9$, and $h(x) = g \circ f(x) = (x^2 + x + 1)^9$. We could find h' by first expanding the ninth power. This would be tedious. Using Chain Rule is much faster. Note that $f'(x) = 2x + 1$ and $g'(x) = 9x^8$. Then

$$h'(x) = g'(f(x))f'(x) = 9(x^2 + x + 1)^8(2x + 1).$$

We can easily extend the power rule to all rational exponents. To do so, we need to first deal with roots. This will be accomplished with the help of the next theorem.

Theorem 4.5.9. *Suppose that f is injective and differentiable on $[a, b]$. If $z \in [a, b]$, and if $f'(z) \neq 0$, then f^{-1} is differentiable at $f(z)$ and*

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}.$$

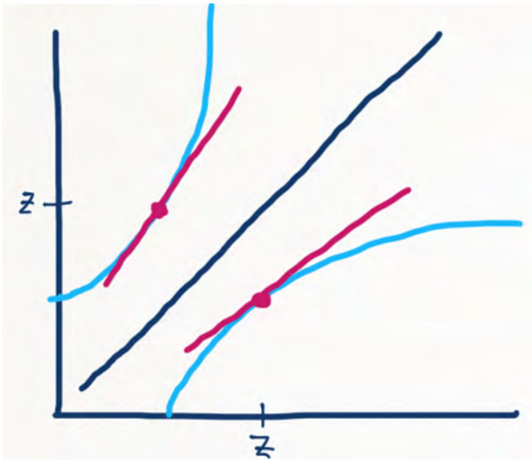

Figure 4.3: The function on the right is f. The function on the left is f^{-1}. The point $(z, f(z))$ is on the graph of f, so the point $(f(z), z)$ is on the graph of f^{-1}. The slope of the tangent to this point on f^{-1} is the reciprocal of the slope of the tangent at the corresponding point on f, that is: $\dfrac{1}{f'(z)}$.

Proof. Notice that by the Extreme Value Theorem and the Intermediate Value Theorem, $f([a, b])$ is a closed interval $[c, d]$, so that $f(z)$ is an accumulation point of $f([a, b])$. Since f is injective, we know that if $x \neq z$, then $f(x) \neq f(z)$. Then, since $f'(z) \neq 0$, we know

$$\lim_{x \to z} \frac{x - z}{f(x) - f(z)} = \frac{1}{f'(z)}.$$

Let $\epsilon > 0$. There is a $\delta_1 > 0$ so that for all $x \in [a, b]$, if $0 < |x - z| < \delta_1$, then

$$\left| \frac{x - z}{f(x) - f(z)} - \frac{1}{f'(z)} \right| < \epsilon.$$

By Theorem 3.7.3, we know that f^{-1} is continuous. Therefore, there is a $\delta > 0$ so that for all $x \in [c, d]$,

$$\text{if } 0 < |x - f(z)| < \delta, \text{ then } |f^{-1}(x) - f^{-1}(f(z))| < \delta_1.$$

Now, $f^{-1}(f(z)) = z$, so

$$\text{if } 0 < |x - f(z)| < \delta, \text{ then } |f^{-1}(x) - z| < \delta_1.$$

Also, by injectivity, we know that since $x \neq f(z)$ then

$$f^{-1}(x) \neq f^{-1}(f(z)) = z.$$

Thus we can actually say

$$\text{if } 0 < |x - f(z)| < \delta, \text{ then } 0 < |f^{-1}(x) - z| < \delta_1.$$

Suppose that $x \in [c, d]$ and that $0 < |x - f(z)| < \delta$. Then we know $0 < |f^{-1}(x) - z| < \delta_1$. Now $f^{-1}(x) \in [a, b]$ and $0 < |f^{-1}(x) - z| < \delta_1$, so

$$\left| \frac{f^{-1}(x) - z}{f(f^{-1}(x)) - f(z)} - \frac{1}{f'(z)} \right| < \epsilon.$$

Note that $z = f^{-1}(f(z))$ and $f(f^{-1}(x)) = x$, so we have that if $x \in [c, d]$ and $0 < |x - f(z)| < \delta$ then

$$\left| \frac{f^{-1}(x) - f^{-1}(f(z))}{x - f(z)} - \frac{1}{f'(z)} \right| < \epsilon.$$

We have established that

$$\lim_{x \to f(z)} \frac{f^{-1}(x) - f^{-1}(f(z))}{x - f(z)} = \frac{1}{f'(z)}.$$

Therefore, f^{-1} is differentiable at z and

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}.$$

$\square$

We can use this theorem about derivatives of inverse functions to prove that functions of the form $x^{1/n}$, where n is a positive integer, are differentiable.

Theorem 4.5.10. *Suppose that n is a positive integer. Let $D = \mathbb{R}$ if n is odd and $D = [0, \infty)$ if n is even. Let $g : D \to D$ be given by $g(x) = x^{1/n}$. Then g is differentiable on $D - \{0\}$ and*

$$g'(x) = \frac{1}{n} x^{1/n - 1}.$$

Proof. Let $f : D \to D$ be given by $f(x) = x^n$. Then $g = f^{-1}$. Let $x \in D - \{0\}$. We will show that g is differentiable at x and that $g'(x)$ has the stated value. Let $z = g(x)$. Then $z \in D - \{0\}$. Let $a, b \in D - \{0\}$ so that $a < z < b$. Then f is differentiable on $[a, b]$ and $f'(z) \neq 0$. By Theorem 4.5.9 $g = f^{-1}$ is differentiable at $f(z) = x$ and

$$g'(x) = (f^{-1})'(f(z)) = \frac{1}{f'(z)} = \frac{1}{nz^{n-1}} =$$

$$= \frac{1}{n} z^{1-n} = \frac{1}{n}(x^{1/n})^{1-n} = \frac{1}{n} x^{1/n-1}.$$

□

We can use Theorem 4.5.10, the Chain Rule, and the Quotient Rule to extend the Power Rule to all rational exponents. For simplicity's sake, we restrict our domain to positive real numbers for this theorem.

Theorem 4.5.11. (Power Rule for Rational Exponents) *Suppose that $m, n \in \mathbb{Z}$ are relatively prime with $n > 0$. Let $D = (0, \infty)$. Let $f : D \to D$ be given by $f(x) = x^{m/n}$. Then f is differentiable on all of D and*

$$f'(x) = \frac{m}{n} x^{m/n-1}.$$

Proof. If $m = 0$, then $f(x) = 1$ and $f'(x) = 0$ as required by the theorem. Suppose that $m > 0$. Then

$$f(x) = (x^m)^{1/n}.$$

Applying the Chain Rule and Theorem 4.5.10 gives

$$f'(x) = \frac{1}{n} (x^m)^{1/n-1} m x^{m-1} = \frac{m}{n} x^{m/n-1}.$$

Now assume that $m < 0$. Then

$$f(x) = \frac{1}{x^{|m|/n}}.$$

We can use what we just proved for positive integers along with the quotient rule to get

$$f'(x) = \frac{0 \cdot x^{|m|/n} - 1 \cdot \frac{|m|}{n} x^{|m|/n-1}}{(x^{|m|/n})^2} = -1 \cdot \frac{-m}{n} \frac{x^{-m/n-1}}{x^{-2m/n}} = \frac{m}{n} x^{m/n-1}.$$

□

This power rule can be extended to any real number, but doing so would require a healthy treatments of arbitrary exponential functions. This will have to wait.

Exercises 4.5

4.5.1 Prove the chain rule (that $(g \circ f)'(a) = g'(f(a))f'(a)$) assuming that f is injective. Hint: See the discussion before the proof of the chain rule.
4.5.2 Use the derivative of the identity function, the product rule, and induction to prove the power rule for natural number exponents.
4.5.3 Suppose that $f : [a, b] \to [c, d]$ is differentiable at $z \in [a, b]$, that $g : [c, d] \to [r, s]$ is differentiable at $f(z)$, and $h : [r, s] \to [t, u]$ is differentiable at $g(f(z))$. Prove that the function $h \circ (g \circ f)$ is differentiable at z and find the derivative.
4.5.4 Suppose that $f : [a, b] \to [c, d]$ and $g : [c, d] \to [r, s]$ are differentiable. Suppose also that f' and g' are also differentiable. Prove that $(g \circ f)'$ is differentiable and find the derivative.
4.5.5 Find the equation of the line tangent to f^{-1} at the point $(3, 1)$ if $f(x) = x^3 + 2x^2 - x + 1$.

4.6 Extrema and the Mean Value Theorem

Definition 4.6.1. A function f has a *local minimum* at c if f is defined on an open interval I around c so that $f(c) \leq f(x)$ for all $x \in I$. A function f has a *local maximum* at c if f is defined on an open interval I around c so that $f(x) \leq f(c)$ for all $x \in I$. A *local extremum* is either a local maximum or a local minimum.

Theorem 4.6.2. (Fermat's Theorem) *If f has a local extremum at c and if f is differentiable at c, then $f'(c) = 0$.*

Proof. Suppose that f has a local maximum at c. Then f is defined on an open interval (a, b) containing c so that $f(x) \leq f(c)$ for all $x \in (a, b)$. Let $\langle x_n \rangle$ be a sequence in (a, c) with $\lim x_n = c$ and let $\langle y_n \rangle$ be a sequence in (c, b) with $\lim y_n = c$. Then

$$\lim \frac{f(x_n) - f(c)}{x_n - c} = \lim \frac{f(y_n) - f(c)}{y_n - c} = f'(c).$$

Now, since f has a local maximum at c, then

$$f(x_n) \leq f(c) \text{ and } f(y_n) \leq f(c)$$

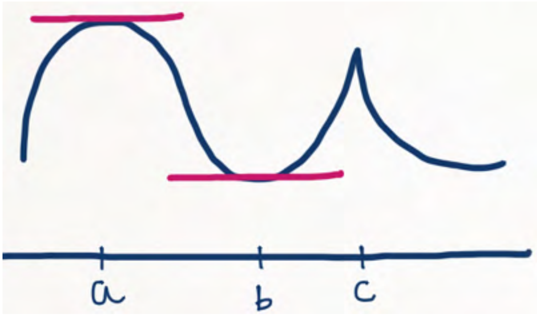

Figure 4.4: Fermat's Theorem tells us that local extrema of f occur either at points where the derivative of f is 0 – such as a and b – or where the derivative is not defined – as at c.

for all n. This means that $f(x_n) - f(c) \leq 0$ and $f(y_n) - f(c) \leq 0$. On the other hand, $x_n - c < 0$ and $y_n - c > 0$ for all n. It follows that for all n

$$0 \leq \frac{f(x_n) - f(c)}{x_n - c} \text{ and } \frac{f(y_n) - f(c)}{y_n - c} \leq 0.$$

But then

$$0 \leq \lim \frac{f(x_n) - f(c)}{x_n - c} = f'(c) = \lim \frac{f(y_n) - f(c)}{y_n - c} \leq 0$$

so $f'(c) = 0$. □

Theorem 4.6.3. (Rolle's Theorem) *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is at least one $z \in (a, b)$ so that $f'(z) = 0$.*

Proof. Since f is continuous on $[a, b]$, by The Extreme Value Theorem (Theorem 3.6.2) there are $u, v \in [a, b]$ with $f(u) \leq f(x) \leq f(v)$ for all $x \in [a, b]$. If $\{u, v\} = \{a, b\}$, then it follows that f is constant on $[a, b]$. Then $f'(x) = 0$ for all $x \in (a, b)$. Suppose then that either u or v is in (a, b). Let z be u or v – whichever is in (a, b). Then f has a local extremum at z. By Fermat's Theorem, $f'(z) = 0$. □

Theorem 4.6.4. (Mean Value Theorem) *Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). There is at least one $z \in (a, b)$ so that $f'(z) = \dfrac{f(b) - f(a)}{b - a}$.*

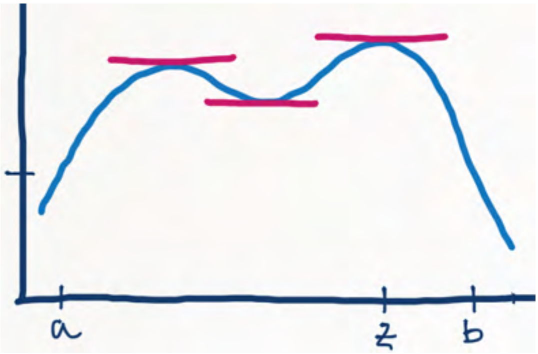

Figure 4.5: Rolle's Theorem tells us that if f is differentiable and $f(a) = f(b)$, then f must have a horizontal tangent line somewhere between a and b. This particular function has three.

Proof. Let

$$m = \frac{f(b) - f(a)}{b - a}.$$

Define $L : [a, b] \to \mathbb{R}$ so that $L(x) = m(x - a) + f(a)$ (so, L is the line connecting the endpoints of f on $[a, b]$). As a linear funciton, L is differentiable and L' is constantly the slope of L, which is m. Define $h : [a, b] \to \mathbb{R}$ by $h(x) = f(x) - L(x)$. Then h is continuous on $[a, b]$ and differentiable on (a, b). Moreover, $h(a) = f(a) - L(a) = 0$ and $h(b) = f(b) - L(b) = 0$. By Rolle's Theorem, there is some $z \in (a, b)$ where $h'(z) = 0$. but then

$$0 = h'(z) = f'(z) - L'(z) = f'(z) - m = f'(z) - \frac{f(b) - f(a)}{b - a}$$

so

$$f'(z) = \frac{f(b) - f(a)}{b - a}.$$

$\square$

Theorem 4.6.5. (Zero Derivative Theorem) *Suppose that f is differentiable on an interval I. If $f'(x) = 0$ for all $x \in I$, then f is constant on I.*

Proof. Suppose that f' is constantly 0 on I. Suppose that $a < b \in I$. We will show that $f(a) = f(b)$. Since f is differentiable on all of I, f is continuous on all of I and, hence, on $[a, b]$. Since f is differentiable

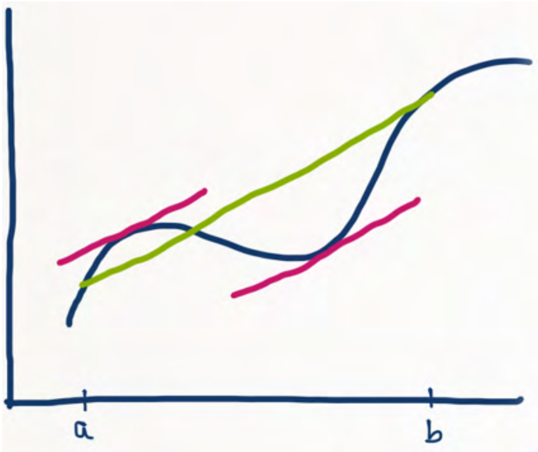

Figure 4.6: The Mean Value Theorem tells us that if f is differentiable, then there is a tangent line between a and b which is parallel to the line connecting the points $(a, f(a))$ and $(b, f(b))$. In this picture, there are two.

on I, f is differentiable on (a, b). By the Mean Value Theorem, there is a $z \in (a, b)$ where

$$f'(z) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(z) = 0$, so

$$0 = \frac{f(b) - f(a)}{b - a}.$$

It now follows that $f(b) = f(a)$. Thus f is constant. □

Theorem 4.6.6. (Constant Difference Theorem) *Suppose that f and g are both differentiable on an interval I and that $f'(x) = g'(x)$ for all $x \in I$. There is some $c \in \mathbb{R}$ so that $f(x) - g(x) = c$ for all $x \in I$.*

Proof. Define $h : I \to \mathbb{R}$ by $h = f - g$. Since $f' = g'$, then h' is constantly 0. By the Zero Derivative Theorem, h is a constant. Therefore, there is a $c \in \mathbb{R}$ so that $f(x) - g(x) = h(x) = c$ for all $x \in I$. □

Exercises 4.6

4.6.1 Suppose that f has a local minimum at c and that $f'(c)$ exists. Prove that $f'(c) = 0$. (This is the other half of the proof of Fermat's Theorem.)

4.6.2 Use Rolle's Theorem to prove that the equation $x^3 - 3x + b = 0$ has at most one solution in $[-1, 1]$.
4.6.3 Suppose that $f : [0, 2] \to \mathbb{R}$ is differentiable, $f(0) = 0$, $f(1) = 2$, and $f(2) = 2$. Prove that

1. There is a $c_1 \in [0, 2]$ where $f'(c_1) = 0$.

2. There is a $c_2 \in [0, 2]$ where $f'(c_2) = 2$.

4.6.4 Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable and that $|f'(x)| \leq M$ for all $x \in (a, b)$. Prove that f is uniformly continuous on (a, b). Give an example of a uniformly continuous and differentiable function on $(-1, 1)$ which has an unbounded derivative.
4.6.5 Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function so that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.
4.6.6 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $1 \leq f'(x) \leq 2$ for all $x \in \mathbb{R}$. Prove that if $f(0) = 0$ then $x \leq f(x) \leq 2x$ for all $x \geq 0$.

4.7 First Derivative Tests

Theorem 4.7.1. *If f is differentiable on an interval I and $f'(x) \neq 0$ for all $x \in \mathbb{R}$, then f is injective on I.*

Proof. Suppose that f is differentiable on I but is not injective. Then there exist $a < b \in I$ so that $f(a) = f(b)$. The function f is continuous on $[a, b]$ and differentiable on (a, b). By Rolle's Theorem, there is a $z \in (a, b)$ where $f'(z) = 0$. Thus, if f is not injective, then f' is sometimes 0. This is the contrapositive of the theorem. $\square$

Definition 4.7.2. Suppose that I is an interval. A function $f : I \to \mathbb{R}$ is *increasing* on I if $f(x) \leq f(y)$ for all $x < y$ in I. The function f is *decreasing* on I if $f(x) \geq f(y)$ for all $x < y$ in I. If f is either increasing or decreasing on I, then f is *monotonic* on I. The function $f : I \to \mathbb{R}$ is *strictly increasing* on I if $f(x) < f(y)$ for all $x < y$ in I. The function f is *strictly decreasing* on I if $f(x) > f(y)$ for all $x < y$ in I.

Theorem 4.7.3. (First Derivative Test for Monotonicity) *Suppose that I is an interval and that $f : I \to \mathbb{R}$ is differentiable on I.*

1. *f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$.*

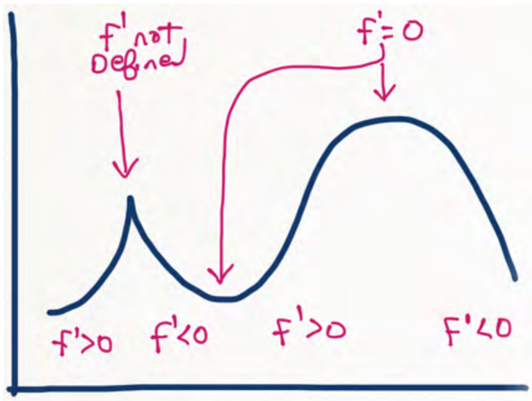

Figure 4.7: The results of this section tell us that the local extrema of f occur at points where the derivative of f is either 0 or not defined. When $f' > 0$, the function f is increasing. When $f' < 0$, the function f is decreasing. This information can be used to analyze a function and locate local extrema.

2. *f is decreasing on I if and only if $f'(x) \leq 0$ for all $x \in I$.*

Proof. We will prove (1). The rest of the theorem is left as an exercise. Suppose first that f is increasing on I. Let $z \in I$, and let $\langle x_n \rangle$ be a sequence in I approaching z so that $x_n > z$ for all n (a special case is needed if z is a right hand endpoint of I). Then $x_n - z > 0$ for all n. Since f is increasing, $f(x_n) \geq f(z)$ so $f(x_n) - f(z) \geq 0$ for all n. It follows that $f'(z) = \lim \dfrac{f(x_n) - f(z)}{x_n - z} \geq 0$.

Suppose now that $f'(x) \geq 0$ on I. We will prove that f is increasing on I. Let $a < b \in I$. We will show that $f(a) \leq f(b)$. The function f is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem, there is some $z \in (a, b)$ so that

$$f'(z) = \frac{f(b) - f(a)}{b - a}.$$

Then $f(b) - f(a) = f'(z)(b - a)$. Since $f'(z) \geq 0$ and $b - a > 0$, then $f(b) - f(a) \geq 0$. Thus $f(a) \leq f(b)$. This holds whenever $a < b$, so f is increasing. $\square$

Notice in this proof that if we instead had assumed that $f'(z) > 0$, then the conclusion would be that $f(b) - f(a) > 0$ so $f(a) < f(b)$.

In this case, f would be *strictly* increasing. Thus the proof is easily modified to prove:

Theorem 4.7.4. *Suppose that I is an interval and that $f : I \to \mathbb{R}$ is differentiable on I.*

1. *If $f'(x) > 0$ for all $x \in I$ then f is strictly increasing on I.*

2. *If $f'(x) < 0$ for all $x \in I$ then f is strictly decreasing on I.* □

Theorem 4.7.5. (First Derivative Test for Local Extrema) *Suppose that $a < b < c$ in $\mathbb{R}$ and that $f : (a, c) \to \mathbb{R}$ is continuous on (a, c) and differentiable on (a, b) and (b, c).*

1. *If $f'(x) > 0$ on (a, b) and $f'(x) < 0$ on (b, c), then f has a local maximum at b.*

2. *If $f'(x) < 0$ on (a, b) and $f'(x) > 0$ on (b, c), then f has a local minimum at b.*

Proof. We will prove part (1). The rest is left as an exercise. Suppose that $a < x < b$. Then f is continuous on $[x, b]$ and differentiable on (x, b). By the Mean Value Theorem, there is some $z \in (x, b)$ so that

$$f'(z) = \frac{f(b) - f(x)}{b - x}.$$

Then $f(b) - f(x) = f'(z)(b - x)$. Since $f'(z) > 0$ and $(b - x) > 0$, then $f(b) - f(x) > 0$. Thus $f(b) > f(x)$ when $x \in (a, b)$.

Now suppose that $b < x < c$. Then f is continuous on $[b, x]$ and differentiable on (b, x). By the Mean Value Theorem, there is some $z \in (b, x)$ so that

$$f'(z) = \frac{f(x) - f(b)}{x - b}.$$

Then $f(x) - f(b) = f'(z)(x - b)$. Since $f'(z) < 0$ and $(x - b) > 0$, then $f(x) - f(b) < 0$. Thus $f(b) > f(x)$ when $x \in (b, c)$. Thus, for any $x \in (a, c)$, $f(x) \leq f(b)$, and f has a local maximum at b. □

Exercises 4.7

4.7.1 Prove part (2) of Theorem 4.7.3.

4.7.2 State and prove the special case which is needed in the proof of part (1) of Theorem 4.7.3 if z is a right hand endpoint of I.

4.7.3 Prove part (1) of Theorem 4.7.4.
4.7.4 Prove part (2) of Theorem 4.7.5.
4.7.5 Let $f, g : [0,1] \to \mathbb{R}$ be differentiable with $f(0) = g(0)$ and $f'(x) > g'(x)$ for all $x \in [0,1]$. Prove $f(x) > g(x)$ for all $x \in (0,1]$.
4.7.6 Suppose that $f : [a,b] \to \mathbb{R}$ is differentiable at $c \in (a,b)$ and that $f'(c) > 0$. Prove that there is an $x \in (c,b)$ so that $f(x) > f(c)$.

4.8 l'Hôpital's Rule

We close this chapter with an application of the derivative to finding special types of limits. This application, known as l'Hôpital's Rule, is one of the fundamental tools for evaluating limits. To prove the theorem related to l'Hôpital's Rule, we first need this somewhat technical extension to the Mean Value Theorem. The proof of this theorem is a quick application of Rolle's Theorem to the function

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

Theorem 4.8.1. (Extended Mean Value Theorem) *Suppose that f and g are continuous on $[a,b]$ and differentiable on (a,b). There is at least one $z \in (a,b)$ where*

$$f'(z)(g(b) - g(a)) = g'(z)(f(b) - f(a)).$$

□

Theorem 4.8.2. (l'Hôpital's Rule) *Suppose that f and g are continuous on $[a,b]$ and differentiable on (a,b) and that $z \in (a,b)$. If*

1. $g'(x) \neq 0$ *for* $x \in (a,b)$,
2. $f(z) = g(z) = 0$, *and*
3. $\lim_{x\to z} f'/g'$ *exists*

Then $\lim_{x\to z} f(x)/g(x)$ *exists and* $\lim_{x\to z} f(x)/g(x) = \lim_{x\to z} f'(x)/g'(x)$.

Proof. Let $L = \lim_{x\to z} f'(x)/g'(x)$. Suppose that $\langle x_n \rangle$ is any sequence in $(a,b) - \{z\}$ with $\lim x_n = z$. By Theorem 4.8.1 there is a sequence $\langle y_n \rangle$ so that for every n, y_n is strictly between x_n and z and

$$f'(y_n)(g(x_n) - g(z)) = g'(y_n)(f(x_n) - f(z)).$$

Notice that by the Squeeze Theorem $\lim y_n = z$. Also notice that since g' is never 0, by Theorem 4.7.1, g is injective. Since $x_n \neq z$ for all n, this means that $g(x_n) - g(z) \neq 0$, so we can divide by this expression to get

$$\frac{f'(y_n)}{g'(y_n)} = \frac{f(x_n) - f(z)}{g(x_n) - g(z)}.$$

Since $f(z) = g(z) = 0$, this becomes

$$\frac{f'(y_n)}{g'(y_n)} = \frac{f(x_n) - f(z)}{g(x_n) - g(z)} = \frac{f(x_n)}{g(x_n)}.$$

Since $\lim\limits_{x \to z} f'(x)/g'(x) = L$ and since $\lim y_n = z$, then $\lim \dfrac{f'(y_n)}{g'(y_n)} = L$. It follows now that

$$\lim \frac{f(x_n)}{g(x_n)} = \lim \frac{f'(x)}{g'(x)} = L.$$

Since this is true for all sequences $\langle x_n \rangle$ approaching z in $(a, b) - \{z\}$, then

$$\lim_{x \to z} \frac{f(x)}{g(x)} = L = \lim_{x \to z} \frac{f'(x)}{g'(x)}.$$

□

Exercises 4.8

4.8.1 Prove the Extended Mean Value Theorem 4.8.1.

4.8.2 In this exercise, we will prove a simplified version of l'Hôpital's Rule. Suppose that $z \in D \subseteq \mathbb{R}$ is an accumulation point of D and that $f, g : D \to \mathbb{R}$ are functions so that:

1. f and g are differentiable at z,
2. $g(x) \neq 0$ for $x \neq z$ in D,
3. $f(z) = g(z) = 0$, and
4. $g'(z) \neq 0$.

Then $\lim\limits_{x \to z} \dfrac{f(x)}{g(x)}$ exists and is equal to $\dfrac{f'(z)}{g'(z)}$.

4.8.3 Find the limit:

1. $\lim\limits_{x \to 1} \dfrac{\ln x}{x - 1}$

2. $\lim_{x \to 0} \dfrac{x}{e^x - 1}$

3. $\lim_{x \to 0} \dfrac{\sin(x)}{x}$

4. $\lim_{x \to 0} \dfrac{x^2 \sin(x)}{\sin(x) - x\cos(x)}$

Chapter 5

Integration

In this chapter, we develop the integral fully enough to prove the Fundamental Theorem of Calculus. For most discussions about the integral, it is best to picture either the process of approximating the area under a non-negative function or the process of approximating the distance traveled on a number line by an object with non-negative velocity.

5.1 Area

Suppose that $f : [a, b] \to \mathbb{R}$ is never negative and that we would like to approximate the area under the curve $y = f(x)$ over the interval $[a, b]$. We begin by *partitioning* the interval $[a, b]$ into smaller *subintervals*. This is accomplished by selecting points

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b$$

in the interval $[a, b]$. On each of the intervals $[x_{i-1}, x_i]$, we will draw a rectangle from the x-axis up to the curve. The height of this rectangle should be determined by the height of the curve $y = f(x)$. For each $i = 1, 2, \ldots, n$, select $c_i \in [x_{i-1}, x_i]$. Then we can draw a rectangle on the interval $[x_{i-1}, x_i]$ with height $f(c_i)$. The width of this rectangle is $\Delta x_i = x_i - x_{i-1}$. The area of this rectangle is $f(c_i)\Delta x_i$. The total area under the curve $y = f(x)$ can then be approximated by adding up the areas of all of these rectangles:

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

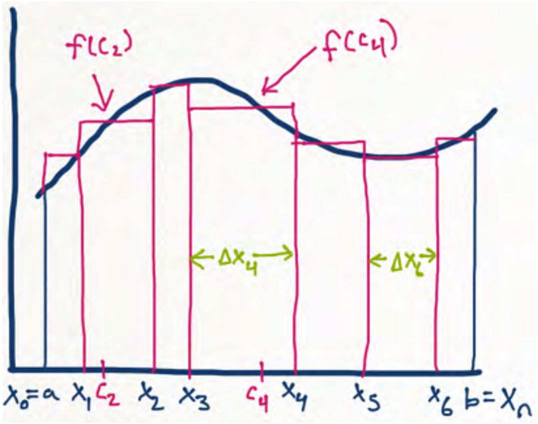

Figure 5.1: This figure depicts an attempt to approximate the area under a curve $y = f(x)$ using rectangles. The points $a = x_0 < x_1 < \cdots < x_n = b$ partition the interval $[a, b]$. Points $c_1, c_2, \ldots, c_n$ are selected in each subinterval. The points c_2 and c_4 are displayed. On each interval $[x_{i-1}, x_i]$, a rectangle is drawn with height $f(c_i)$. The width of this rectangle is $\Delta x_i = x_i - x_{i-1}$. The widths Δx_4 and Δx_6 are displayed.

Whether or not this is a good approximation of the area depends on properties of the function $f(x)$ and the partition.

Example 5.1.1. We illustrate this process with $f : [-1, 1] \to \mathbb{R}$ given by $f(x) = 1 - x^2$. Let n be a positive integer. We will partition $[-1, 1]$ into n intervals of the same width. Since the width of our interval is $1-(-1) = 2$, the width of each smaller interval is $\Delta x = 2/n$. We select points $x_0 = -1 < x_1 < x_2 < \cdots < x_n = 1$ which divide $[-1, 1]$ into n subintervals of width Δx. This requires that for $i = 0, 1, \ldots, n$ we should have $x_i = -1 + i\Delta x = -1 + 2i/n$. To simplify notation here, we will choose c_i in $[x_{i-1}, x_i]$ to be the right hand endpoint x_i. The rectangle we draw on $[x_{i-1}, x_i]$ has width $\Delta x = 2/n$ and height

$$f(x_i) = f\left(-1 + \frac{2i}{n}\right) = 1 - \left(-1 + \frac{2i}{n}\right)^2 = \frac{4i}{n} - \frac{4i^2}{n^2}.$$

The area of this rectangle is

$$f(x_i)\Delta x = \left(\frac{4i}{n} - \frac{4i^2}{n^2}\right)\frac{2}{n} = \frac{8i}{n^2} - \frac{8i^2}{n^3}.$$

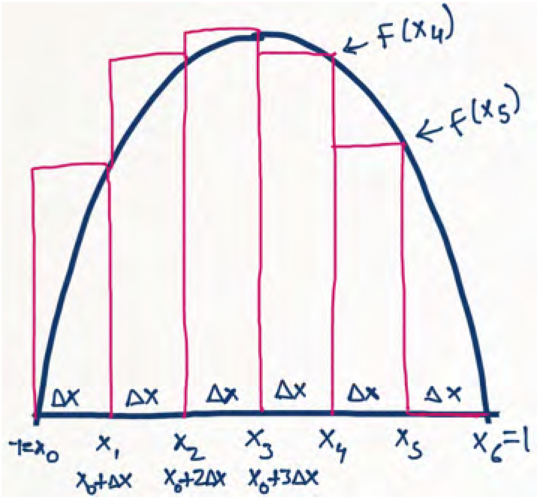

Figure 5.2: This figure depicts an arrangement of six rectangles as described in Example 5.1.1. In this case, we use a regular partition with all rectangles of equal width Δx, and rectangles are drawn to the height of the function at the right hand endpoints of the intervals.

The total area under $y = f(x)$ over $[-1, 1]$ is approximated by the sum of the areas of these rectangles:

$$\begin{aligned}\text{Area} &\approx \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \sum_{i=1}^{n} \left(\frac{8i}{n^2} - \frac{8i^2}{n^3} \right) \\ &= \frac{8}{n^2} \sum_{i=1}^{n} i - \frac{8}{n^3} \sum_{i=1}^{n} i^2.\end{aligned}$$

Using the familiar formulas for summations

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

this becomes

$$\begin{aligned}\text{Area} &\approx \frac{8}{n^2} \cdot \frac{n(n+1)}{2} - \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= \frac{4(n+1)}{n} - \frac{4(n+1)(2n+1)}{3n^2}.\end{aligned}$$

This gives us a sequence of approximations to the area under $y = f(x)$ using n rectangles for each $n \in \mathbb{N}$. Presumably, this approximation is better if we use more, narrower rectangles. Perhaps

$$\text{Area} = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \left(\frac{4(n+1)}{n} - \frac{4(n+1)(2n+1)}{3n^2} \right) = \frac{4}{3}.$$

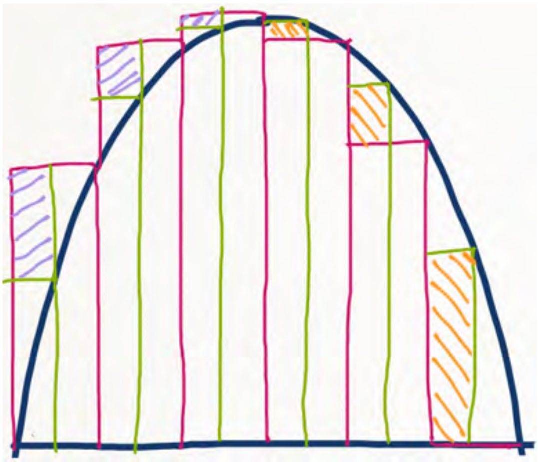

Figure 5.3: It appears that as we use more and more rectangles, an approximation at area using a regular partition improves. In this figure, we have used twice as many rectangles to approximate area as in Figure 5.2. The shaded regions indicate improvement in the approximation. In the left half of the picture, the shaded regions were over-approximations before. In the right half of the picture, the shaded regions indicate what were under-approxiations.

There are at least two potential flaws with the process we just followed. First, we partitioned the interval $[-1, 1]$ in a very precise way so that all subintervals were the same width. This is called a *regular* partition. It could be that this choice of partition affected the limiting process in some way. Second, we selected values in the subintervals to be the right hand endpoints. This might also have affected the limiting process. Perhaps we would have found a different result with left hand endpoints or midpoints or the highest or lowest points on the interval. The formalities we develop below must avoid all such potential errors.

5.2 Distance

Suppose that an object is moving on the y-axis and that its velocity is given by a function v. Suppose also that $v(t) \geq 0$ for all times t in an interval $[a, b]$. We would like to approximate the distance traveled by the object from time $t = a$ to time $t = b$. As before, we begin by partitioning the time interval $[a, b]$ into smaller subintervals by selecting points $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ in $[a, b]$. For each i, let $\Delta t_i = t_i - t_{i-1}$ and select $c_i \in [t_{i-1}, t_i]$. We will approximate the distance traveled by the object over each small interval $[t_{i-1}, t_i]$ and then add these smaller approximations together. Now, on an interval $[t_{i-1}, t_i]$, the velocity of the moving object cannot change too much (velocity is continuous), so since $c_i \in [t_{i-1}, t_i]$, then $v(t) \approx v(c_i)$ on the entire interval $[t_{i-1}, t_i]$. This means that the distance traveled from $t = t_{i-1}$ to $t = t_i$ is approximately $v(c_i)\Delta t_i$ (distance is rate times time). The total distance is then approximated by summing over all of the intervals:

$$\text{Distance} \approx \sum_{i=1}^{n} v(c_i)\Delta t_i.$$

Notice the similarity between this sum and the sum we found considering area in Section 5.1.

5.3 Riemann Sums

The sums encountered in the previous discussions are called *Riemann Sums*. We can consider such sums for any function $f : [a, b] \to \mathbb{R}$ without consideration of the interpretation of f and without consideration of whether or not f is positive or negative. When we consider such sums with narrower and narrower subintervals, then the values of the sums might tend to get closer to each other. This should surely happen in the case of the velocity function since there should be less variation in velocity over smaller and smaller time intervals. Let the *mesh* of a partition $x_0 = a < x_1 < x_2 < \cdots < x_n = b$ be the maximum value of $\Delta x_i = x_i - x_{i-1}$. We can consider a limit of Riemann Sums of the form:

$$\lim_{\text{mesh}\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

taken over all partitions of $[a, b]$, over all ways of selecting the c_i's, as the mesh of the partitions gets smaller and smaller. When such a limit exists, we call this limit the *Riemann Integral* of f over $[a, b]$ and denote it as $\int_a^b f$. As indicated above, this value has meaning for applications related to area and distance (and a host of other applications).

To simplify the limit process in the description of the Riemann Integral, Darboux suggested replacing each $f(c_i)$ with either

$$\sup\{f(c_i) : c_i \in [x_{i-1}, x_i]\}$$

(which corresponds in some sense to drawing rectangles as *tall* as possible) or

$$\inf\{f(c_i) : c_i \in [x_{i-1}, x_i]\}$$

(which corresponds in some sense to drawing rectangles as *short* as possible). All Riemann Sums for a given partition will fall between

$$\sum_{i=1}^{n} \sup\{f(c_i) : c_i \in [x_{i-1}, x_i]\}\Delta x_i$$

and

$$\sum_{i=1}^{n} \inf\{f(c_i) : c_i \in [x_{i-1}, x_i]\}\Delta x_i.$$

Using these types of sums to simplify the limiting process will lead us to the *Darboux Integral*. This notion of the integral is equivalent to the Riemann Integral.

5.4 The Fundamental Theorems and Why They are Obvious

There are (at least) two theorems called the Fundamental Theorem of Calculus. They are listed below as Theorems 5.9.4 and 5.9.6. The first roughly states that to calculate an integral of a derivative of a function f, we simply evaluate f at the endpoints of the interval and subtract:

$$\int_a^b f' = f(b) - f(a).$$

Suppose that $f(t)$ gives the position of an object moving on a number line at time t. Then the derivative f' gives the velocity of the object.

If $f' \geq 0$, then the discussion above indicates that $\int_a^b f'$ is the distance traveled by the object. Clearly, one way to calculate the distance traveled by an object is to take its final position $f(b)$ and subtract its initial position $f(a)$.

The other Fundamental Theorem considers a function of the form

$$F(x) = \int_a^x f'.$$

The theorem says that F is continuous, and if f' is continuous then F is differentiable with $F' = f'$. Again, if we interpret f as position and f' as velocity, then F is distance traveled. Surely distance traveled should be continuous. Moreover, the derivative of distance traveled with respect to time is velocity, so F' should be the same as f'.

This may all be well and good, but interpretations can be misleading. We begin now the process of formally defining the integral and proving the Fundamental Theorems of Calculus in a manner that is independent of any interpretation as area or position or distance.

5.5 Partitions and Sums

Here we make formal the idea of a partition and the upper and lower sums mentioned above. Notice that since we will be employing suprema and infima, we insist that our functions are bounded. This is the only restriction on the functions.

Definition 5.5.1. A *partition* of a closed interval $[a, b]$ is a finite subset P of $[a, b]$ which includes a and b. We will usually number the elements of P in an increasing manner so that $P = \{x_0, x_1, \ldots, x_n\}$ where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

The *mesh* of this partition P is

$$|P| = \max\{(x_i - x_{i-1}) : i = 1, 2, \ldots, n\}.$$

If $f : [a, b] \to \mathbb{R}$ is a bounded function, then the *upper sum* of f for partition P is

$$U(f, P) = \sum_{i=1}^{n} \sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}).$$

The *lower sum* of f for partition P is

$$L(f,P) = \sum_{i=1}^{n} \inf\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}).$$

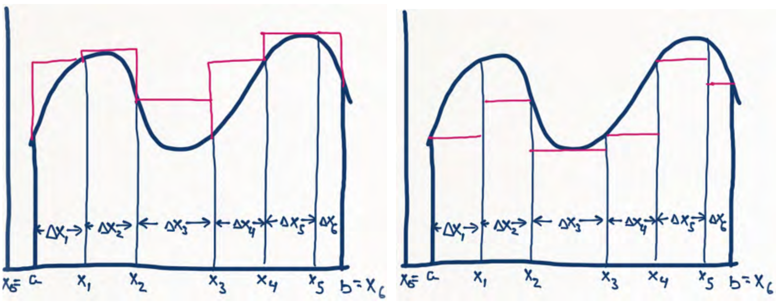

Figure 5.4: Upper (left) and lower (right) sums for the same function with the same partition.

The notation in this definition is a bit cumbersome. We will usually make the following notational conventions for a bounded function $f : [a,b] \to \mathbb{R}$ and a partition $P = \{x_0, x_1, \ldots, x_n\}$ where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

First, we will abuse notation in defining partitions. We can denote this partition P as

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}.$$

Next, we let $\Delta x_i = x_i - x_{i-1}$. Finally, for any $i = 1, 2, \ldots, n$, let

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} \text{ and } M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

We will refer to this notation as the *standard notation* for partitions. With the standard notation,

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i \text{ and } U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i.$$

Example 5.5.2. Suppose $f : [-1,1] \to \mathbb{R}$ is given by $f(x) = 1 - x^2$. Let

$$P = \{-1, -1/2, 0, 1/4, 1/2, 3/4, 1\}.$$

This table summarizes a computation of $L(f,P)$ and $U(f,P)$:

i	x_i	Δx_i	m_i	M_i	$m_i\Delta x_i$	$M_i\Delta x_i$
0	-1					
1	$-1/2$	$1/2$	0	$3/4$	0	$3/8$
2	0	$1/2$	$3/4$	1	$3/8$	$1/2$
3	$1/4$	$1/4$	$15/16$	1	$15/64$	$1/4$
4	$1/2$	$1/4$	$3/4$	$15/16$	$3/16$	$15/64$
5	$3/4$	$1/4$	$7/16$	$3/4$	$7/64$	$3/16$
6	1	$1/4$	0	$7/16$	0	$7/64$
					$L(f,P) = 58/64$	$U(f,P) = 106/64$

Notice that the mesh of P is $|P| = 1/2$ and that $L(f,P) < U(f,P)$. This computation does not do much for us in the way of approximating an area. The area under $y = 1-x^2$ over $[-1,1]$ appears to be somewhere between $58/64$ and $106/64$. For a better approximation, we either need to use a partition with a finer mesh, or we need to use a more general partition.

Example 5.5.3. Suppose that $c \in \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ is given by $f(x) = c$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be any partition of $[a,b]$. Notice that on each interval $[x_{i-1}, x_i]$ the maximum and minimum values of f are both c, so $m_i = M_i = c$ for all i. Then $L(f,P)$ is

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} m_i \Delta x_i \\ &= \sum_{i=1}^{n} c(x_i - x_{i-1}) \\ &= c\sum_{i=1}^{n} (x_i - x_{i-1}) \\ &= c(x_1 - x_0 + x_2 - x_1 + x_3 - x_2 + \cdots + x_n - x_{n-1}) \\ &= c(x_n - x_0) \\ &= c(b-a). \end{aligned}$$

Notice how the sum here "telescopes" down to $(b-a)$. The arithmetic for $U(f,P)$ works identically since each $m_i = M_i$. For this function, every lower sum and every upper sum is $c(b-a)$.

Example 5.5.4. Suppose that $f : [2,5] \to \mathbb{R}$ is given by $f(x) = x$. Let $P = \{2 = x_0 < x_1 < \cdots < x_n = 5\}$ be any parition of $[2,5]$. Notice that since f is increasing, on each interval $[x_i, x_{i-1}]$ the minimum value is $m_i = f(x_{i-1}) = x_{i-1}$ and the maximum value is $M_i = f(x_i) = x_i$. Then

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1})$$

and

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} x_i(x_i - x_{i-1}).$$

These expressions do not mean much as they are. With a little more information later, we will be able to combine these sums into a summation that also telescopes.

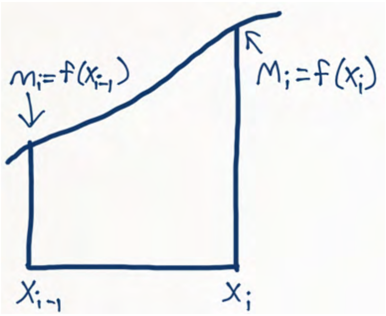

Figure 5.5: If f is increasing on $[x_{i-1}, x_i]$, then the maximum value of f occurs at x_i and the minimum value occurs at x_{i-1}.

Example 5.5.5. Suppose that $f : [0,1] \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

Let $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be any partition of $[0,1]$. Since every interval $[x_{i-1}, x_i]$ contains a rational number, each M_i is 1. Since every interval $[x_{i-1}, x_i]$ also contains an irrational number, each m_i is 0. It follows that

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 \Delta x_i = 0$$

and

$$U(f,P)=\sum_{i=1}^{n} M_i\Delta x_i=\sum_{i=1}^{n}\Delta x_i=1.$$

Notice that this last summation again telescopes down to the width of the interval $[0,1]$. For any partition P then,

$$L(f,P)=0<1=U(f,P).$$

Example 5.5.6. Suppose that $f,g:[2,5]\to\mathbb{R}$ are given by $f(x)=3x^2$ and $g(x)=x^3$. Note that $g'(x)=f(x)$. We will address upper and lower sums for a partition $P=\{2=x_0<x_1<\cdots<x_n=5\}$ of $[2,5]$. Using the standard notation,

$$L(f,P)=\sum_{i=1}^{n} m_i\Delta x_i \text{ and } U(f,P)=\sum_{i=1}^{n} M_i\Delta x_i.$$

If we apply the Mean Value Theorem to g on $[x_i,x_{i-1}]$, then there is a $t_i\in[x_i,x_{i-1}]$ so that

$$f(t_i)=g'(t_i)=\frac{g(x_i)-g(x_{i-1})}{x_i-x_{i-1}}.$$

It follows that $g(x_i)-g(x_{i-1})=f(t_i)(x_i-x_{i-1})=f(t_i)\Delta x_i$. Then

$$\sum_{i=1}^{n} f(t_i)\Delta x_i=\sum_{i=1}^{n}(g(x_i)-g(x_{i-1}))=$$

$$=g(x_n)-g(x_0)=g(5)-g(2)=125-8=117.$$

(Notice the telescoping.) Moreover, since $m_i\le f(t_i)\le M_i$ for all i, we know that $L(f,P)\le 117\le U(f,P)$. This holds for all partitions P.

We now need to establish some order theorems about upper and lower sums which will allow us to define the integral.

Lemma 5.5.7. *Suppose that $f:[a,b]\to\mathbb{R}$ is a bounded function and P is a partition of $[a,b]$. Further suppose that $M\in\mathbb{R}$ so that $|f(x)|\le M$ for all $x\in[a,b]$. Then*

$$-M(b-a)\le L(f,P)\le U(f,P)\le M(b-a).$$

Proof. Suppose that $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ and use the standard notation for partitions. Note that

$$\sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} (x_i - x_{i-1}) = (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}).$$

This summation "telescopes" and collapses to $x_n - x_0 = b - a$. Note also that $-M \leq m_i \leq M_i \leq M$ for each i. Now

$$-M(b-a) = -M \sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} -M\Delta x_i \leq \sum_{i=1}^{n} m_i \Delta x_i = L(f, P)$$

and

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i = U(f, P)$$

and

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i \leq \sum_{i=1}^{n} M \Delta x_i = M \sum_{i=1}^{n} \Delta x_i = M(b-a).$$

□

As a consequence of this lemma, the sets of upper and lower sums for a given function are bounded. This will allow us to apply the Completeness Axiom to these sets to define upper and lower integrals later. The lemma also tells us that the lower sum for a particular partition is less than or equal to the upper sum for that partition. What might be more surprising is that every lower sum is less than or equal to every upper sum. To prove this, we need the following lemma. This result tells us that if we add points to a partition, then the lower sum may only increase, and the upper sum may only decrease. Thus, when partitions are *refined* by adding points, the upper and lower sums move closer toward each other (if they move at all).

Lemma 5.5.8. (Partition Refinement Lemma) *If $P \subseteq Q$ are partitions of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is bounded then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Proof. Suppose that $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Since partitions are by definition finite, Q can be constructed by adding one point at a time to P. We will prove that if we add one point to P, then the lower sum may increase and the upper sum may decrease. The result will then follow by induction. Suppose that $z \in [a, b] - P$. Then there is some j so that $x_{j-1} < z < x_j$. Let $P' = P \cup \{z\}$. Use the standard notation for partitions in reference to P. We will calculate $U(f, P) - U(f, P')$ and see that this difference is non-negative. It will follow that $U(f, P) \geq U(f, P')$. The situation described in this proof is displayed in Figure 5.6.

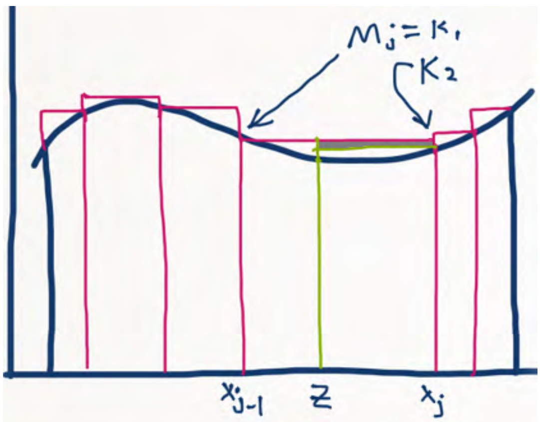

Figure 5.6: This figure depicts what is going on in the proof of Lemma 5.5.8. Outside of the interval $[x_{j-1}, x_j]$ the partitions P and P' are identical. These regions contribute nothing to the difference $U(f, P) - U(f, P')$. Within the interval $[x_{j-1}, x_j]$, P' has one additional point z. Note that K_1 (the supremum on the left) and K_2 (the supremum on the right) are both less than or equal to M_j. The difference $U(f, P) - U(f, P')$ is the thin shaded region.

Define

$$K_1 = \sup\{f(x) : x \in [x_{j-1}, z]\} \text{ and } K_2 = \sup\{f(x) : x \in [z, x_j]\}.$$

Since we have $[x_{j-1}, z] \subseteq [x_{j-1}, x_j]$ and $[z, x_j] \subseteq [x_{j-1}, x_j]$, it follows that $K_1 \leq M_j$ and $K_2 \leq M_j$. If we separate out the interval $[x_{j-1}, x_j]$ in the summations for $U(f, P)$ and $U(f, P')$ then we get

$$U(f, P) = \sum_{i=1}^{j-1} M_i \Delta x_i + M_j(x_j - x_{j-1}) + \sum_{i=j+1}^{n} M_i \Delta x_i$$

and

$$U(f,P') = \sum_{i=1}^{j-1} M_i \Delta x_i + K_1(z - x_{j-1}) + K_2(x_j - z) + \sum_{i=j+1}^{n} M_i \Delta x_i.$$

Then

$$\begin{aligned} U(f,P) - U(f,P') &= M_j(x_j - x_{j-1}) - [K_1(z - x_{j-1}) + K_2(x_j - z)] \\ &= M_j(x_j - x_{j-1}) - K_1(z - x_{j-1}) - K_2(x_j - z) \\ &\geq M_j(x_j - x_{j-1}) - M_j(z - x_{j-1}) - M_j(x_j - z) \\ &= M_j(x_j - x_{j-1} - z + x_{j-1} - x_j + z) \\ &= 0. \end{aligned}$$

Since $U(f,P) - U(f,P') \geq 0$, we know that $U(f,P) \geq U(f,P')$. A similar argument with infima will show that $L(f,P) \leq L(f,P')$. We have now shown that if we add one point to P to create P', then

$$L(f,P) \leq L(f,P') \leq U(f,P') \leq U(f,P).$$

Thus adding one point to a partition moves the lower and upper sums "toward the middle." We can now construct Q from P by adding finitely many points one at a time. Applying what we have shown to each step (by induction) gives the desired result. □

A consequence of this refinement lemma is that every lower sum is less than or equal to every upper sum.

Theorem 5.5.9. *Suppose that P and Q are partitions of $[a,b]$ and that $f : [a,b] \to \mathbb{R}$ is bounded. Then $L(f,P) \leq U(f,Q)$.*

Proof. Note that $P \cup Q$ is also a partition of $[a,b]$. Since $P \subseteq P \cup Q$, Lemma 5.5.8 gives that $L(f,P) \leq L(f,P \cup Q)$. Since $Q \subseteq P \cup Q$, Lemma 5.5.8 also gives $U(f,P \cup Q) \leq U(f,Q)$. Putting these together gives

$$L(f,P) \leq L(f,P \cup Q) \leq U(f,P \cup Q) \leq U(f,Q).$$

□

5.6 Integrals

By Lemma 5.5.8, we know that when points are added to a partition, lower sums increase. Keeping in mind the limiting process discussed for Riemann Sums earlier, we might like to know if these sums increase toward a particular limit as partitions are refined. They do, and this limit is merely the least upper bound of all lower sums. This least upper bound exists because Lemma 5.5.7 tells us that the set of all lower sums of a bounded function on a closed interval is bounded. Similarly, the set of all upper sums of a fixed function has a greatest lower bound.

Definition 5.6.1. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. The *upper integral* of f over $[a, b]$ is

$$\overline{\int_a^b} f = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

The *lower integral* of f over $[a, b]$ is

$$\underline{\int_a^b} f = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

Theorem 5.6.2. *Suppose that P and Q are partitions of $[a, b]$ and that $f : [a, b] \to \mathbb{R}$ is bounded. Let $M \in \mathbb{R}$ so that $|f(x)| \le M$ for all $x \in [a, b]$. Then*

$$-M(b-a) \le L(f, P) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f, Q) \le M(b-a).$$

Proof. Let A and B be these sets:

$$A = \{L(f, R) : R \text{ is a partition of } [a, b]\}$$

and

$$B = \{U(f, R) : R \text{ is a partition of } [a, b]\}.$$

If R' is any partition of $[a, b]$, then we know that $L(f, R) \le U(f, R')$ for all partitions R of $[a, b]$. This means that $U(f, R')$ is an upper bound of A, so $U(f, R') \ge \sup A = \underline{\int_a^b} f$. Thus, for all partitions R' of $[a, b]$, we have $\underline{\int_a^b} f \le U(f, R')$. This means that $\underline{\int_a^b} f$ is a lower bound of B, so $\underline{\int_a^b} f \le \inf B = \overline{\int_a^b} f$. The result now follows from Lemma 5.5.7. $\square$

The lower integral can be considered an attempt at calculating the area under a function looking only at the shortest possible rectangles that follow the function. The upper integral is comparable to working with the tallest rectangles that follow the function. If these two approaches do not agree, then we would have no good idea for what the area under a function (or the integral) should be. To avoid this dilemma, we isolate our attention to those functions for which the upper and lower integrals agree.

Definition 5.6.3. Suppose that $f : [a,b] \to \mathbb{R}$ is bounded. If

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

then f is *integrable* on $[a,b]$. The *integral* of f over $[a,b]$ is

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f.$$

Example 5.6.4. Recall that in Example 5.5.3 we considered a constant function $f(x) = c$ on a closed interval $[a,b]$. We saw in that example that $L(f,P) = U(f,P) = c(b-a)$ for any partition P of $[a,b]$. It follows that $\underline{\int_a^b} f = \overline{\int_a^b} f = c(b-a)$, so f is integrable on $[a,b]$ and $\int_a^b f = c(b-a)$.

Example 5.6.5. Recall that in Example 5.5.5 we considered the function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

on the interval $[0,1]$. We saw there that for any partition P of $[0,1]$, the lower sum was $L(f,P) = 0$ and the upper sum was $U(f,P) = 1$. It follows that $\underline{\int_a^b} f = 0 < 1 = \overline{\int_a^b} f$. Therefore, f is not integrable on $[0,1]$.

Example 5.6.6. In Example 5.5.6 we considered $f(x) = 3x^2$ on the interval $[2,5]$. We saw there that $L(f,P) \le 117 \le U(f,P)$ for any partition P. It follows that $\underline{\int_2^5} f \le 117 \le \overline{\int_2^5} f$. We still do not know if this function is integrable, but if it is, the value of the integral must be 117.

Exercises 5.6

5.6.1 Mimic Example 5.1.1 with the function $f(x) = x^3$ on $[0, 1]$.
5.6.2 Mimic Example 5.1.1 with the function $f(x) = x + x^2$ on $[1, 4]$.
5.6.3 Let P be a regular partition of $[2, 5]$ into n subintervals. Let $f(x) = x$. Calculate the difference $U(f, P) - L(f, P)$. Refer to Example 5.5.4.
5.6.4 Repeat the previous exercise on the interval $[0, 1]$ with the function from Example 5.5.5.
5.6.5 Mimic the procedure in Example 5.5.6 with $f(x) = 2x$ on the interval $[0, 1]$.
5.6.6 Give an example of a function $f : [0, 1] \to \mathbb{R}$ which is not integrable on $[0, 1]$ so that $|f|$ is integrable on $[0, 1]$.

5.7 Conditions for Integrability

It is rare that we can tell directly from sums whether or not a function is integrable as easily as in Example 5.6.4. Our next order of business to to develop a general method of telling when a function might be integrable.

Theorem 5.7.1. (The ϵ-Partition Integrability Condition) *A bounded function $f : [a, b] \to \mathbb{R}$ is integrable if and only if for every $\epsilon > 0$ there is a partition P of $[a, b]$ so that $U(f, P) - L(f, P) < \epsilon$.*

Proof. Suppose first that f is integrable. Then $\underline{\int_a^b} f = \overline{\int_a^b} f$. Let $\epsilon > 0$. Since $\underline{\int_a^b} f$ is the supremum of all lower sums, there is a partition P_1 so that $\underline{\int_a^b} f - L(f, P_1) < \epsilon/2$. Similarly, since $\overline{\int_a^b} f$ is the infimum of all upper sums, there is a partition P_2 so that $U(f, P_2) - \overline{\int_a^b} < \epsilon/2$. Let $P = P_1 \cup P_2$. Then by Lemma 5.5.8

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2).$$

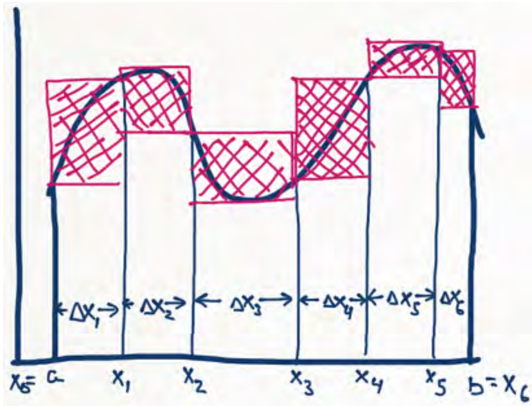

Figure 5.7: Regardless of whether or not f is positive, the difference $U(f,P) - L(f,P)$ is an area as depicted in this figure. Theorem 5.7.1 declares that f is integrable if and only if this area can be made smaller than any positive ϵ.

It follows that

$$
\begin{aligned}
U(f,P) - L(f,P) &\le U(f,P_2) - L(f,P_1) \\
&= U(f,P_2) - \int_a^b f + \int_a^b f - L(f,P_1) \\
&= U(f,P_2) - \overline{\int_a^b} f + \underline{\int_a^b} f - L(f,P_1) \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}
$$

Now suppose that such a partition exists for every positive ϵ. We will show that f is integrable. To do so, we must show that the lower and upper integrals of f are equal. We will accomplish this by showing that their difference is less than every positive real number. Let $\epsilon > 0$. There is a partition P so that $U(f,P) - L(f,P) < \epsilon$. Since

$$
L(f,P) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f,P)
$$

it follows that $0 \le \overline{\int_a^b} f - \underline{\int_a^b} f < \epsilon$. Since this is true for all $\epsilon > 0$, we have that $\overline{\int_a^b} f = \underline{\int_a^b} f$ so f is integrable. $\square$

Example 5.7.2. In Example 5.5.4 we considered the function $f(x) = x$ on the interval $[2, 5]$. We will show that f is integrable on $[2, 5]$. Let $\epsilon > 0$. Suppose that P is a partition of $[2, 5]$ with $|P| < \epsilon/3$. Using the standard notation, we saw in Example 5.5.4 that

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} x_{i-1} \Delta x_i$$

and

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} x_i \Delta x_i.$$

Therefore

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} (x_i - x_{i-1}) \Delta x_i \\
&< \sum_{i=1}^{n} (x_i - x_{i-1})(\epsilon/3) \\
&= (\epsilon/3) \sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= (\epsilon/3)(x_n - x_0) \\
&= (\epsilon/3)3 \\
&= \epsilon.
\end{aligned}$$

By the ϵ-partition integrability condition, f is integrable on $[2, 5]$.

The ϵ-partition integrability condition allows us to see when some functions are (or are not) integrable. It does not do much good for finding the actual value of the integral. This is taken care of by the next theorem.

Theorem 5.7.3. *Suppose that $f : [a, b] \to \mathbb{R}$ is integrable and that $R \in \mathbb{R}$ so that $L(f, P) \leq R \leq U(f, P)$ for all partitions P of $[a, b]$. Then $\int_a^b f = R$.*

Proof. Suppose that f is integrable and that such an R exists. We will show that $\int_a^b f = R$ by showing that $\left|\int_a^b f - R\right|$ is less than every positive real number. Suppose that $\epsilon > 0$. There is a partition P so that $U(f,P) - L(f,P) < \epsilon$. Since $L(f,P) \leq R \leq U(f,P)$ and $L(f,P) \leq \int_a^b f \leq U(f,P)$, it follows that $\left|\int_a^b f - R\right| < \epsilon$. Since this is true for all ϵ, $\int_a^b f = R$. □

The proofs of the previous two theorems can be combined to prove:

Theorem 5.7.4. *Suppose that $f : [a,b] \to \mathbb{R}$ is bounded and that $R \in \mathbb{R}$. If for every $\epsilon > 0$ there is a partition P of $[a,b]$ so that $U(f,P) - L(f,P) < \epsilon$ and $L(f,P) \leq R \leq U(f,P)$, then f is integrable on $[a,b]$ and $\int_a^b f = R$.* □

Example 5.7.5. In Example 5.5.6 we considered $f(x) = 3x^2$ on the interval $[2,5]$. We saw there that for any partition P, the number 117 was between $L(f,P)$ and $U(f,P)$. If we can show that f is integrable, then this will be the value of the integral. Let $\epsilon > 0$. Let P be a partition of $[2,5]$ with $|P| < \epsilon/63$. We will use the standard notation for partitions. Note that since f is increasing $m_i = f(x_{i-1})$ and $M_i = f(x_i)$ for each i. This means that

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=1}^{n}(M_i - m_i)\Delta x_i \\
&= \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))\Delta x_i \\
&< \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))\epsilon/63 \\
&= (\epsilon/63)\sum_{i=1}^{n}(f(x_i) - f(x_{i-1})) \\
&= (\epsilon/63)(f(5) - f(2)) \\
&= (\epsilon/63)63 \\
&= \epsilon.
\end{aligned}$$

Thus, by the ϵ-partition integrability condition, f is integrable on $[2,5]$. Since $L(f,P) \le 117 \le U(f,P)$ for all partitions P (as shown in Example 5.5.6), $\int_2^5 f = 117$.

The ϵ-partition integrability condition allows us quickly to conclude that monotonic and continuous functions are integrable.

Theorem 5.7.6. *If $f : [a,b] \to \mathbb{R}$ is bounded and monotonic, then f is integrable.*

Proof. Assume that f is increasing and bounded on $[a,b]$. The case when f is decreasing is left as an exercise. Let $\epsilon > 0$. If $f(b) = f(a)$, then f is constant and, hence, integrable by Example 5.6.4. Suppose then that $f(b) \neq f(a)$. Let $\delta = \epsilon/(f(b) - f(a))$. Suppose that

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

is any partition of $[a,b]$ with $|P| < \delta$. Using the standard partition notation, note that $M_i = f(x_i)$ and $m_i = f(x_{i-1})$ because f is increasing. Then

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i \\ &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\ &< \sum_{i=1}^{n} (M_i - m_i)\delta \\ &= \delta \sum_{i=1}^{n} (M_i - m_i) \\ &= \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &= \delta(f(x_n) - f(x_0)) \\ &= \delta(f(b) - f(a)) \\ &= \epsilon. \end{aligned}$$

By the ϵ-partition integrability condition, f is integrable. $\square$

Theorem 5.7.7. *If $f : [a,b] \to \mathbb{R}$ is continuous, then f is integrable.*

Proof. Let $\epsilon > 0$. Since f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$. This means that there is a δ so that if $x, y \in [a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon/(b - a)$. Suppose that

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

is any partition of $[a, b]$ with $|P| < \delta$. We will use the standard partition notation. For each i, since f is continuous on $[x_{i-1}, x_i]$, the Extreme Value Theorem guarantees the existence of $u_i, v_i \in [x_{i-1}, x_i]$ with $f(u_i) = m_i$ and $f(v_i) = M_i$. Note also that since $x_i - x_{i-1} < \delta$, then

$$M_i - m_i = f(v_i) - f(u_i) < \epsilon/(b - a).$$

Then

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i \\
&= \sum_{i=1}^{n} (M_i - m_i) \Delta x_i \\
&= \sum_{i=1}^{n} (f(v_i) - f(u_i)) \Delta x_i \\
&< \sum_{i=1}^{n} (\epsilon/(b - a)) \Delta x_i \\
&= (\epsilon/(b - a)) \sum_{i=1}^{n} \Delta x_i \\
&= (\epsilon/(b - a))(b - a) \\
&= \epsilon.
\end{aligned}$$

By the ϵ-partition integrability condition, f is integrable. □

We close this section with results that allow us to "take apart" and "piece together" integrable functions to create more integrable functions. The first result shows that integrability is preserved when intervals are restricted.

Theorem 5.7.8. *Suppose that $f : [a, b] \to \mathbb{R}$ is integrable and*

$$a \le u < v \le b.$$

Then f is integrable on $[u, v]$.

Proof. Let $\epsilon > 0$. Since f is integrable on $[a, b]$, there is a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ so that

$$U(f, P) - L(f, P) < \epsilon.$$

By Lemma 5.5.8, we can assume that $u, v \in P$ so that there are j and k so that $u = x_j$ and $v = x_k$. We can consider $Q = \{x_j, x_{j+1}, \ldots, x_k\}$ as a partition of $[u, v]$. Then

$$\begin{aligned} U(f, Q) - L(f, Q) &= \sum_{i=j+1}^{k} (M_i - m_i)\Delta x_i \\ &\leq \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\ &= U(f, P) - L(f, P) \\ &< \epsilon. \end{aligned}$$

By the ϵ-partition integrability condition, f is integrable on $[u, v]$. □

This next result allows us to piece together functions which are integrable on adjacent intervals.

Theorem 5.7.9. *Suppose that $f : [a, c] \to \mathbb{R}$ is bounded and $a < b < c$. If f is integrable on $[a, b]$ and on $[b, c]$, then f is integrable on $[a, c]$. Moreover,* $\displaystyle\int_a^c f = \int_a^b f + \int_b^c f.$

Proof. Suppose that f is integrable on $[a, b]$ and on $[b, c]$. Let $\epsilon > 0$. There is a partition P_1 of $[a, b]$ so that $U(f, P_1) - L(f, P_1) < \epsilon/2$. There is a partition P_2 of $[b, c]$ so that $U(f, P_2) - L(f, P_2) < \epsilon/2$. Let $P = P_1 \cup P_2$, so that P is a partition of $[a, c]$ which contains b. Use the standard notation for partitions with P and assume that $x_k = b$. Note that

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{k} M_i \Delta x_i + \sum_{i=k+1}^{n} M_i \Delta x_i = U(f, P_1) + U(f, P_2).$$

Similarly, $L(f, P) = L(f, P_1) + L(f, P_2)$. Therefore

$$\begin{aligned} U(f, P) - L(f, P) &= (U(f, P_1) + U(f, P_2)) - (L(f, P_1) + L(f, P_2)) \\ &= (U(f, P_1) - L(f, P_1)) + (U(f, P_2) - L(f, P_2)) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Furthermore, since

$$L(f,P_1) \leq \int_a^b f \leq U(f,P_1) \text{ and } L(f,P_2) \leq \int_b^c f \leq U(f,P_2)$$

then

$$L(f,P) = L(f,P_1)+L(f,P_2) \leq \int_a^b f+\int_b^c f \leq U(f,P_1)+U(f,P_2) = U(f,P).$$

Now by Theorem 5.7.4, f is integrable on $[a,c]$ and

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

□

This next result allows us to change the value of an integrable function at an endpoint without affecting integrability or the value of the integral. Combined with the previous two theorems, it follows that we can alter an integrable function at finitely many points without changing integrability. This will be one of the more technical proofs of this section. The main idea is to select partitions with points near enough to the endpoint that Δx is small enough to minimize the effect of the value of the function at the endpoint.

Theorem 5.7.10. *Suppose that $f,g : [a,b] \to \mathbb{R}$ are bounded. If $f(x) = g(x)$ for all $x \in [a,b)$ and if f is integrable on $[a,b]$, then g is integrable on $[a,b]$ and $\int_a^b f = \int_a^b g$.*

Proof. Let $\epsilon > 0$. Since f is integrable on $[a,b]$, there is a partition P_1 of $[a,b]$ so that $U(f,P_1) - L(f,P_1) < \epsilon/6$. Let $M \in \mathbb{R}$ so that $|f(x)| < M$ and $|g(x)| < M$ for all $x \in [a,b]$. Select $z \in (a,b)$ so that $M(b-z) < \epsilon/12$. Let $P = P_1 \cup \{z\}$. Then $U(f,P)-L(f,P) < \epsilon/6$. We will use the standard notation for partitions for $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ with the modification that we will affix superscripts to the M_i's and m_i's (such as M_i^f or m_i^g) to indicate which function we are considering. Notice that for $i < n$, $M_i^f = M_i^g$ so that

$$U(f,P) - U(g,P) = (M_n^f - M_n^g)\Delta x_n.$$

Also, note that $\Delta x_n < \epsilon/(12M)$ since $z \in P$. Then

$$\begin{aligned}
|U(f,P) - U(g,P)| &= |(M_n^f - M_n^g)\Delta x_n| \\
&< |M_n^f - M_n^g|\epsilon/(12M) \\
&\le (|M_n^f| + |M_n^g|)\epsilon/(12M) \\
&\le (|M| + |M|)\epsilon/(12M) \\
&= \epsilon/6.
\end{aligned}$$

Similarly, $|L(f,P) - L(g,P)| < \epsilon/6$. It follows that

$$\begin{aligned}
|U(g,P) - L(g,P)| &= |U(g,P) - U(f,P) + U(f,P) - L(f,P) \\
&\qquad + L(f,P) - L(g,P)| \\
&\le |U(g,P) - U(f,P)| + |U(f,P) - L(f,P)| \\
&\qquad + |L(f,P) - L(g,P)| \\
&< \epsilon/6 + \epsilon/6 + \epsilon/6 \\
&= \epsilon/2. \\
&< \epsilon
\end{aligned}$$

At this point, we know by the ϵ-partition integrability condition that g is integrable.

We now want to show that $\displaystyle\int_a^b g = \int_a^b f$. To do so, we show that the difference between these values is less than every positive ϵ. We let $\epsilon > 0$ and then repeat the discussion above. Now consider the difference

$$\begin{aligned}
\left|\int_a^b g - \int_a^b f\right| &= \left|\int_a^b g - U(g,P) + U(g,P) - U(f,P) + U(f,P) - \int_a^b f\right| \\
&\le \left|\int_a^b g - U(g,P)\right| + |U(g,P) - U(f,P)| + \left|U(f,P) - \int_a^b f\right| \\
&< \epsilon/2 + \epsilon/6 + \epsilon/6 \\
&= 5\epsilon/6 \\
&< \epsilon.
\end{aligned}$$

Since $\displaystyle\left|\int_a^b g - \int_a^b f\right| < \epsilon$ for all $\epsilon > 0$, these two integrals are equal.

$\square$

As a consequence of the previous theorems, we have:

Theorem 5.7.11. *If $f : [a,b] \to \mathbb{R}$ is piecewise integrable on $[a,b]$, then f is integrable $[a,b]$.* □

This implies that if a function on a closed interval is bounded, and if it is piecewise continuous or monotonic, then the function is integrable. This accounts for most functions encountered in a basic calculus class.

Exercises 5.7

5.7.1 Use Theorem 5.7.4 to prove that $f(x) = x^3$ is integrable on $[1,3]$.
5.7.2 Use Theorem 5.7.4 to prove that $f(x) = x^2$ is integrable on $[-1,1]$.
5.7.3 Define $f : [0,2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 2 & 1 \leq x \leq 2 \end{cases}$$

Prove that f is integrable on $[0,2]$.
5.7.4 Prove that if $f : [a,b] \to \mathbb{R}$ is decreasing, then f is integrable.
5.7.5 Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and integrable, that $f(x) \geq 0$ for all $x \in \mathbb{R}$, and that $\displaystyle\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.
5.7.6 Suppose that $f : [0,1] \to \mathbb{R}$ is continuous and that $\displaystyle\int_0^x f = \int_x^1 f$ for all $x \in (0,1]$. Prove that $f(x) = 0$ for all $x \in [0,1]$.
5.7.7 Let $f : [a,b] \to \mathbb{R}$ be bounded. Suppose that there are sequences $\langle U_n \rangle$ and $\langle L_n \rangle$ of upper and lower sums for f so that $\lim(U_n - L_n) = 0$.

a. Prove that f is integrable on $[a,b]$.

b. Prove that $\langle U_n \rangle$ and $\langle L_n \rangle$ converge to a common limit L.

c. Prove that $\displaystyle\int_a^b f = L$.

5.7.8 Use monotonicity as in Examples 5.5.4 and 5.7.2 to prove that $f(x) = x^5$ is integrable on $[0,1]$.
5.7.9 Mimic Examples 5.5.6, 5.6.6, and 5.7.5 to prove that $f(x) = x^4$ is integrable on $[0,1]$ and find the value of the integral.
5.7.10 Mimic Examples 5.5.6, 5.6.6, and 5.7.5 to prove $f(x) = \cos(x)$ is integrable on $[0, \pi/2]$ and find the value of the integral. (You may

assume the standard properties of $\cos(x)$, but list what you actually need to know about $\cos(x)$ to solve this problem.)

5.7.11 Suppose that $f, g : [a, b] \to \mathbb{R}$ are bounded. Define

$$M^f = \sup\{f(x) : x \in [a, b]\}$$

$$M^g = \sup\{g(x) : x \in [a, b]\}$$

$$M^{f+g} = \sup\{f(x) + g(x) : x \in [a, b]\}.$$

Prove that $M^{f+g} \le M^f + M^g$.

5.7.12 Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Define

$$M^f = \sup\{f(x) : x \in [a, b]\}$$

$$M^{|f|} = \sup\{|f(x)| : x \in [a, b]\}$$

$$m^f = \inf\{f(x) : x \in [a, b]\}$$

$$m^{|f|} = \inf\{|f(x)| : x \in [a, b]\}$$

Prove that $M^{|f|} - m^{|f|} \le M^f - m^f$.

5.8 Algebraic Properties of the Integral

The integral satisfies many algebraic properties that simplify the process of finding integrals. We prove enough of them here to get us through the Fundamental Theorem of Calculus. The first two theorems show that the integral interacts nicely with some arithmetic operations. To prove the first, we need to recall (from Exercise 1.6.16) that for any bounded set A and for any $k > 0$

$$\sup\{kx : x \in A\} = k \sup\{x : x \in A\} = k \sup A$$

and

$$\inf\{kx : x \in A\} = k \inf\{x : x \in A\} = k \inf A.$$

On the other hand, if $k < 0$, then

$$\sup\{kx : x \in A\} = k \inf\{x : x \in A\} = k \inf A$$

and

$$\inf\{kx : x \in A\} = k \sup\{x : x \in A\} = k \sup A.$$

Theorem 5.8.1. *If $f : [a, b] \to \mathbb{R}$ is integrable and $k \in \mathbb{R}$, then kf is integrable on $[a, b]$ and* $\displaystyle\int_a^b (kf) = k \int_a^b f$.

Proof. If $k = 0$, then $kf = 0$ is integrable by Example 5.6.4 and $\displaystyle\int_a^b (0f) = 0 = 0 \int_a^b f$. Suppose then that $k > 0$. The case when $k < 0$ will be left as an exercise. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be any partition of $[a, b]$. We will prove that $U(kf, P) = kU(f, P)$. We will use the standard notation for partitions, except that we will use superscripts (such as M_i^k and m_i^k) to indicate when we are addressing kf rather than f. Note that by the discussion above, since $k > 0$, then $M_i^k = kM_i$ and $m_i^k = km_i$. Then

$$\begin{aligned} U(kf, P) &= \sum_{i=1}^{n} M_i^k \Delta x_i \\ &= \sum_{i=1}^{n} kM_i \Delta x_i \\ &= k \sum_{i=1}^{n} M_i \Delta x_i \\ &= kU(f, P). \end{aligned}$$

Since $U(kf, P) = kU(f, P)$ for all partitions P, then

$$\begin{aligned} \overline{\int_a^b} (kf) &= \inf\{U(kf, P) : P \text{ a partition of } [a, b]\} \\ &= \inf\{kU(f, P) : P \text{ a partition of } [a, b]\} \\ &= k \inf\{U(f, P) : P \text{ a partition of } [a, b]\} \\ &= k \overline{\int_a^b} f. \end{aligned}$$

We can similarly show that $\displaystyle\underline{\int_a^b} (kf) = k \underline{\int_a^b} f$. Since f is integrable, we now have

$$\underline{\int_a^b} (kf) = k \underline{\int_a^b} f = k \overline{\int_a^b} f = \overline{\int_a^b} (kf)$$

so kf is integrable. The value of this integral is

$$\underline{\int_a^b} (kf) = k \underline{\int_a^b} f = k \int_a^b f.$$

$\square$

We now address integrals of sums and differences.

Theorem 5.8.2. *Suppose that $f, g : [a, b] \to \mathbb{R}$ are integrable. Then $f + g$ and $f - g$ are integrable on $[a, b]$ and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g \text{ and } \int_a^b (f-g) = \int_a^b f - \int_a^b g.$$

Proof. Let $\epsilon > 0$. There are partitions P_1 and P_2 so that

$$U(f, P_1) - L(f, P_1) < \epsilon/2 \text{ and } U(g, P_2) - L(g, P_2) < \epsilon/2.$$

Let $P = P_1 \cup P_2$. Then

$$U(f, P) - L(f, P) < \epsilon/2 \text{ and } U(g, P) - L(g, P) < \epsilon/2.$$

Assume that $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, and we will again modify the standard notation with superscripts. Note that for each i, $M_i^{f+g} \le M_i^f + M_i^g$ and that $m_i^{f+g} \ge m_i^f + m_i^g$ (by Exercise 5.7.11). It follows that

$$\begin{aligned} U(f+g, P) &= \sum_{i=1}^n M_i^{f+g} \Delta x_i \\ &\le \sum_{i=1}^n (M_i^f + M_i^g) \Delta x_i \\ &= \sum_{i=1}^n M_i^f \Delta x_i + \sum_{i=1}^n M_i^g \Delta x_i \\ &= U(f, P) + U(g, P). \end{aligned}$$

Similarly, $L(f+g, P) \ge L(f, P) + L(g, P)$. Then

$$\begin{aligned} U(f+g, P) - L(f+g, P) &\le [U(f, P) + U(g, P)] - [L(f, P) + L(g, P)] \\ &= [U(f, P) - L(f, P)] + [U(g, P) - L(g, P)] \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

By the ϵ-partition integrability condition, $f + g$ is integrable.

To prove that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$, we will show that the difference between these values is less than every positive ϵ. Let $\epsilon > 0$

and repeat the work we just performed to establish integrability. We now have

$$[U(f,P)+U(g,P)]-[L(f,P)+L(g,P)]<\epsilon$$

and

$$[L(f,P)+L(g,P)] \leq \int_a^b f + \int_a^b g \leq [U(f,P)+U(g,P)]$$

and also

$$[L(f,P)+L(g,P)] \leq L(f+g,P) \leq \int_a^b (f+g) \leq U(f+g,P) \leq [U(f,P)+U(g,P)].$$

Since $\int_a^b f + \int_a^b g$ and $\int_a^b (f+g)$ are both in an interval with width less than ϵ, it follows that $\left|\int_a^b f + \int_a^b g - \int_a^b (f+g)\right| < \epsilon$. Since this is true for all $\epsilon > 0$, then $\int_a^b f + \int_a^b g = \int_a^b (f+g)$ as desired.

The work for differences now follows by noting that

$$f-g = f+[(-1)g]$$

and applying the theorems for constant multiples and sums. $\square$

We now turn from arithmetic operations to order relations. These are the final ingredients that we will need to prove the Fundamental Theorem of Calculus.

Lemma 5.8.3. *If $f : [a,b] \to \mathbb{R}$ is integrable and $k \in \mathbb{R}$, so that $k \leq f(x)$ for all $a \in [a,b]$, then $k(b-a) \leq \int_a^b f$.*

Proof. Consider the most trivial partition of $[a,b]$

$$P = \{a = x_0 < x_1 = b\}.$$

Using the standard notation, we see

$$k(b-a) \leq m_1(b-a) = L(f,P).$$

Since $k(b-a)$ is less than or equal to one lower sum, it is less than or equal to the supremum of all lower sums, $k(b-a) \leq \underline{\int_a^b} f = \int_a^b f$. $\square$

Theorem 5.8.4. *Suppose that $f, g : [a,b] \to \mathbb{R}$ are integrable and that $f(x) \le g(x)$ for all $x \in [a,b]$. Then $\int_a^b f \le \int_a^b g$.*

Proof. Let $h(x) = g(x) - f(x)$. Then $h(x) \ge 0$ for all $x \in [a,b]$. Also, by Theorem 5.8.2, h is integrable on $[a,b]$. By Theorem 5.8.2 and Lemma 5.8.3 we now have

$$\int_a^b g - \int_a^b f = \int_a^b h \ge 0(b-a) = 0.$$

It follows that $\int_a^b g \ge \int_a^b f$ as desired. $\square$

Theorem 5.8.5. *If $f : [a,b] \to \mathbb{R}$ is integrable then $|f|$ is integrable on $[a,b]$ and $\left|\int_a^b f\right| \le \int_a^b |f|$.*

Proof. We first show that $|f|$ is integrable. Let $\epsilon > 0$. There is a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$ so that $U(f,P) - L(f,P) < \epsilon$. We again use the standard notation modified with superscripts. Note that for each i, $M_i^{|f|} - m_i^{|f|} \le M_i^f - m_i^f$ (by Exercise 5.7.12). Therefore

$$\begin{aligned} U(|f|,P) - L(|f|,P) &= \sum_{i=1}^n (M_i^{|f|} - m_i^{|f|})\Delta x_i \\ &\le \sum_{i=1}^n (M_i^f - m_i^f)\Delta x_i \\ &= U(f,P) - L(f,P) \\ &< \epsilon. \end{aligned}$$

By the ϵ-partition integrability condition, $|f|$ is integrable on $[a,b]$.

Since $-|f| \le f \le |f|$, Theorems 5.8.4 and 5.8.1 now give

$$-\int_a^b |f| = \int_a^b -|f| \le \int_a^b f \le \int_a^b |f|.$$

Thus, $\left|\int_a^b f\right| \le \int_a^b |f|$. $\square$

Exercises 5.8

5.8.1 Prove that $\int_a^b (kf) = k \int_a^b f$ when $k < 0$.

5.8.2 Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and that $|f(x)| < B$ for all $x \in [a, b]$.

a. Prove that

$$U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)]$$

for all partitions P of $[a, b]$. Hint: Difference of squares.

b. Prove that if f is integrable on $[a, b]$, then f^2 is also integrable on $[a, b]$.

5.8.3 Let f and g be integrable functions on an interval $[a, b]$.

a. Prove that fg is integrable. Hint: Use that

$$4fg = (f + g)^2 - (f - g)^2.$$

b. Prove that $\max(f, g)$ and $\min(f, g)$ are integrable on $[a, b]$.

5.8.4 Prove that if h is continuous on $[a, b]$ and $\int_a^b h = 0$ then there is some $x \in [a, b]$ with $h(x) = 0$.

5.8.5 Suppose that f and g are continuous on $[a, b]$ and that

$$\int_a^b f = \int_a^b g.$$

Prove that there exists $x \in [a, b]$ with $f(x) = g(x)$.

5.9 The Fundamental Theorem of Calculus

There are two related theorems that are graced with the title "Fundamental Theorem of Calculus." One theorem is an evaluation theorem which allows one to evaluate integrals by doing derivatives backwards. The other theorem declares that the derivative of an integral is the original function. These theorems are frequently numbered I and II or called "First" and "Second," but the order varies from text to text. We will avoid such terminology because we do not wish to imply some philosophy that holds one of the theorems above the other. In the end though, one theorem has to be printed first. We choose to present first the theorem with the simpler proof.

We begin by defining some notation which will simplify our arguments by allowing us to be slightly less careful with the order of the endpoints of intervals.

Definition 5.9.1. If f is a function defined at a real number a, then

$$\int_a^a f = 0.$$

Suppose that $a < b$ in $\mathbb{R}$ and that $f : [a, b] \to \mathbb{R}$ is integrable. Then

$$\int_b^a f = -\int_a^b f.$$

This notation allows some of the results of the previous sections to be extended to integrals of the form $\displaystyle\int_a^b f$ where we do not know which, if either, of a and b is larger. In particular, this notation allows us the following extension of Theorem 5.7.9.

Theorem 5.9.2. *Suppose that f is integrable on an interval containing a, b, and c. Then* $\displaystyle\int_a^c f = \int_a^b f + \int_b^c f$. $\square$

In the first version of the Fundamental Theorem of Calculus that we will encounter, we will refer to a function defined on (a, b) being integrable on $[a, b]$. Here is the definition of this notion.

Definition 5.9.3. Suppose that $f : (a, b) \to \mathbb{R}$ is a bounded function. We say that f is integrable on $[a, b]$ if an extension of f to $[a, b]$ is integrable. Note that by Theorem 5.7.10 the values at the endpoints really are irrelevant, so this means that every extension of f to $[a, b]$ is integrable (and the integrals are all equal).

Now we finally approach one of the Fundamental Theorems of Calculus. Before addressing this Theorem, you might want to review Examples 5.5.6, 5.6.6, and 5.7.5.

Theorem 5.9.4. (Fundamental Theorem of Calculus) *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If f' is integrable on $[a, b]$, then*

$$\int_a^b f' = f(b) - f(a).$$

Proof. We will prove that $f(b) - f(a)$ is between every pair of upper and lower sums and then apply Theorem 5.7.3. Let

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

be any partition of $[a, b]$. We use the standard notation for partitions. Apply the Mean Value Theorem on each interval $[x_{i-1}, x_i]$ to find some $t_i \in (x_{i-1}, x_i)$ so that

$$f'(t_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}.$$

It follows that $f'(t_i)\Delta x_i = f(x_i) - f(x_{i-1})$. Now

$$f(b) - f(a) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \sum_{i=1}^{n} f'(t_i)\Delta x_i.$$

Since $m_i \leq f(t_i) \leq M_i$ for each i, we have

$$L(f', P) = \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} f'(t_i)\Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i = U(f', P).$$

Therefore

$$L(f', P) \leq f(b) - f(a) \leq U(f', P).$$

Since this is true for all partitions P, and since f' is integrable, Theorem 5.7.3 now tells us that

$$\int_a^b f' = f(b) - f(a).$$

□

Example 5.9.5. In Example 5.1.1 we attempted to calculate $\displaystyle\int_{-1}^{1} f$ where $f : [-1, 1] \to \mathbb{R}$ is given by $f(x) = 1 - x^2$. It happens to be that f is the derivative of $g(x) = x - \dfrac{1}{3}x^3$. Therefore,

$$\int_{-1}^{1} f = g(1) - g(-1) = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}.$$

This agrees with our work in Example 5.1.1

Theorem 5.9.6. (Fundamental Theorem of Calculus) *Suppose that $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ and that $F : [a, b] \to \mathbb{R}$ is defined by*

$$F(x) = \int_a^x f.$$

Then F is continuous on $[a, b]$. If f is continuous at $z \in (a, b)$, then F is differentiable at z and $F'(z) = f(z)$.

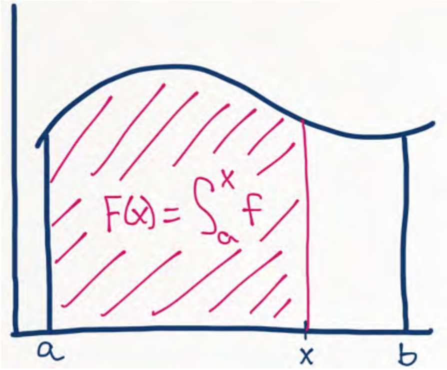

Figure 5.8: If $f \geq 0$, then the function F in Theorem 5.9.6 gives the area under f between a and x. If f is actually a velocity function, then F gives displacement (which is the same as distance if $f \geq 0$).

Proof. First we prove that F is (uniformly) continuous on $[a, b]$. Let $\epsilon > 0$. Let $M > 0$ be an upper bound of $|f|$ on $[a, b]$. Let $0 < \delta < \epsilon/M$.

Suppose that $x < y \in [a, b]$ with $|x - y| < \delta$. Then

$$\begin{aligned}
|F(y) - F(x)| &= \left| \int_a^y f - \int_a^x f \right| \\
&= \left| \int_a^y f + \int_x^a f \right| \\
&= \left| \int_x^y f \right| \\
&\le \int_x^y |f| \\
&\le \int_x^y M \\
&= M(y - x) \\
&< M\delta \\
&< M\epsilon/M \\
&= \epsilon.
\end{aligned}$$

Thus F is uniformly continuous on $[a, b]$.

Now suppose that f is continuous at $z \in (a, b)$. We will prove that F is differentiable at z and that $F'(z) = f(z)$. We do so applying the definition of the limit to the definition of the derivative. Let $\epsilon > 0$. There is a δ so that if $x \in [a, b]$ and $|x - z| < \delta$ then $|f(x) - f(z)| < \epsilon$. Suppose that $x \in [a, b]$ and that $0 < |x - z| < \delta$. We will show that

$$\left| \frac{F(x) - F(z)}{x - z} - f(z) \right| < \epsilon.$$

There are two cases, either $x < z$ or $z < x$. We will address the case when $x < z$. In this case

$$\begin{aligned}
\frac{F(x)-F(z)}{x-z} &= \frac{1}{x-z}\left(\int_a^x f - \int_a^z f\right) \\
&= \frac{1}{x-z}\left(\int_a^x f + \int_z^a f\right) \\
&= \frac{1}{x-z}\int_z^x f \\
&= \frac{1}{x-z}\left(-\int_x^z f\right) \\
&= \frac{1}{z-x}\int_x^z f
\end{aligned}$$

In the next part of the proof, we will use a trick to combine a constant $f(z)$ into an integral. Since $f(z)$ is constant, $\displaystyle\int_x^z f(z) = (z-x)f(z)$. Dividing gives $f(z) = \dfrac{1}{z-x}\displaystyle\int_x^z f(z)$. Now consider

$$\begin{aligned}
\left|\frac{F(x)-F(z)}{x-z} - f(z)\right| &= \left|\left(\frac{1}{z-x}\int_x^z f\right) - f(z)\right| \\
&= \left|\left(\frac{1}{z-x}\int_x^z f\right) - \left(\frac{1}{z-x}\int_x^z f(z)\right)\right| \\
&= \left|\frac{1}{z-x}\int_x^z [f - f(z)]\right| \\
&\leq \frac{1}{|z-x|}\int_x^z |f - f(z)|
\end{aligned}$$

Now, since $|x-z| < \delta$, then $|f - f(z)| < \epsilon$ on the interval $[x, z]$. It follows that

$$\begin{aligned}
\left|\frac{F(x)-F(z)}{x-z} - f(z)\right| &\leq \frac{1}{|z-x|}\int_x^z |f - f(z)| \\
&\leq \frac{1}{|z-x|}\int_x^z \epsilon \\
&= \frac{1}{|z-x|}(z-x)\epsilon \\
&= \epsilon.
\end{aligned}$$

We have satisfied the definition of the limit that

$$\lim_{x \to z} \frac{F(x) - F(z)}{x - z} = f(z).$$

Thus, $F'(z) = f(z)$. $\square$

Exercises 5.9

5.9.1 Prove Theorem 5.9.2

5.9.2 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 4 & 1 < x \end{cases}$$

a. Find the function $F(x) = \int_0^x f$.

b. Sketch the graph of F. Where is F continuous?

c. Where is F differentiable? Find F'.

5.9.3 Let $f : \mathbb{R} \to \mathbb{R}$ be the greatest integer function.

a. Find the function $F(x) = \int_0^x f$.

b. Sketch the graph of F. Where is F continuous?

c. Where is F differentiable? Find F'.

5.9.4 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x < 0 \\ x^2 + 1 & 0 \le x \le 2 \\ 0 & 2 < x \end{cases}$$

a. Find the function $F(x) = \int_0^x f$.

b. Sketch the graph of F. Where is F continuous?

c. Where is F differentiable? Find F'.

5.9.5 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and define

$$F(x) = \int_{x-1}^{x+1} f$$

for all $x \in \mathbb{R}$. Prove that F is differentiable on $\mathbb{R}$ and find F'.
5.9.6 Suppose that $g : [0,1] \to [0,1]$ is a strictly increasing bijection. Give a geometric argument for the equality

$$\int_0^1 g + \int_0^1 g^{-1} = 1.$$

5.9.7 Use Theorem 5.9.6 to prove Theorem 5.9.4 in the case when f' is continuous.

Chapter 6

Series

6.1 Definitions and Basic Properties

First, an example:

Example 6.1.1. A man is asked to walk a mile. First, he walks half a mile. Then, he walks half of the remaining distance, or a quarter of a mile. Next, he walks half of the remaining distance, or one eighth of a mile. He continues in this manner indefinitely, at each step walking half the remaining distance. After n steps, he has walked

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}$$

miles. A quick induction argument shows that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}=1-\frac{1}{2^n}.$$

The distances traveled by the man after n steps for some values of n are:

n	distance
1	1/2
2	3/4
3	7/8
4	15/16
5	31/32
6	63/64

Even though the man does not ever reach a full mile, the fact that $\lim \left(1 - \frac{1}{2^n}\right) = 1$ tells us that he does get arbitrarily close to a mile, and if he could take infinitely many steps, maybe we could say the man would reach one mile. In fact, we might want to write

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

The topic of this section is this sort of infinite summation.

Definition 6.1.2. A *series* is a formal expression of the form

$$\sum_{k=m}^{\infty} a_k$$

where is $\langle a_k \rangle$ is any sequence defined for $k \geq m$. This series may also be written as

$$a_m + a_{m+1} + a_{m+2} + a_{m+3} + \cdots$$

For integers $n \geq m$, the n^{th} *partial sum* of this series is

$$s_n = \sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + a_{m+3} + \cdots + a_n.$$

Note that the partial sums of the series form a sequence $\langle s_n \rangle$.

Remark 6.1.3. The initial value of the index (subscript) may vary from series to series. We will write most of our theorems assuming the initial value is $k = 1$. It should be clear though that the results hold for other initial values.

Example 6.1.4. The sum discussed in Example 6.1.1 is the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$. Some partial sums of this series are listed in the table in that example. Notice how these numbers seem to get closer and closer to 1 as n gets larger and larger. We will say that the series converges to 1.

Example 6.1.5. For the sum $\sum_{k=1}^{\infty} k$, the partial sums look like

$$1 + 2 + \cdots + n.$$

As n increases, these partial sums are unbounded, so the sequence of partial sums cannot converge.

Definition 6.1.6. Suppose that $\sum_{k=1}^{\infty} a_k$ is any series and that $\langle s_n \rangle$ is the sequence of partial sums of that series. We say that $\sum_{k=1}^{\infty} a_k$ *converges* to a real number L if the sequence of partial sums $\langle s_n \rangle$ converges to L. In this case, we write $\sum_{k=1}^{\infty} a_k = L$. If a series does not converge, then that series *diverges*.

Example 6.1.7. Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

Some quick partial fractions work gives us that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

so

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

This makes it easy to calculate partial sums. For any n

$$\begin{aligned} s_n &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Now

$$\lim s_n = \lim \left(1 - \frac{1}{n+1}\right) = 1$$

so our series converges to 1. We can write

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

For the next example, we need to recall how to factor a difference of n^{th} powers. For any positive integer n,

$$1 - r^{n+1} = (1-r)(1 + r + r^2 + r^3 + \cdots + r^n).$$

A consequence of this factoring is that

$$(1 + r + r^2 + r^3 + \cdots + r^n) = \frac{1 - r^{n+1}}{1-r} \text{ if } r \neq 1.$$

Definition 6.1.8. A *geometric series* is a series of the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{k=0}^{\infty} ar^k$$

where a and r are real numbers with $a \neq 0$.

Theorem 6.1.9. (Geometric Series) *Consider the geometric series*

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{k=0}^{\infty} ar^k.$$

- *If* $|r| < 1$ *then the series converges to* $\dfrac{a}{1-r}$.
- *If* $|r| \geq 1$ *then the series diverges.*

Proof. If $r = 1$, then the series is $\sum_{k=0}^{\infty} a$, and its partial sums are given by $s_n = (n+1)a$. In this case, $\langle s_n \rangle$ is unbounded and cannot converge. Thus the series diverges.

Suppose now that $r \neq 1$. Then the partial sums of the geometric series are given by

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^n \\ &= a(1 + r + r^2 + \cdots + r^n) \\ &= a\frac{1 - r^{n+1}}{1-r}. \end{aligned}$$

It follows that if $|r| < 1$ then

$$\lim s_n = \frac{a}{1-r}.$$

In this case, the series converges to $\dfrac{a}{1-r}$. Otherwise, $\langle s_n \rangle$ is unbounded and the series does not converge. $\square$

Example 6.1.10. The series

$$\sum_{k=0}^{\infty} \frac{5}{3^n}$$

is a geometric series with $a = 5$ and $r = \dfrac{1}{3}$. Since $|r| < 1$, the series converges to

$$\frac{a}{1-r} = \frac{5}{1-\frac{1}{3}} = \frac{15}{2}.$$

Example 6.1.11. The series

$$\sum_{k=2}^{\infty} \frac{2 \cdot 3^k}{4^{k+1}}$$

is also a geometric series. We can see the geometric nature of the series if we rearrange things a bit:

$$\sum_{k=2}^{\infty} \frac{2 \cdot 3^k}{4^{k+1}} = \sum_{k=2}^{\infty} \frac{2}{4}\frac{3^k}{4^k} = \sum_{k=2}^{\infty} \frac{1}{2}\left(\frac{3}{4}\right)^k.$$

Thus we have a geometric series with $r = \dfrac{3}{4}$. Such a series must converge. However, we have to be careful before we use the geometric series formula. The formula assumes that the first exponent on r in the sum is 0. In our case, the first exponent is 2. We can rearrange things a bit to make the first exponent 0:

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{1}{2}\left(\frac{3}{4}\right)^k &= \frac{1}{2}\left(\frac{3}{4}\right)^2 + \frac{1}{2}\left(\frac{3}{4}\right)^3 + \frac{1}{2}\left(\frac{3}{4}\right)^4 + \frac{1}{2}\left(\frac{3}{4}\right)^5 + \cdots \\
&= \frac{1}{2}\left(\frac{3}{4}\right)^2 + \frac{1}{2}\left(\frac{3}{4}\right)^2\left(\frac{3}{4}\right)^1 + \frac{1}{2}\left(\frac{3}{4}\right)^2\left(\frac{3}{4}\right)^2 + \cdots \\
&= \frac{9}{32} + \frac{9}{32}\left(\frac{3}{4}\right)^1 + \frac{9}{32}\left(\frac{3}{4}\right)^2 + \frac{9}{32}\left(\frac{3}{4}\right)^3 + \cdots
\end{aligned}$$

This is now a geometric series with $a = \dfrac{9}{32}$ and $r = \dfrac{3}{4}$. It converges to

$$\frac{a}{1-r} = \frac{\frac{9}{32}}{1-\frac{3}{4}} = \frac{9}{8}.$$

Suppose that $\sum_{k=1}^{\infty} a_k$ is any convergent series and that $\langle s_n \rangle$ is its sequence of partial sums. There is some number L so that $\lim s_n = L$. It is also the case that $\lim s_{n-1} = L$. Therefore,

$$\lim a_n = \lim(s_n - s_{n-1}) = \lim s_n - \lim s_{n-1} = L - L = 0.$$

That is, the terms of any convergent series must converge to 0. The contrapositive of this statement can be useful to identify some series as divergent:

Theorem 6.1.12. (Term Test for Divergence) *If* $\lim a_k \neq 0$ *then* $\sum_{k=1}^{\infty} a_k$ *diverges.* □

Example 6.1.13. Consider the series $\sum_{k=1}^{\infty} \frac{n+1}{n}$. Since

$$\lim \frac{n+1}{n} = 1 \neq 0$$

the series diverges by the Term Test.

For series whose terms are not negative, convergence is equivalent to being bounded. Suppose that $a_k \geq 0$ for all k and let $\langle s_n \rangle$ be the sequence of partial sum of $\sum_{k=1}^{\infty} a_k$. For any n, because a_n is not negative, $s_{n+1} = s_n + a_{n+1} \geq s_n$. Thus $\langle s_n \rangle$ is an increasing sequence. Such a sequence converges if and only if it is bounded.

Theorem 6.1.14. *If* $a_k \geq 0$ *for all* k*, then* $\sum_{k=1}^{\infty} a_k$ *converges if and only if its sequence of partial sums is bounded.*

As a result, convergence for series of nonnegative terms reduces to a discussion of "bigness" and "smallness." For the series to converge, the partial sums must stay small (bounded). For the series to diverge, the partial sums must get large (unbounded).

Before we can address the next example, we need to consider the integral $\int_1^n \frac{1}{x^p} dx$ for various values of p. It is not hard to see that:

$$\int_1^n \frac{1}{x^p} dx = \begin{cases} \frac{1}{1-p}\left(n^{1-p}-1\right) & p<1 \\ \ln(n) & p=1 \\ \frac{1}{p-1}\left(1-\frac{1}{n^{p-1}}\right) & p>1 \end{cases}$$

Consider the sequence $\langle I_n \rangle$ given by $I_n = \int_1^n \frac{1}{x^p} dx$. In every case, $\langle I_n \rangle$ is an increasing sequence. The sequence is unbounded if $p \leq 1$ and converges to $\frac{1}{1-p}$ if $p > 1$.

Theorem 6.1.15. **(p-series)** *The p-series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

converges if $p > 1$ and diverges if $p \leq 1$.

Proof. Let $\langle I_n \rangle$ be as in the discussion before the theorem, and let $\langle s_n \rangle$ be the sequence of partial sums of the series in question. If $p \leq 0$, then the terms of the series do not converge to 0, so the series diverges by the Term Test. Suppose then that $p > 0$. Let $f(x) = \frac{1}{x^p}$. Consider the regular partition P of $[1, n+1]$ with $\Delta x = 1$. We will consider the upper sum associated to f and the partition P. See Figure 6.1. Since f is decreasing, f has a maximum value at the left hand enpoint of each interval in the partition. Therefore,

$$\begin{aligned} I_{n+1} &= \int_1^{n+1} \frac{1}{x^p} dx \\ &\leq U(f, P) \\ &= f(1)\Delta x + f(2)\Delta x + \cdots + f(n)\Delta x \\ &= \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} \\ &= s_n \end{aligned}$$

so $I_{n+1} \leq s_n$.

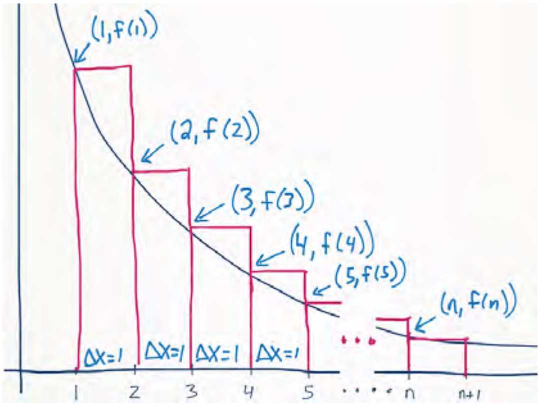

Figure 6.1: The integral of f over $[1, n+1]$ is less than the upper sum of f which is $f(1) + f(2) + \cdots + f(n)$.

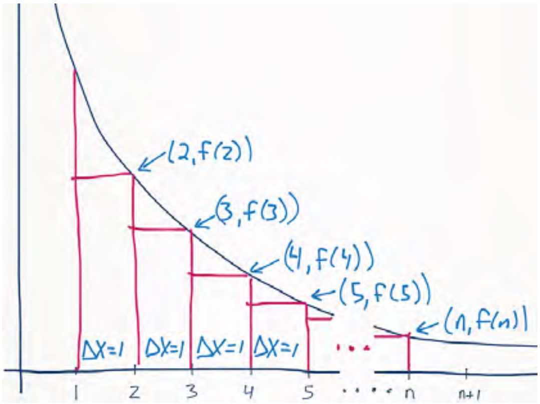

Figure 6.2: The integral of f over $[1, n]$ is greater than the lower sum of f which is $f(2) + f(3) + \cdots + f(n)$.

Now suppose that P is the regular partition of $[1, n]$ with $\Delta x = 1$. We will consider the lower sum associated to f and the partition P. See Figure 6.2. Since f is decreasing, f has a minimum value at the right hand endpoint of each interval of the partition. Therefore,

$$\begin{aligned}
I_n &= \int_1^n \frac{1}{x^p} dx \\
&\geq L(f, P) \\
&= f(2)\Delta x + f(3)\Delta x + \cdots + f(n)\Delta x \\
&= \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} \\
&= s_n - \frac{1}{1^p}
\end{aligned}$$

so $I_n \geq s_n - 1$ or $s_n \leq 1 + I_n$.

We now have that $I_{n+1} \leq s_n \leq 1 + I_n$. Suppose now that $p \leq 1$. Then we know that the sequence $\langle I_{n+1} \rangle$ is unbounded, so the sequence $\langle s_n \rangle$ of partial sums must be unbounded and cannot converge. In this case, the series in question must diverge. On the other hand, suppose that $p > 1$. This means that the sequence $\langle I_n \rangle$ must converge and must be bounded. Therefore, the sequence $\langle s_n \rangle$ of partial sums is also bounded. Now the sequence $\langle s_n \rangle$ is increasing (since we are adding only positive terms), so $\langle s_n \rangle$ is a bounded increasing sequence. As such, $\langle s_n \rangle$ must converge, so the series converges. □

Example 6.1.16. Harmonic Series The harmonic series is the p-series

$$\sum_{k=1}^{\infty} \frac{1}{k}.$$

This series diverges, but it does so very slowly. Here are decimal approximations of a few partial sums of the series:

n	s_n
10	2.93
100	5.19
1000	7.49
10000	9.79

Example 6.1.17. The p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. It happens to converge to $\dfrac{\pi^2}{6}$. Values of the summations of p-series for even p are known.

The value even for $p = 3$ is unknown.

Convergent sequence can be combined algebraically to form new convergent sequences.

Theorem 6.1.18. *Suppose that* $\sum_{k=1}^{\infty} a_k$ *converges to* L*, that* $\sum_{k=1}^{\infty} b_k$ *converges to* M*, and that* $c \in \mathbb{R}$*.*

1. $\sum_{k=1}^{\infty} (ca_k)$ *converges to* cL*.*

2. $\sum_{k=1}^{\infty} (a_k + b_k)$ *converges to* $L + M$*.*

3. $\sum_{k=1}^{\infty} (a_k - b_k)$ *converges to* $L - M$*.*

We express (1) in this theorem by saying we can factor a constant out of a series. We may even write $\sum_{k=1}^{\infty} (ca_k) = c\sum_{k=1}^{\infty} a_k$. We may express (2) as

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

Proof. We will prove (1). The other parts are left as exercises. Let $\langle s_n \rangle$ be the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$, and let $\langle t_n \rangle$ be the sequence of partial sums of $\sum_{k=1}^{\infty} (ca_k)$. Note that for any n we have

$$t_n = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = cs_n.$$

It follows that

$$\lim t_n = \lim cs_n = cL$$

so that $\sum_{k=1}^{\infty} (ca_k)$ converges to cL. $\square$

Example 6.1.19. Consider the series

$$\sum_{k=0}^{\infty} \frac{2^k - 1}{3^k 4^{k+1}}.$$

We can break the terms of this series into two fractions like so:

$$\sum_{k=0}^{\infty} \frac{2^k - 1}{3^k 4^{k+1}} = \sum_{k=0}^{\infty} \left(\frac{2^k}{3^k 4^{k+1}} - \frac{1}{3^k 4^{k+1}} \right).$$

When we do so, it looks like we are addressing the difference of two geometric series:

$$\sum_{k=0}^{\infty} \left(\frac{2^k}{3^k 4^{k+1}} - \frac{1}{3^k 4^{k+1}} \right) = \sum_{k=0}^{\infty} \frac{2^k}{3^k 4^{k+1}} - \sum_{k=0}^{\infty} \frac{1}{3^k 4^{k+1}}.$$

We can manipulate these two geometric series to make the values of a and r more explicit:

$$\sum_{k=0}^{\infty} \frac{2^k}{3^k 4^{k+1}} - \sum_{k=0}^{\infty} \frac{1}{3^k 4^{k+1}} = \sum_{k=0}^{\infty} \frac{1}{4} \left(\frac{2}{12} \right)^k - \sum_{k=0}^{\infty} \frac{1}{4} \left(\frac{1}{12} \right)^k.$$

In both cases, $|r| < 1$ so the series converge. Using the geometric series formula we have:

$$\sum_{k=0}^{\infty} \frac{1}{4} \left(\frac{2}{12} \right)^k - \sum_{k=0}^{\infty} \frac{1}{4} \left(\frac{1}{12} \right)^k = \frac{\frac{1}{4}}{1 - \frac{1}{6}} - \frac{\frac{1}{4}}{1 - \frac{1}{12}} = \frac{3}{110}.$$

Remark 6.1.20. Notice that our work in this last example is backwards. We cannot apply Theorem 6.1.18 to decompose a series algebraically into smaller series until we already know the smaller series converge. If we had gone through all of this work only to find a smaller series that diverged, then our results would in most cases be inconclusive. Our work for this problem should probably look like this:

$$\begin{aligned}
\frac{3}{110} &= \frac{\frac{1}{4}}{1-\frac{1}{6}} - \frac{\frac{1}{4}}{1-\frac{1}{12}} \\
&= \sum_{k=0}^{\infty} \frac{1}{4}\left(\frac{2}{12}\right)^k - \sum_{k=0}^{\infty} \frac{1}{4}\left(\frac{1}{12}\right)^k \\
&= \sum_{k=0}^{\infty} \frac{2^k}{3^k 4^{k+1}} - \sum_{k=0}^{\infty} \frac{1}{3^k 4^{k+1}} \\
&= \sum_{k=0}^{\infty} \left(\frac{2^k}{3^k 4^{k+1}} - \frac{1}{3^k 4^{k+1}}\right) \\
&= \sum_{k=0}^{\infty} \frac{2^k - 1}{3^k 4^{k+1}}
\end{aligned}$$

But almost no one does it that way.

Exercises 6.1

6.1.1 Find the sum of each of these series.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+2k}$
2. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k}$
3. $\displaystyle\sum_{k=2}^{\infty} \frac{3\cdot 4^{k+1}}{5^k}$
4. $\displaystyle\sum_{k=0}^{\infty} \frac{2^k-3^k}{4^k}$
5. $\displaystyle\sum_{k=0}^{\infty} \frac{6}{4k^2-1}$
6. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{k^2+k}\right)$

6.1.2 Determine if each of these series converges or diverges. Support your answers.

1. $\displaystyle\sum_{k=0}^{\infty} \cos(k\pi/2)$
2. $\displaystyle\sum_{k=0}^{\infty} \frac{2^k-1}{3^k}$
3. $\displaystyle\sum_{k=0}^{\infty} (\sqrt{2})^k$
4. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{x^k}$

5. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

6. $\sum_{k=0}^{\infty} e^{-k}$

7. $\sum_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^k$

8. $\sum_{k=1}^{\infty} \left(1-\frac{1}{k}\right)^k$

9. $\sum_{k=1}^{\infty} (\sqrt{k}-\sqrt{k+1})$

10. $\sum_{k=1}^{\infty} \frac{3+k}{k^3}$

11. $\sum_{k=0}^{\infty} \frac{k!}{2^k}$

12. $\sum_{k=0}^{\infty} \frac{\cos(k\pi)}{2^k}$

6.1.3 Prove part (2) of Theorem 6.1.18

6.1.4 Suppose that $\sum_{k=1}^{\infty} a_k$ is any series and c is a nonzero real number. Prove that $\sum_{k=1}^{\infty} a_k$ diverges if and only if $\sum_{k=1}^{\infty} ca_k$ diverges.

6.1.5 Provide an induction argument for

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}=1-\frac{1}{2^n}.$$

6.1.6 Suppose that $0 \leq a_k \leq b_k$ for all k, that $\sum_{k=1}^{\infty} a_k$ converges to A, and that $\sum_{k=1}^{\infty} b_k$ converges to B. Prove that $A \leq B$.

6.1.7 The method of proof of Theorem 6.1.15 hints at a more general test for convergence. State and prove this test.

6.1.8 Use the theorem you stated in Exercise 6.1.7 to determine the values of p for which the series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$ converges.

6.1.9 We will need a couple of common limits in this chapter that we have not yet encountered. We derive them in this exercise.

1. For $k \geq 1$, let $x_k = k^{1/k} - 1$. Solve for k in this expression and use the Binomial Theorem to prove that $\binom{k}{2} x_k^2 \leq k$ for $k \geq 2$

2. Use the previous result to conclude that $\lim x_k = 0$.

3. Use the previous result to find $\lim k^{1/k}$.

4. Use the Squeeze Theorem to find $\lim a^{1/k}$ where $a > 1$.

5. Use the previous result to find $\lim a^{1/k}$ where $0 < a < 1$.

6.1.10 We prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2^x$ is continuous. Suppose that $x \in \mathbb{R}$ and that $\langle x_n \rangle$ is any sequence converging to x. We need to prove that the sequence $\langle 2^{x_n} \rangle$ converges to 2^x. Let $\epsilon > 0$.

1. Explain why there is an $N > 0$ so that if $n > N$ then
$$\left|2^{1/n} - 1\right| < \frac{\epsilon}{2^x}.$$

2. Explain why there is an M so that if $n > M$ then
$$|x_n - x| < \frac{1}{N}.$$

3. Suppose that n is greater than N and M. Explain why
$$|2^{x_n} - 2^x| < \epsilon.$$
Hint: $|2^{x_n} - 2^x| = 2^x \left|2^{x_n - x} - 1\right|$.

6.2 Tests for Convergence

In general, it is very difficult to tell what the limit of a convergent series is. Geometric series and telescoping series are very special series in this regard. Usually, the question facing us for a series is not what the limit is but whether or not the series converges. In this section, we develop some basic tests for convergence of series.

Since the convergence of a series with nonnegative terms reduces to the question of boundedness of the sequence of partial sums, we first define a notion relating the convergence of a generic series to the convergence of a series of nonnegative terms.

Definition 6.2.1. A series $\displaystyle\sum_{k=1}^{\infty} a_k$ *converges absolutely* if the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges. A series $\displaystyle\sum_{k=1}^{\infty} a_k$ *converges conditionally* if the series $\displaystyle\sum_{k=1}^{\infty} a_k$ converges but the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ diverges.

We will see in a little while that if a series is absolutely convergent, then it is convergent. In an absolutely convergent series, the signs of the terms do not affect convergence of the series (although they will affect the limit if the series does converge). In a conditionally convergent series, the signs of the terms do affect convergence of the series.

Example 6.2.2. Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series, we know that $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k^2} \right|$ converges, so $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges absolutely.

Example 6.2.3. Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$. Since $\sum_{k=1}^{\infty} \frac{1}{k}$ is a divergent p-series, we know that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ does not converge absolutely. However, we will learn below (after the Alternating Series Test) that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ does converge. Since this series is convergent but not absolutely convergent, it is conditionally convergent.

Theorem 6.2.4. (Direct Comparison Test) *Suppose that* $\sum_{k=1}^{\infty} a_k$ *is a series of nonnegative terms and that* $\sum_{k=1}^{\infty} b_k$ *is any series.*

- *If* $\sum_{k=1}^{\infty} a_k$ *converges and* $|b_k| \leq a_k$ *for all* k*, then* $\sum_{k=1}^{\infty} b_k$ *converges absolutely.*

- *If* $\sum_{k=1}^{\infty} a_k$ *diverges and* $a_k \leq b_k$ *for all* k*, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

Proof. Suppose first that $\sum_{k=1}^{\infty} a_k$ converges and $|b_k| \leq a_k$ for all k. Let $\langle s_n \rangle$ be the sequence of partial sums for $\sum_{k=1}^{\infty} a_k$ and let $\langle t_n \rangle$ be the sequence of partial sums for $\sum_{k=1}^{\infty} |b_k|$ (note the absolute values). Since $a_k \geq 0$ and $|b_k| \geq 0$ for all k, $\langle s_n \rangle$ and $\langle t_n \rangle$ are increasing sequences. Since $\langle s_n \rangle$ converges, $\langle s_n \rangle$ is bounded. Since $|b_k| \leq a_k$ for all k, we know that $t_n \leq s_n$ for all n. This implies that $\langle t_n \rangle$ is also bounded. As a bounded increasing sequence, $\langle t_n \rangle$ must converge. Hence, $\sum_{k=1}^{\infty} |b_k|$ converges, so $\sum_{k=1}^{\infty} b_k$ converges absolutely.

Now suppose that $\sum_{k=1}^{\infty} a_k$ diverges and $a_k \leq b_k$ for all k. Again, let $\langle s_n \rangle$ be the sequence of partial sums for $\sum_{k=1}^{\infty} a_k$ and let $\langle t_n \rangle$ be the sequence of partial sums for $\sum_{k=1}^{\infty} b_k$ (note the lack of absolute values). Since $0 \leq a_k \leq b_k$ for all k, we know that that $s_n \leq t_n$ for all n. Since $\sum_{k=1}^{\infty} a_k$ diverges, $\langle s_n \rangle$ diverges. As a divergent increasing sequence, $\langle s_n \rangle$ must be unbounded. Since $s_n \leq t_n$ for all n, the sequence $\langle t_n \rangle$ is also unbounded and cannot converge. Thus $\sum_{k=1}^{\infty} b_k$ diverges. □

Remark 6.2.5. Note in the Comparison Test, we do not really need that $|b_k| \leq a_k$ for all k. We just need this inequality to hold *eventually*. That is, we only need that $|b_k| \leq a_k$ for all $k \geq N$ for some N. This is enough to bound the partial sums since the first few terms of the series cannot affect whether or not the partial sums are bounded.

We commented above that convergence of nonnegative series amounts to a consideration of "bigness." In the Comparison Test, if $\sum_{k=1}^{\infty} a_k$ converges, then the partial sums of the series stay small. If $|b_k| \leq a_k$, then the partial sums of $\sum_{k=1}^{\infty} |b_k|$ must also stay small. If $\sum_{k=1}^{\infty} a_k$ diverges, then the partial sums get large. If $a_k \leq b_k$ then the partial sums for $\sum_{k=1}^{\infty} b_k$ are even larger.

We can use the Comparison Test to prove that absolutely convergent series are convergent.

Theorem 6.2.6. *If* $\sum_{k=1}^{\infty} a_k$ *converges absolutely, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

Proof. Suppose that $\sum_{k=1}^{\infty} a_k$ converges absolutely. For every k, we know

$$-|a_k| \leq a_k \leq |a_k|.$$

Adding $|a_k|$ gives

$$0 \leq a_k + |a_k| \leq 2|a_k|.$$

Now, since $\sum_{k=1}^{\infty} |a_k|$ converges, Theorem 6.1.18 tells us that $\sum_{k=1}^{\infty} 2|a_k|$ also converges. The Comparison Test tells us that $\sum_{k=1}^{\infty}(a_k + |a_k|)$ converges absolutely. However, for all k, $a_k + |a_k| \geq 0$ so this really means that $\sum_{k=1}^{\infty}(a_k + |a_k|)$ converges. (The absolute values do not matter for nonnegative terms.)

We now know that $\sum_{k=1}^{\infty} |a_k|$ and $\sum_{k=1}^{\infty}(a_k + |a_k|)$ both converge. By Theorem 6.1.18 again, we know that their difference converges. But their difference is:

$$\sum_{k=1}^{\infty}(a_k + |a_k|) - \sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty}(a_k + |a_k| - |a_k|) = \sum_{k=1}^{\infty} a_k.$$

Thus $\sum_{k=1}^{\infty} a_k$ converges. □

Example 6.2.7. Consider the series $\sum_{k=1}^{\infty} \frac{2^k}{5^k + k^5}$. Since $5^k + k^5 > 5^k$, we know that $\frac{2^k}{5^k + k^5} < \frac{2^k}{5^k} = \left(\frac{2}{5}\right)^k$. Since $\sum_{k=1}^{\infty} \left(\frac{2}{5}\right)^k$ is a convergent geometric series, the series $\sum_{k=1}^{\infty} \frac{2^k}{5^k + k^5}$ converges (absolutely, but the series is positive) by the Direct Comparison Test.

Example 6.2.8. Consider the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2k-1}$. Since $2k - 1 < 2k$, we know that $\displaystyle\frac{1}{2k-1} > \frac{1}{2k}$. The series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2k}$ is a constant multiple of (part of) the Harmonic Series. As such, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2k}$ diverges. By the Direct Comparison Test, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2k-1}$ must also diverge.

Example 6.2.9. Consider the series $\displaystyle\sum_{k=2}^{\infty} \frac{k^2}{k^3-1}$. Since $k^3 - 1 < k^3$, we have $\displaystyle\frac{k^2}{k^3-1} > \frac{k^2}{k^3} = \frac{1}{k}$. Since $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k}$ is a divergent p-series, $\displaystyle\sum_{k=2}^{\infty} \frac{k^2}{k^3-1}$ diverges by the Direct Comparison Test.

Example 6.2.10. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$. Since $|\sin(k)| \leq 1$ for all k, we know that $\displaystyle\left|\frac{\sin(k)}{k^2}\right| \leq \frac{1}{k^2}$ for all k. Since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series, the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ converges absolutely.

Example 6.2.11. Consider the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2-1}$. Since $k^2 - 1 < k^2$, we know that $\displaystyle\frac{1}{k^2-1} > \frac{1}{k^2}$. But wait! $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges, so all we know about the series in question is that it is larger than a convergent series. It may converge, and it may diverge. This inequality is facing the wrong direction. The comparison test gives us no information in this case.

Example 6.2.12. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{2^k+k^2}$. When we try the Direct Comparison Test, we see that $\displaystyle\frac{3^k}{2^k+k^2} < \frac{3^k}{2^k}$. The series $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{2^k}$ is a divergent geometric series. Again, our inequality is facing the wrong direction to continue with the Direct Comparison Test.

As the previous two examples indicate, the Direct Comparison Test does not always work out quite the way we want it to. We have the following adaptation of the Comparison Test that often works when Direct Comparison does not.

Theorem 6.2.13. (Limit Comparison Test) *Suppose that* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *are series of positive terms. If* $\lim \frac{a_k}{b_k}$ *exists and is not* 0*, then* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *both converge or both diverge.*

Proof. Let $c = \lim \frac{a_k}{b_k}$. We apply the definition of convergence of a sequence with $\epsilon = c/2$ (which must be positive). There is a real number N so that for all integers $k > N$ we have $\left|\frac{a_k}{b_k} - c\right| < \frac{c}{2}$. For $k > N$ this implies that

$$-\frac{c}{2} < \frac{a_k}{b_k} - c < \frac{c}{2}.$$

Adding c gives

$$\frac{c}{2} < \frac{a_k}{b_k} < \frac{3c}{2}.$$

This then implies that for $k > N$

$$\frac{c}{2} b_k < a_k < \frac{3c}{2} b_k.$$

Now, if $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} \frac{3c}{2} b_k$ converges. In this case, $\sum_{k=1}^{\infty} a_k$ converges by the Direct Comparison Test. On the other hand, if $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} \frac{c}{2} b_k$ diverges. In this case, $\sum_{k=1}^{\infty} a_k$ diverges by the Direct Comparison Test. □

Example 6.2.14. Let us reconsider Example 6.2.11. Consider the series $\sum_{k=2}^{\infty} \frac{1}{k^2 - 1}$. The -1 on bottom is surely insignificant if k is large.

Therefore, these terms are about the same size as $\frac{1}{k^2}$. Therefore, we do a Limit Comparison with $\sum_{k=1}^{\infty} \frac{1}{k^2}$ (a convergent p-series). We set up the limit of the ratio of terms:

$$\lim \frac{\frac{1}{k^2-1}}{\frac{1}{k^2}} = \lim \frac{k^2}{k^2-1} = 1.$$

Since we arrived at a limit of 1, both series converge or both diverge. Since the second series is a convergent p-series, the series $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ converges.

Example 6.2.15. We revisit the failed Direct Comparison in Example 6.2.12. Consider the series $\sum_{k=1}^{\infty} \frac{3^k}{2^k+k^2}$. We deem the k^2 on the bottom to be a distraction. It is surely dominated by 2^k as k gets large. Therefore, we do a Limit Comparison with $\sum_{k=1}^{\infty} \frac{3^k}{2^k}$ (which is a divergent geometric series). We set up a limit of the ratio of the terms (it does not matter which terms go on top):

$$\lim \frac{\frac{3^k}{2^k+k^2}}{\frac{3^k}{2^k}} = \lim \frac{3^k 2^k}{(2^k+k^2)3^k} = \lim \frac{2^k}{2^k+k^2}.$$

A few applications of L'Hôpital's Rule tell us that this limit is equal to

$$\begin{aligned}
\lim \frac{2^k}{2^k+k^2} &= \lim \frac{2^k \ln 2}{2^k \ln 2 + 2k} \\
&= \lim \frac{2^k \ln 2 \ln 2}{2^k \ln 2 \ln 2 + 2} \\
&= \lim \frac{2^k \ln 2 \ln 2 \ln 2}{2^k \ln 2 \ln 2 \ln 2} \\
&= 1.
\end{aligned}$$

Since we arrived at a limit of 1, both series converge or both diverge. Since the second series is a divergent geometric series, the series $\sum_{k=1}^{\infty} \frac{3^k}{2^k+k^2}$ diverges.

Let $a_k = ar^k$ for some real numbers r and $a \neq 0$. The series $\sum_{k=0}^{\infty} a_k$ is a geometric series. Notice that for all k,

$$\frac{a_{k+1}}{a_k} = r = |a_k|^{1/k}.$$

This series converges if this common value is less than 1 and diverges if it is greater than 1. Considering the same ratios and roots for general series will allow us to compare many series to geometric series. This is the basis for the next two theorems which are our two most powerful tests for convergence.

Theorem 6.2.16. (Ratio Test) *Suppose that* $\sum_{k=1}^{\infty} a_k$ *is a series of nonzero terms and that* $\rho = \lim \left|\frac{a_{k+1}}{a_k}\right|$ *exists.*

1. *If* $\rho < 1$ *then* $\sum_{k=1}^{\infty} a_k$ *converges.*

2. *If* $\rho > 1$ *then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

3. *If* $\rho = 1$*, then this test gives no information.*

Proof. Suppose first that $\rho = \lim \left|\frac{a_{k+1}}{a_k}\right| < 1$. Let $\rho < r < 1$. By the definition of convergence of a sequence, there must be some N so that if $k > N$ then $\left|\frac{a_{k+1}}{a_k}\right| < r$. This implies that if $k > N$ then $|a_{k+1}| < r|a_k|$. Let M be the least integer greater than N. Then we have these inequalities:

$$\begin{aligned}
&|a_{M+1}| < r|a_M| \\
&|a_{M+2}| < r|a_{M+1}| < r^2|a_M| \\
&|a_{M+3}| < r|a_{M+2}| < r^3|a_M| \\
&|a_{M+4}| < r|a_{M+3}| < r^4|a_M| \\
&\quad\vdots
\end{aligned}$$

In general, $|a_{M+k}| < r^k|a_M|$ for $k \geq 1$. Since $0 < r < 1$, the series $\sum_{k=1}^{\infty} a_{M+k} = \sum_{k=M+1}^{\infty} a_k$ converges absolutely by comparison with the geometric series $\sum_{k=1}^{\infty} r^k|a_M|$. Since the first M terms of a series cannot affect convergence, this implies that $\sum_{k=1}^{\infty} a_k$ converges absolutely.

Now suppose $\rho = \lim \left|\frac{a_{k+1}}{a_k}\right| > 1$. This proof is similar to the previous, except we bound the ratios above 1 rather than below 1. Let $\rho > r > 1$. By the definition of convergence of a sequence, there must be some N so that if $k > N$ then $\left|\frac{a_{k+1}}{a_k}\right| > r$. This implies that if $k > N$ then $|a_{k+1}| > r|a_k|$. Let M be the least integer greater than N. Then we have these inequalities:

$$\begin{aligned}
&|a_{M+1}| > r|a_M| \\
&|a_{M+2}| > r|a_{M+1}| < r^2|a_M| \\
&|a_{M+3}| > r|a_{M+2}| < r^3|a_M| \\
&|a_{M+4}| > r|a_{M+3}| < r^4|a_M| \\
&\qquad\qquad \vdots
\end{aligned}$$

In general, $|a_{M+k}| > r^k|a_M|$ for $k \geq 1$. Since $r > 1$, the series $\sum_{k=1}^{\infty} a_{M+k} = \sum_{k=M+1}^{\infty} a_k$ diverges by comparison with the geometric series $\sum_{k=1}^{\infty} r^k|a_M|$. Since the first M terms of a series cannot affect convergence, this implies that $\sum_{k=1}^{\infty} a_k$ diverges.

To demonstrate that the test gives no information if $\rho = 1$, we simply note that the two series

$$\sum_{k=1}^{\infty} \frac{1}{n} \text{ and } \sum_{k=1}^{\infty} \frac{1}{n^2}$$

both have $\rho = 1$. However, the first diverges and the second converges. $\square$

The Ratio Test can be particularly useful for series involving factorials and exponents. When working with factorials, we will regularly use the trick that $(k+1)! = (k+1)k!$.

Example 6.2.17. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2}$. We apply the Ratio Test. If $a_k = \dfrac{(2k)!}{(k!)^2}$, then $a_{k+1} = \dfrac{(2(k+1))!}{((k+1)!)^2}$. Then

$$\begin{aligned}
\rho &= \lim \left|\frac{a_{k+1}}{a_k}\right| \\
&= \lim \frac{\frac{(2(k+1))!}{((k+1)!)^2}}{\frac{(2k)!}{(k!)^2}} \\
&= \lim \frac{(2(k+1))!}{((k+1)!)^2} \frac{(k!)^2}{(2k)!} \\
&= \lim \frac{(2k+2)!}{(k+1)!(k+1)!} \frac{k!k!}{(2k)!} \\
&= \lim \frac{(2k+2)(2k+1)(2k)!}{(k+1)k!(k+1)k!} \frac{k!k!}{(2k)!} \\
&= \lim \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \\
&= 4
\end{aligned}$$

Since $\rho = 4 > 1$ this series diverges.

Example 6.2.18. We apply the ratio test to the series $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(2k)!}$. If

$a_k = \dfrac{2^k k!}{(2k)!}$ then $a_{k+1} = \dfrac{2^{k+1}(k+1)!}{(2(k+1))!}$. Calculating ρ gives

$$\begin{aligned}
\rho &= \lim \left| \frac{a_{k+1}}{a_k} \right| \\
&= \lim \frac{\frac{2^{k+1}(k+1)!}{(2(k+1))!}}{\frac{2^k k!}{(2k)!}} \\
&= \lim \frac{2^k 2(k+1)k!}{(2k+2)(2k+1)(2k)!} \frac{(2k)!}{2^k k!} \\
&= \lim \frac{2(k+1)}{(2k+2)(2k+1)} \\
&= 0
\end{aligned}$$

Since $\rho = 0$, this series converges (absolutely, but the terms are positive).

Example 6.2.19. We apply the Ratio Test to the series

$$\sum_{k=0}^{\infty} \frac{1}{2^{k+(-1)^k}} = \frac{1}{2} + \frac{1}{1} + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \cdots .$$

For this series, $a_k = \dfrac{1}{2^{k+(-1)^k}}$. If k is even, then $k+1$ is odd and

$$\frac{a_{k+1}}{a_k} = \frac{\frac{1}{2^{(k+1)-1}}}{\frac{1}{2^{k+1}}} = \frac{2^{k+1}}{2^{(k+1)-1}} = \frac{2^{k+1}}{2^k} = 2.$$

On the other hand, if k is odd, then $k+1$ is even and

$$\frac{a_{k+1}}{a_k} = \frac{\frac{1}{2^{(k+1)+1}}}{\frac{1}{2^{k-1}}} = \frac{2^{k-1}}{2^{k+2}} = \frac{1}{8}.$$

Since the sequence $\left\langle \left| \dfrac{a_{k+1}}{a_k} \right| \right\rangle$ alternates between 2 and $\dfrac{1}{8}$, the limit required by the Ratio Test cannot exist. The Ratio Test fails.

Theorem 6.2.20. (Root Test) *Suppose that* $\sum_{k=1}^{\infty} a_k$ *is any series and that* $\rho = \lim |a_k|^{1/k}$ *exists.*

1. *If* $\rho < 1$ *then* $\sum_{k=1}^{\infty} a_k$ *converges.*

2. *If* $\rho > 1$ *then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

3. *If* $\rho = 1$*, then this test gives no information.*

Proof. Suppose first that $\rho = \lim |a_k|^{1/k} < 1$. Let $\rho < r < 1$. By the definition of convergence of a sequence, there must be some N so that if $k > N$ then $|a_k|^{1/k} < r$. This implies that if $k > N$ then $|a_k| < r^k$. Since $0 < r < 1$, the series $\sum_{k=1}^{\infty} a_k$ converges absolutely by comparison with the geometric series $\sum_{k=1}^{\infty} r^k$.

Now suppose that $\rho = \lim |a_k|^{1/k} > 1$. Let $\rho > r > 1$. By the definition of convergence of a sequence, there must be some N so that if $k > N$ then $|a_k|^{1/k} > r$. This implies that if $k > N$ then $|a_k| > r^k$. Since $r > 1$, the series $\sum_{k=1}^{\infty} a_k$ diverges by comparison with the geometric series $\sum_{k=1}^{\infty} r^k$.

The two examples from the proof of Theorem 6.2.16 demonstrate that the Root Test can give no information if $\rho = 1$. $\square$

Example 6.2.21. We revisit Example 6.2.19 in which the Ratio Test failed. The series in question is

$$\sum_{k=0}^{\infty} \frac{1}{2^{k+(-1)^k}} = \frac{1}{2} + \frac{1}{1} + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \cdots .$$

Using the ratio test yields:

$$\rho = \lim |a_k|^{1/k} = \lim \left| \frac{1}{2^{k+(-1)^k}} \right|^{1/k} = \lim \left| \frac{1}{2^{(k+(-1)^k)/k}} \right| .$$

The exponent on 2 in the last fraction is either $\dfrac{k+1}{k}$ or $\dfrac{k-1}{k}$ depending on whether k is even or odd. In both cases, as k increases, the exponent approaches 1. Therefore, $\rho = \dfrac{1}{2}$. Since $\rho < 1$, this series converges (absolutely).

The Root Test can be quite useful for series which involve exponents.

Example 6.2.22. Consider the series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k+1}\right)^{2k}$. Applying the Root Test gives

$$\rho = \lim |a_k|^{1/k} = \lim \left|\left(\frac{1}{k+1}\right)^{2k}\right|^{1/k} = \lim \frac{1}{(k+1)^2} = 0.$$

Since $\rho < 1$, The series converges (absolutely).

Example 6.2.23. Consider the series $\displaystyle\sum_{k=1}^{\infty}\frac{(-2)^k}{k^2}$. Applying the Root Test gives

$$\rho = \lim |a_k|^{1/k} = \lim \left|\frac{(-2)^k}{k^2}\right|^{1/k} = \lim \frac{2}{(k^{1/k})^2} = 2.$$

Since $\rho > 1$, The series diverges.

Theorem 6.2.24. (Alternating Series Test) *Suppose that $\langle a_k \rangle$ is a decreasing sequence which converges to 0. The series $\displaystyle\sum_{k=0}^{\infty}(-1)^k a_k$ converges.*

Proof. The situation described in this proof is depicted in Figure 6.3. Let $\langle s_n \rangle$ be the sequence of partial sums of $\displaystyle\sum_{k=0}^{\infty}(-1)^k a_k$. We consider the even indexed terms s_{2n} and the odd indexed terms s_{2n+1} of $\langle s_n \rangle$ separately. Note that

$$\begin{aligned} s_{2n+3} &= (a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} - a_{2n+1}) + (a_{2n+2} - a_{2n+3}) \\ &= s_{2n+1} + (a_{2n+2} - a_{2n+3}). \end{aligned}$$

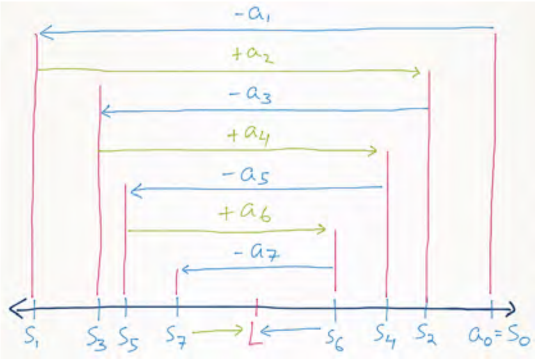

Figure 6.3: The odd terms of $\langle s_n \rangle$ form an increasing sequence which approaches a limit L from the left. The even terms form a decreasing sequence which approaches L from the right. Note that the distance between any term s_n and L is less than a_{n+1}.

Since $\langle a_k \rangle$ is decreasing, $(a_{2n+2} - a_{2n+3}) \geq 0$. This means that for all n we have $s_{2n+1} \leq s_{2n+3}$, so $\langle s_{2n+1} \rangle$ is an increasing sequence. Also

$$s_{2n+1} = a_0 - (a_1 - a_2) - (a_3 - a_4) + \cdots - (a_{2n-1} - a_{2n}) - a_{2n+1}$$

Since the differences in parenthesis are nonnegative, $s_{2n+1} \leq a_0$ for all n. Thus, $\langle s_{2n+1} \rangle$ is an increasing sequence which is bounded above. As such, $\langle s_{2n+1} \rangle$ converges to a number L.

On the other hand, for all $n \geq 1$, $s_{2n} = s_{2n-1} + a_{2n}$. Since $\lim s_{2n-1} = L$ and $\lim a_n = 0$, we have

$$\lim s_{2n} = \lim s_{2n-1} + \lim a_{2n} = L + 0 = L.$$

Since the odd terms of $\langle s_n \rangle$ converge to L and the even terms also converge, to L, we conclude that $\lim s_n = L$ and that the series converges. $\square$

Example 6.2.25. The Alternating Harmonic Series Since $\left\langle \dfrac{1}{k} \right\rangle$ is decreasing and converges to 0, the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by the Alternating Series Test. Note that since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the alternating harmonic series converges conditionally. (If you are wondering, this series converges to $\ln(2)$. We will have tools to approach this limit after our discussion of power series.)

Example 6.2.26. Consider the series $\sum_{k=1}^{\infty} \frac{(-k)^k}{k+1}$. The signs of this series do alternate, however $\langle |a_k| \rangle$ is unbounded, so the Alternating Series Test does not apply. This series diverges by the Term Test.

Example 6.2.27. Consider the series $\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{k!}$. The expression $\cos(k\pi)$ is just a fancy way of making the series alternate. The value of $\cos(k\pi)$ alternates between 1 and -1. The absolute values of the terms decrease toward 0. This series converges by the Alternating Series Test.

Exercises 6.2

6.2.1 Determine if each of these series converges or diverges. Support your answers.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k+\sqrt{k}}$
2. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k)}{k^3}$
3. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k}{2k+1}\right)^k$
4. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$
5. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}}$
6. $\displaystyle\sum_{k=0}^{\infty} \frac{2k+3^k}{3^k+4^k}$
7. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{2^k}$
8. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k+1)^k}$
9. $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{(2k)!}$
10. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$
11. $\displaystyle\sum_{k=0}^{\infty} \frac{\cos(k\pi)}{\sqrt{k}+1}$
12. $\displaystyle\sum_{k=0}^{\infty} \frac{(k+3)!}{k!+3!}$
13. $\displaystyle\sum_{k=1}^{\infty} e^{-k}k^{-e}$
14. $\displaystyle\sum_{k=1}^{\infty} \frac{e^k}{k^e}$

6.2.2 Suppose that $\sum_{k=1}^{\infty} a_k$ converges absolutely and that $\langle b_k \rangle$ is bounded. Prove that $\sum_{k=1}^{\infty}(a_k b_k)$ converges absolutely.

6.2.3 Suppose that $\sum_{k=1}^{\infty} a_k$ is a convergent series of nonnegative terms. Prove that $\sum_{k=1}^{\infty} a_k^2$ converges.

6.2.4 Suppose that $\langle c_k \rangle$ is a decreasing sequence that converges to 0. Let s be the sum of the series $\sum_{k=0}^{\infty}(-1)^k c_k$, and let $\langle s_n \rangle$ be the sequence of partial sums of this series. Derive from the proof of the Alternating Series Test an upper bound for $|s_n - s|$.

6.2.5 Suppose $\sum_{k=1}^{\infty} a_k$ converges absolutely. Prove this infinite version of the Triangle Inequality:

$$\left|\sum_{k=1}^{\infty} a_k\right| \leq \sum_{k=1}^{\infty} |a_k|.$$

6.3 Power Series

We begin this section with an example that motivates the topic we are about to study.

Example 6.3.1. Consider the geometric series $\sum_{k=0}^{\infty} x^k$ which includes a variable x. There are two obvious questions to ask about this series:

1. For which values of x does the series converge?

2. Where the series does converge, what does it converge to?

Our knowledge of geometric series allows us to answer these questions quickly. The series converges exactly for those x with $|x| < 1$, and it converges to $\frac{1}{1-x}$. Thus, we might write

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \cdots \text{ for } x \in (-1,1).$$

Thus, at least on the interval $(-1, 1)$, we can replace the function $\frac{1}{1-x}$ with the "infinite polynomial" $1 + x + x^2 + x^3 + x^4 + \cdots$. Polynomials are easy to do calculus with. If these infinite versions of polynomials are just as easy to do calculus with, then it may be that we can replace some complicated functions with these special series in order to simplify calculations.

Definition 6.3.2. A series of the form

$$\sum_{k=m}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

is a *power series around* 0.

Remark 6.3.3. In these sections we isolate our attention to power series around 0. It should be noted that all of the results we approach can easily be extended to power series around a for any real number a. These power series look like:

$$\sum_{k=m}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Example 6.3.4. Consider the power series $\sum_{k=0}^{\infty} k! x^k$. If $x = 0$, then for $k > 0$ all of the terms of this series are 0, and the series converges (this is common to all of the series that we will look at). On the other hand, if $x \neq 0$, then the terms of this series are unbounded, so the series diverges by the Term Test. This power series converges only for $x = 0$.

Example 6.3.5. Consider the power series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. To determine where this power series converges, we apply the Ratio Test. First,

$$\rho = \lim \left| \frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}} \right| = \lim \left| \frac{x^{k+1} k!}{(k+1)k! x^k} \right| = \lim \frac{x}{k+1} = 0.$$

Since $\rho = 0 < 1$ no matter what x is, this series converges for all x by the Ratio Test. Moreover, because we are using the Ratio Test, the series converges everywhere absolutely.

Example 6.3.6. Consider the power series $\sum_{k=1}^{\infty} \frac{x^k}{k2^k}$. To determine where this power series converges, we will apply the Root Test. First

$$\rho = \lim \left| \frac{x^k}{k2^k} \right|^{1/k} = \lim \frac{|x|}{k^{1/k}2} = \frac{|x|}{2}.$$

For this series to converge, we need $\rho < 1$. This happens if $\frac{|x|}{2} < 1$ or if $-2 < x < 2$. On the interval $(-2, 2)$ the series converges absolutely. At $x = \pm 2$, $\rho = 1$, and the Root Test gives no information. We have to consider these values of x separately. At $x = 2$, the series is $\sum_{k=1}^{\infty} \frac{1}{k}$, which diverges. At $x = -2$, this series is $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, which converges. Therefore, our original power series converges on the interval $[-2, 2)$.

Example 6.3.7. Consider the power series $\sum_{k=1}^{\infty} \frac{x^k}{k}$. We apply the Root Test to determine where this series converges. First, we find ρ:

$$\rho = \lim \left| \frac{x^k}{k} \right|^{1/k} = \lim \frac{|x|}{k^{1/k}} = |x|.$$

The series converges absolutely when $|x| = \rho < 1$ – which is on the interval $(-1, 1)$. We consider the endpoints of this interval separately. At $x = 1$, this series is $\sum_{k=1}^{\infty} \frac{1}{k}$. This is the harmonic series, which diverges. At $x = -1$, the series is $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$. This is the alternating harmonic series, which converges. Our series converges for x in the interval $[-1, 1)$.

Notice in these examples each power series may converge everywhere, at a point, or on a bounded interval (which might be open, closed, or half open and half closed). This is typical for power series. We will prove this in the next theorem. First, we need a lemma.

Lemma 6.3.8. *If the power series* $\sum_{k=0}^{\infty} c_k x^k$ *converges for* $x = b \neq 0$, *then the power series converges absolutely for all* x *with* $|x| < |b|$.

Proof. Since $\sum_{k=0}^{\infty} c_k b^k$ converges, we know that $\lim \left(c_k b^k\right) = 0$. This implies that there is a real number N so that if $k > N$ then $|c_k b^k| < 1$. Then $|c_k| < \dfrac{1}{|b^k|}$ for $k > N$. Suppose now that $|x| < |b|$. Then $\left|\dfrac{x}{b}\right| < 1$ so the geometric series $\sum_{k=0}^{\infty} \left|\dfrac{x}{b}\right|^k$ converges. Since $|c_k| < \dfrac{1}{|b^k|}$ for $k > N$, then

$$|c_k x^k| = |c_k||x^k| < \frac{|x^k|}{|b^k|} = \left|\frac{x}{b}\right|^k$$

for $k > N$. It follows that $\sum_{k=0}^{\infty} c_k x^k$ converges absolutely by the Comparison Test. $\square$

Suppose now that $\sum_{k=0}^{\infty} c_k x^k$ is any power series. Let S be the collection of $x \in \mathbb{R}$ for which the series converges absolutely. Note that S is not empty since the power series converges absolutely at $x = 0$. It may be that S is unbounded above. If this is the case, then it has to be that $S = \mathbb{R}$. To see this, suppose that $a \in \mathbb{R}$. Since S is unbounded above, there is a $b \in S$ with $|a| < b$. Since $b \in S$, the series converges for $x = b$. By the Lemma, the series must converge absolutely for $x = a$. Thus, $\mathbb{R} \subseteq S \subseteq \mathbb{R}$ so it must be that $S = \mathbb{R}$.

It might also be that S is bounded above. By the Completeness Axiom, $R = \sup S$ exists in $\mathbb{R}$. Since $0 \in S$, $R \geq 0$. If $R = 0$, then the power series converges only at $x = 0$. Suppose that $R > 0$. We claim that $\sum_{k=0}^{\infty} c_k x^k$ converges absolutely for $x = a$ where $|a| < R$ and diverges for $x = b$ where $|b| > R$. Suppose that $|a| < R$. Then there is a $b \in S$ with $|a| < b < R$. Since $b \in S$, $\sum_{k=0}^{\infty} c_k x^k$ converges at $x = b$. By the Lemma, the series converges absolutely for $x = a$. Now suppose by way of contradiction $|b| > R$ and that $\sum_{k=0}^{\infty} c_k x^k$ converges for $x = b$. Since $|b| > R$, there is a real number a with $R < a < |b|$. By the Lemma, $\sum_{k=0}^{\infty} c_k x^k$ converges absolutely for $x = a$ so $a \in S$. However, this contradicts the fact that $R = \sup S$. Thus, if $|b| > R$, then $\sum_{k=0}^{\infty} c_k x^k$ must diverge at $x = b$. Thus we have:

Theorem 6.3.9. *Suppose that* $\sum_{k=0}^{\infty} c_k x^k$ *is any power series. Exactly one of these three statements is true:*

1. *The series converges only for* $x = 0$.
2. *There is a real number* $R > 0$ *so that the series converges absolutely when* $|x| < R$ *and diverges when* $|x| > R$.
3. *The series converges absolutely for all real numbers.* $\square$

Definition 6.3.10. In case (1) of Theorem 6.3.9 we say that the power series has a radius of convergence of 0. In case (2), we say the series has a radius of convergence R. In case (3), the radius of convergence of the series is ∞. In (2) and (3) the set of all x for which the power series converges is called the *interval of convergence* of the power series.

Example 6.3.11. To find the radius and interval of convergence of a power series, we first apply either the Root Test or Ratio Test. We then check the endpoints individually.

1. In Example 6.3.1 the interval of convergence is $(-1, 1)$. The radius of convergence is 1.
2. In Example 6.3.4 the radius of convergence is 0.
3. In Example 6.3.5 the interval of convergence is $(-\infty, \infty)$. The radius of convergence is ∞.
4. In Example 6.3.6 the interval of convergence is $[-2, 2)$. The radius of convergence is 2.
5. In Example 6.3.7 the interval of convergence is $[-1, 1)$. The radius of convergence is 1.

Exercises 6.3

6.3.1 Find the interval of convergence of each of these power series.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{(3x)^k}{k}$
2. $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^3 3^k}$
3. $\displaystyle\sum_{k=0}^{\infty} \frac{2^k x^k}{k!}$
4. $\displaystyle\sum_{k=0}^{\infty} \frac{(-x)^k}{\sqrt{k}+1}$
5. $\displaystyle\sum_{k=0}^{\infty} \frac{kx^k}{3^k}$
6. $\displaystyle\sum_{k=0}^{\infty} \frac{kx^k}{k+1}$

6.3.2 Find the interval of convergence of each of these power series.

1. $\displaystyle\sum_{k=0}^{\infty} (3x-2)^k$
2. $\displaystyle\sum_{k=0}^{\infty} \frac{(x+1)^k}{2k}$

6.3.3 Suppose that $\displaystyle\sum_{k=0}^{\infty} c_k x^k$ has a finite radius of convergence R, that $c_k \geq 0$ for all k, and that the power series converges at $x = R$. Prove that the power series converges also at $x = -R$.

6.3.4 Suppose that $\displaystyle\sum_{k=0}^{\infty} c_k x^k$ is a power series so that $\lim \left| \dfrac{c_{k+1}}{c_k} \right|$ and $\lim |c_k|^{1/k}$ both exist. Prove these limits are equal.

6.4 Properties of Power Series

Consider the power series $\displaystyle\sum_{k=0}^{\infty} c_k x^k$. On the interior of its interval of convergence, this power series converges to a nice function which we will call f for this discussion. It is continuous (and, hence, integrable on bounded intervals) and differentiable. Moreover, derivatives and integrals of power series are as easy to compute as derivatives and integrals of polynomials. In this section, we prove that if

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

then on the interior of its interval of convergence $f(x)$ is continuous and differentiable and

$$f'(x) = \sum_{k=1}^{\infty} kc_k x^{k-1}.$$

We will call this series the *term-by-term derivative* of $f(x)$. Moreover, on the interior of the interval of convergence of $f(x)$ this power series

$$\sum_{k=0}^{\infty} \frac{c_k}{k+1} x^{k+1}$$

converges to a function whose derivative is $f(x)$. We will call this series the *term-by-term integral* of $f(x)$. We prove these statements in this section. We begin with the continuity of $f(x)$. First, we need a lemma that says that we can bound the size of the tail of a power series across any bounded closed interval on which it converges.

Lemma 6.4.1. *Suppose that $R > 0$ is less than the radius of convergence of the power series $\sum_{k=0}^{\infty} c_k x^k$. For every $\epsilon > 0$ there is an integer $N > 0$ so that if $n > N$ and $x \in [-R, R]$ then $\left|\sum_{k=n}^{\infty} c_k x^k\right| < \epsilon$*

Proof. Since R is less than the radius of convergence of $\sum_{k=0}^{\infty} c_k x^k$, we know that $\sum_{k=0}^{\infty} |c_k R^k|$ converges to a number L. This implies that there is an integer N so that the partial sum $\sum_{k=0}^{N} |c_k R^k|$ is within ϵ of L. Then

$$\sum_{k=N+1}^{\infty} |c_k R^k| = \left|\sum_{k=0}^{\infty} |c_k R^k| - \sum_{k=0}^{N} |c_k R^k|\right| = \left|L - \sum_{k=0}^{N} |c_k R^k|\right| < \epsilon.$$

Now suppose $n > N$ and $|x| \leq R$. Then

$$\left|\sum_{k=n}^{\infty} c_k x^k\right| \leq \sum_{k=n}^{\infty} |c_k x^k| \leq \sum_{k=N+1}^{\infty} |c_k x^k| \leq \sum_{k=N+1}^{\infty} |c_k R^k| < \epsilon.$$

□

Now we can prove that any power series with a positive radius of convergence converges to a continuous function.

Theorem 6.4.2. *Any power series converges to a continuous function on the interior of its interval of convergence.*

Proof. Suppose that z is in the interior of the interval of convergence of $f(x) = \sum_{k=0}^{\infty} c_k x^k$. We will prove that f is continuous at z. There is an $R > 0$ which is less than the radius of convergence of the power series so that $z \in [-R, R]$. Let $\epsilon > 0$. By Lemma 6.4.1 there is an N so that $\left| \sum_{k=N+1}^{\infty} c_k x^k \right| < \epsilon/3$ for all $x \in [-R, R]$. Let $p(x) = \sum_{k=0}^{N} c_k x^k$. Note that $p(x)$ is simply a polynomial. Also note that on $[-R, R]$ we have

$$\begin{aligned} |f(x) - p(x)| &= \left| \sum_{k=0}^{\infty} c_k x^k - \sum_{k=0}^{N} c_k x^k \right| \\ &= \left| \sum_{k=N+1}^{\infty} c_k x^k \right| \\ &< \epsilon/3 \end{aligned}$$

Since p is a polynomial, p is continuous at z. Therefore, there is a $\delta > 0$ so that if $x \in [-R, R]$ and if $|x - z| < \delta$, then $|p(x) - p(z)| < \epsilon/3$. Now, suppose that $x \in [-R, R]$ and if $|x - z| < \delta$. Then

$$\begin{aligned} |f(x) - f(z)| &= |f(x) - p(x) + p(x) - p(z) + p(z) - f(z)| \\ &\leq |f(x) - p(x)| + |p(x) - p(z)| + |p(z) - f(z)| \\ &= |f(x) - p(x)| + |p(x) - p(z)| + |f(z) - p(z)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon. \end{aligned}$$

Thus f is continuous at z. This is true for all $z \in [-R, R]$. □

The power series $f(x)$ is also continuous at the endpoints of its interval of convergence if it converges at the endpoints. This is known as *Abel's Theorem.*

Theorem 6.4.3. (Abel's Theorem) *Any power series with a finite radius of convergence which converges at an endpoint of its interval of convergence is continuous at that endpoint.*

Proof. Suppose first that $f(x) = \sum_{k=0}^{\infty} c_k x^k$ has a radius of convergence of 1 and that f converges at 1. We show f is continuous at $x = 1$. Since $f(x)$ converges at $x = 1$, the series $\sum_{k=0}^{\infty} c_k$ converges to some number $L = f(1)$. Let $\langle s_n \rangle$ be the sequence of partial sums of $\sum_{k=0}^{\infty} c_k$. Let $x \in (0, 1)$. Since $s_0 = c_0$ and since $c_k = s_k - s_{k-1}$, we have

$$\begin{aligned}
\sum_{k=0}^{n} c_k x^k &= c_0 + \sum_{k=1}^{n} c_k x^k \\
&= s_0 + \sum_{k=1}^{n} (s_k - s_{k-1}) x^k \\
&= s_0 + \sum_{k=1}^{n} s_k x^k - \sum_{k=1}^{n} s_{k-1} x^k \\
&= s_0 + \sum_{k=1}^{n} s_k x^k - x \sum_{k=1}^{n} s_{k-1} x^{k-1} \\
&= s_0 + \sum_{k=1}^{n} s_k x^k - x \sum_{k=0}^{n-1} s_k x^k \\
&= s_0 + \sum_{k=1}^{n-1} s_k x^k + s_n x^n - x \sum_{k=1}^{n-1} s_k x^k - s_0 x \\
&= (1 - x) s_0 + s_n x^n + \sum_{k=1}^{n-1} s_k (1 - x) x^k \\
&= s_n x^n + \sum_{k=0}^{n-1} s_k (1 - x) x^k
\end{aligned}$$

For every n we have

$$\sum_{k=0}^{n} c_k x^k = s_n x^n + \sum_{k=0}^{n-1} s_k (1 - x) x^k.$$

Taking the limit we now have

$$f(x) = \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} s_k(1-x)x^k.$$

Since $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$, we have $1 = \displaystyle\sum_{k=0}^{\infty}(1-x)x^k$ so

$$f(1) = L = L \cdot 1 = L\sum_{k=0}^{\infty}(1-x)x^k = \sum_{k=0}^{\infty} L(1-x)x^k.$$

Now we consider the difference $f(1) - f(x)$ (because we want f to be continuous at 1):

$$f(1) - f(x) = \sum_{k=0}^{\infty}(L - s_k)(1-x)x^k.$$

Let $\epsilon > 0$. Since $\lim s_n = L$, there is a real number N so that if $n > N$ then $|L - s_n| < \epsilon/2$. Then

$$\begin{aligned}
|f(1) - f(x)| &= \left|\sum_{k=0}^{\infty}(L - s_k)(1-x)x^k\right| \\
&= \left|\sum_{k=0}^{N}(L - s_k)(1-x)x^k + \sum_{k=N+1}^{\infty}(L - s_k)(1-x)x^k\right| \\
&\le \sum_{k=0}^{N}|L - s_k|(1-x)x^k + \sum_{k=N+1}^{\infty}|L - s_k|(1-x)x^k \\
&\le \sum_{k=0}^{N}|L - s_k|(1-x)x^k + \sum_{k=N+1}^{\infty}\frac{\epsilon}{2}(1-x)x^k \\
&< \sum_{k=0}^{N}|L - s_k|(1-x)x^k + \frac{\epsilon}{2}.
\end{aligned}$$

The sum $\displaystyle\sum_{k=0}^{N}|L - s_k|(1-x)x^k$ is a polynomial. Call it h. Then for all $x \in (0,1)$ we have $|f(1) - f(x)| < h(x) + \epsilon/2$. Note that $h(1) = 0$.

Since h is a polynomial, it is continuous at $x = 1$, and there is a $\delta > 0$ so that if $x \in (1 - \delta, 1)$ then

$$0 \leq h(x) = |h(x) - h(0)| < \frac{\epsilon}{2}.$$

It follows than that if $x \in (1 - \delta, 1)$ then

$$|f(1) - f(x)| < h(x) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, f is continuous at $x = 1$.

So far, we have proven any power series with a radius of convergence of 1 which converges at 1 is continuous at 1. Suppose now that f is a power series with a radius of convergence of 1 which converges at -1. Then $f(-x)$ is a power series with a radius of convergence of 1 which converges at $x = 1$. As such, $f(-x)$ is continuous at $x = 1$. It follows that $f(x)$ is continuous at $x = -1$.

We now have that any power series with a radius of convergence of 1 which converges at an endpoint of its interval of convergence must be continuous at that endpoint. Suppose that f is a power series with a finite radius of convergence $R > 0$. Then $f(x/R)$ is a power series with a radius of convergence of 1. If f converges at an endpoint, then $f(x/R)$ converges at the corresponding endpoint and is continuous there. It follows that f is also continuous at the endpoint in question.

We have (finally) observed that any power series with a finite radius of convergence which converges at an endpoint of its interval of convergence is continuous at that endpoint. $\square$

We turn now to derivatives and integrals. We must first prove that the series claimed at the beginning of the section to give the derivative and integral of a power series actually converge where we want them to. First, we approach the term-by-term integral.

Lemma 6.4.4. *The power series*

$$g(x) = \sum_{k=0}^{\infty} \frac{c_k}{k+1} x^{k+1}$$

converges on the interior of the interval of convergence of the power series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Proof. Note that for any x, the series $g(x)$ and $G(x) = \sum_{k=0}^{\infty} \frac{c_k}{k+1} x^k$ are multiples of each other so both converge or both diverge. We need only prove that $G(x)$ converges on the interior of the interval of convergence of $f(x)$. If z is in the interior of the interval of convergence of $f(x)$ then $f(x)$ converges absolutely at $x = z$. This means that the series $\sum_{k=0}^{\infty} \left|c_k z^k\right|$ converges. Since $\left|\frac{c_k}{k+1} z^k\right| \leq \left|c_k z^k\right|$, it follows that $G(x)$ converges (absolutely) at $x = z$ by the Comparison Test. As stated above, this implies that $g(x)$ must also converge for $x = z$. Thus, $g(x)$ converges at every point of the interior of the interval of convergence of $f(x)$. $\square$

Lemma 6.4.5. *The power series*

$$h(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

converges on the interior of the interval of convergence of the power series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Proof. Note that for any x, $h(x)$ and $H(x) = \sum_{k=1}^{\infty} k c_k x^k$ are multiples of each other and therefore both converge or both diverge. We need only show that $H(x)$ converges for all x in the interior of the interval of convergence of $f(x)$. Let z be in the interior of the interval of convergence of $f(x)$. Then there is some $R > 0$ in the interior of the interval of convergence of $f(x)$ with $z \in (-R, R)$. Since R is in the interior of the interval of convergence of $f(x)$, it must be that $f(x)$ converges absolutely for $x = R$. This means that $\sum_{k=0}^{\infty} \left|c_k R^k\right|$ converges so $\lim \left|c_k R^k\right| = 0$. This implies that there is some N so that if $k > N$ then $\left|c_k R^k\right| < 1$. It follows that if $k > N$ then $|c_k| < \frac{1}{R^k}$.

For each $k > N$,

$$\left|k c_k z^k\right| \leq \left|k \frac{1}{R^k} z^k\right| = k \left|\frac{z}{R}\right|^k.$$

The series $\sum_{k=1}^{\infty} k \left|\frac{z}{R}\right|^k$ converges by the Root Test since

$$\lim \left| k \left|\frac{z}{R}\right|^k \right|^{1/k} = \lim \left| k^{1/k} \frac{z}{R} \right| = \left|\frac{z}{R}\right| < 1.$$

The series $H(x)$ then converges (absolutely) at $x = z$ by the Comparison Test. As stated above, this implies that $h(x)$ must also converge for $x = z$. Thus, $h(x)$ converges at every point of the interior of the interval of convergence of $f(x)$. $\square$

Suppose that f, g, and h are as in the previous two Lemmas. The series $g(x)$ is the term-by-term integral of $f(x)$. Lemma 6.4.4 says that the term-by-term integral of a power series converges on the interior of the interval of convergence of the power series. The series $h(x)$ is the term-by-term derivative of $f(x)$. Lemma 6.4.5 says that the term-by-term derivative of a power series converges on the interior of the interval of convergence of the power series. Let I be in interior of the interval of convergence of $f(x)$. Let J be the interior of the interval of convergence of $g(x)$, and let K be the interior of the interval of convergence of $h(x)$. Lemma 6.4.4 implies that $I \subseteq J$. Lemma 6.4.5 implies that $I \subseteq K$. Now, $f(x)$ is the term-by-term derivative of $g(x)$. By 6.4.5 $f(x)$ must converge on the interior of the interval of convergence of $g(x)$. This means that $J \subseteq I$. Similarly, $f(x)$ is the term-by-term integral of $h(x)$. By 6.4.4 $f(x)$ must converge on the interior of the interval of convergence of $h(x)$. This means $K \subseteq I$. Putting this all together, we have $I = J = K$. This establishes:

Theorem 6.4.6. *The power series*

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \text{ and } g(x) = \sum_{k=0}^{\infty} \frac{c_k}{k+1} x^{k+1} \text{ and } h(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

all have the same radius of convergence. $\square$

Now that we know our term-by-term derivatives and integrals have the same radius of convergence as our original power series, we can actually prove they give use the derivative and integral of our power series. First we approach integrals. In the proof of this theorem, we will use Calculus I notation $\int_a^b f(x)dx$ for integration to make it easier to keep track of which variable is the variable of integration.

Theorem 6.4.7. *Suppose that* $f(x) = \sum_{k=0}^{\infty} c_k x^k$ *is any power series and that* z *is in the interior of the interval of convergence of* f*. Then*

$$\int_0^z f(t)dt = \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1}.$$

Proof. Let z be in the interior of the interval of convergence of f. We will address the case when $z > 0$. There is an $R > 0$ less than the radius of convergence of f so that $z \in [-R, R]$. Note that by Theorem 6.4.2 f is continuous on $[-R, R]$, so $\int_0^z f(t)dt$ exists. We want to prove that

$$\lim \sum_{k=0}^{n} \frac{c_k}{k+1} z^{k+1} = \int_0^z f(t)dt.$$

We will use the definition of the limit of a sequence. Let $\epsilon > 0$. By Lemma 6.4.1 there is an N so that $\left|\sum_{k=n}^{\infty} c_k x^k\right| < \epsilon/R$ for $n > N$ and for $x \in [-R, R]$. Let $n > N$ then

$$\begin{aligned}
\left|\sum_{k=0}^{n} \frac{c_k}{k+1} z^{k+1} - \int_0^z f(t)dt\right| &= \left|\sum_{k=0}^{n} \int_0^z c_k t^k dt - \int_0^z f(t)dt\right| \\
&= \left|\int_0^z \sum_{k=0}^{n} c_k t^k dt - \int_0^z f(t)dt\right| \\
&= \left|\int_0^z \left(\sum_{k=0}^{n} c_k t^k - f(t)\right) dt\right| \\
&= \left|\int_0^z \left(\sum_{k=0}^{n} c_k t^k - \sum_{k=0}^{\infty} c_k t^k\right) dt\right| \\
&= \left|\int_0^z \left(-\sum_{k=n+1}^{\infty} c_k t^k\right) dt\right| \\
&\le \int_0^z \left|\sum_{k=n+1}^{\infty} c_k t^k\right| dt \\
&\le \int_0^z \epsilon/R dt \\
&= z\epsilon/R \\
&\le \epsilon.
\end{aligned}$$

This last inequality follows from the fact that $z \leq R$. By the definition of convergence of a sequence,

$$\sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1} = \lim \sum_{k=0}^{n} \frac{c_k}{k+1} z^{k+1} = \int_0^z f.$$

□

Theorem 6.4.8. *Suppose that* $f(x) = \sum_{k=0}^{\infty} c_k x^k$ *is any power series. Then* f *is differentiable on the interior of its interval of convergence, and for* x *in the interior of* f*'s interval of convergence*

$$f'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}.$$

Proof. Consider the power series $g(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}$. By Theorem 6.4.7 (with some reindexing) we know that for x in the interior of the interval of convergence of g (which is the same as the interior of the interval of convergence of f)

$$\int_0^x g(t)dt = \sum_{k=1}^{\infty} \frac{k c_k}{k} x^k = \sum_{k=1}^{\infty} c_k x^k = f(x) - c_0.$$

Using the Fundamental Theorem of Calculus to differentiate, we find $g = f'$. This is exactly what we want. □

Using what we know about derivatives and integrals of power series, we can now use the geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \text{ for } |x| < 1$$

to find power series for other functions.

Example 6.4.9. Since

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \text{ for } |x| < 1$$

we can replace x by $-x$ to get

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty}(-x)^k = \sum_{k=0}^{\infty}(-1)^k x^k \text{ for } |x| < 1.$$

Now, if $f(x) = \ln(1+x)$ then $f'(x) = \dfrac{1}{1+x}$. It follows from the Zero Derivative Theorem that the term-by-term integral of our power series differs from $f(x)$ by a constant C:

$$\ln(x+1) = C + \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}.$$

We can find C by plugging in an appropriate value for x. We use $x = 0$. This gives

$$\ln(1) = C + \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} 0^{k+1} = C + 0.$$

Since $\ln(1) = 0$, then $C = 0$ and we have

$$\ln(x+1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

We know that this equality holds at least for x inside the interval $(-1, 1)$. At $x = 1$, we are looking at the alternating harmonic series, which converges. By Abel's Theorem and a consideration of continuity, the equality must also hold at $x = 1$. At $x = -1$, we have the harmonic series, which diverges. Therefore, we have

$$\ln(x+1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} \text{ for } x \text{ in } (-1, 1].$$

Notice that equality at 1 tells us that the alternating harmonic series converges to $\ln(2)$.

Exercises 6.4

6.4.1 Suppose that $\langle c_k \rangle$ is a bounded sequence. Prove that $\sum_{k=0}^{\infty} c_k x^k$ converges on $(-1, 1)$ (at least).

6.4.2 Suppose that $\langle c_k \rangle$ is a sequence which decreases to 0. Prove that $\sum_{k=0}^{\infty} c_k x^k$ converges on $[-1,1)$ (at least).

6.4.3 Suppose that $\sum_{k=0}^{\infty} c_k$ converges absolutely. Prove that $\sum_{k=0}^{\infty} c_k x^k$ converges on $[-1,1]$ (at least).

6.4.4 Beginning with the power series for $\dfrac{1}{1-x}$, find a power series for $\dfrac{x}{(1-x)^2}$.

6.4.5 Find the sum of the series $\sum_{k=0}^{\infty} \dfrac{k}{2^k}$.

6.4.6 Let

$$s(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \text{ and } c(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

1. Find the interval of convergence of $c(x)$.
2. Find the interval of convergence of $s(x)$.
3. Show that $c' = -s$.
4. Show that $s' = c$.
5. Use the Zero Derivative Theorem to show that c^2+s^2 is constant.
6. Find that constant.

6.4.7 Let

$$E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Find $E'(x)$.

6.4.8 Begin with the power series for $\dfrac{1}{1-x}$.

1. Find a power series for $\dfrac{1}{1+x^2}$.
2. Use the power series you just found to find a power series for $\tan^{-1}(x)$ with radius of convergence 1.
3. Prove that the power series you just found converges at $x = 1$.

4. Use Abel's Theorem to explain why this power series is equal to $\tan^{-1}(x)$ at $x = 1$.

5. Consider $\tan^{-1}(1)$ and this power series to find a series that converges to π.

6.5 Taylor Series

We saw in the previous sections that some common functions such as $\dfrac{1}{1-x}$ and $\ln(1+x)$ can be replaced by power series. In this section, we address the issue of how to find a power series for a function and how to prove that power series actually converges to the function desired.

Definition 6.5.1. Suppose that f is any function. Define $f^{(0)} = f$ and for $n > 0$ define $f^{(n)}$ to be the n^{th} derivative of f (assuming f has an n^{th} derivative).

Example 6.5.2. Suppose that $f(x) = \displaystyle\sum_{k=0}^{\infty} c_k x^k$ converges on an open interval containing 0. We know from Theorem 6.4.8 that f has derivatives of all orders. Let us calculate the derivatives of f at 0 (because we can).

$$\begin{aligned}
f^{(0)}(x) &= \sum_{k=0}^{\infty} c_k x^k & f^{(0)}(0) &= c_0 \\
f^{(1)}(x) &= \sum_{k=1}^{\infty} k c_k x^{k-1} & f^{(1)}(0) &= c_1 \\
f^{(2)}(x) &= \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} & f^{(1)}(0) &= 2c_2 \\
f^{(3)}(x) &= \sum_{k=3}^{\infty} k(k-1)(k-2) c_k x^{k-3} & f^{(3)}(0) &= 3 \cdot 2c_3 \\
f^{(4)}(x) &= \sum_{k=4}^{\infty} k(k-1)(k-2)(k-3) c_k x^{k-4} & f^{(4)}(0) &= 4 \cdot 3 \cdot 2c_4
\end{aligned}$$

If we continue this way, we see that it appears as if

$$f^{(k)}(0) = k(k-1)(k-2)\cdots 2c_k = k!c_k.$$

If we maintain the convention that $0! = 1$, then this equality even holds for $k = 0$. The interesting thing here is that we can now solve for c_k:

$$c_k = \frac{f^{(k)}(0)}{k!}.$$

Therefore, we know exactly what the power series for f looks like:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Thus, given any function f for which we might want to find a power series, we know exactly what that power series should be.

Definition 6.5.3. Suppose that f is a function defined on an open interval containing 0 and that f has derivatives of all orders at 0. The *Taylor series for f around* 0 is the power series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

For any positive integer n, the n^{th} *degree Taylor polynomial for f at* 0 is the n^{th} partial sum of the Taylor series for f:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n.$$

The n^{th} *remainder term for f* is the difference

$$R_n(x) = f(x) - P_n(x).$$

Remark 6.5.4. Since we are only addressing power series around 0, we will simply call these polynomials and series Taylor polynomials and Taylor series. The Taylor series around 0 is often called the Maclaurin Series.

Example 6.5.5. In this example, we find some Taylor polynomials and the Taylor series for $f(x) = e^x$. First, we need several derivatives of $f(x)$ and their values at $x = 0$. This is easy for $f(x) = e^x$:

$$
\begin{array}{ll}
f(x) = e^x & f(0) = 1 \\
f'(x) = e^x & f'(0) = 1 \\
f''(x) = e^x & f''(0) = 1 \\
f'''(x) = e^x & f'''(0) = 1 \\
f^{(4)}(x) = e^x & f^{(4)}(0) = 1 \\
\vdots & \\
f^{(k)}(x) = e^x & f^{(k)}(0) = 1
\end{array}
$$

Using the formula $c_k = \dfrac{f^{(k)}}{k!}$, we see that

$$c_0 = 1,\ c_1 = 1,\ c_2 = \frac{1}{2},\ c_3 = \frac{1}{6}, c_4 = \frac{1}{24} \ldots, c_k = \frac{1}{k!}.$$

A few of the Taylor polynomials for $f(x) = e^x$ are

$$P_1(x) = 1 + x.$$

This is the line tangent to $f(x) = e^x$ at $x = 0$. P_1 agrees with f and its first derivative at $x = 0$. Next,

$$P_2(x) = 1 + x + \frac{1}{2}x^2.$$

We might call P_2 a tangent parabola. It agrees with f and its first two derivatives at $x = 0$. Also

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \text{ and } P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4.$$

Graphs of $f(x)$ along with P_1, P_2, P_3, and P_4 are in Figure 6.4. Notice how the higher degree polynomials approximate $f(x)$ better on a wider interval. The Taylor series for $f(x) = e^x$ is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Example 6.5.6. We find a few Taylor polynomials and the Taylor series for $f(x) = \cos(x)$. First, we need some derivatives and their values at $x = 0$.

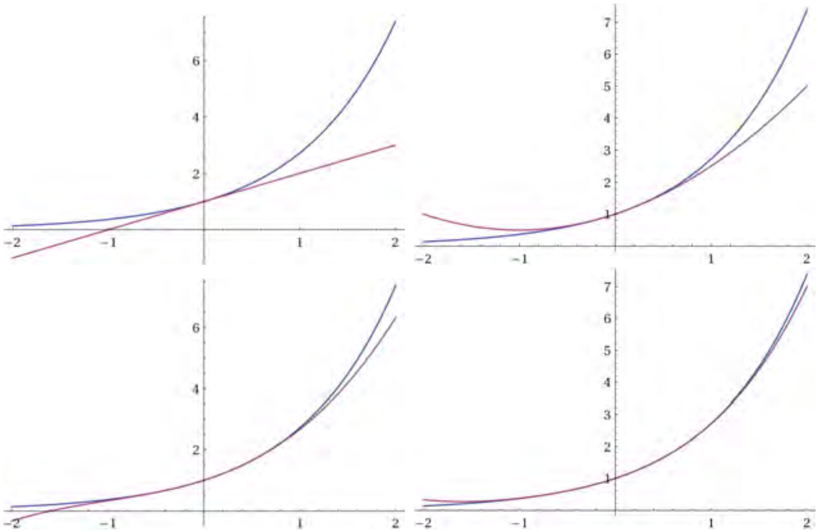

Figure 6.4: The graphs of $f(x) = e^x$ along with P_1, P_2, P_3, P_4 on the interval $[-2, 2]$. Notice how the higher degree polynomials approximate the function better on a wider interval.

$$
\begin{array}{ll}
f(x) = \cos(x) & f(0) = 1 \\
f'(x) = -\sin(x) & f'(0) = 0 \\
f''(x) = -\cos(x) & f''(0) = -1 \\
f'''(x) = \sin(x) & f'''(0) = 0 \\
f^{(4)}(x) = \cos(x) & f^{(4)}(0) = 1 \\
f^{(5)}(x) = -\sin(x) & f^{(5)}(0) = 0 \\
f^{(6)}(x) = -\cos(x) & f^{(6)}(0) = -1 \\
f^{(7)}(x) = \sin(x) & f^{(7)}(0) = 0 \\
\vdots &
\end{array}
$$

Notice that the derivatives seem to follow the pattern 1, 0, -1, 0, 1, 0, -1, 0,.... The first few even degree Taylor polynomials are:

$$P_0(x) = 1$$

$$P_2(x) = 1 - \frac{x^2}{2!}$$

$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

These polynomials are graphed along with $f(x) = \cos(x)$ in Figure 6.5.

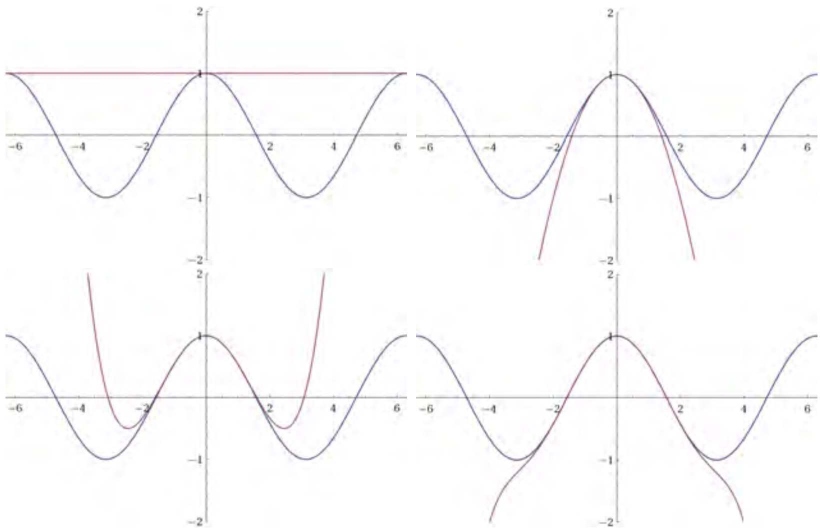

Figure 6.5: The graphs of $f(x) = \cos(x)$ along with P_0, P_2, P_4, P_6 on the interval $[-2\pi, 2\pi]$. Notice how the higher degree polynomials approximate the function better on a wider interval.

To write the Taylor series for $\cos(x)$, we have to think a little harder than we did for e^x. First, the signs alternate, beginning with a positive, so we should have a factor of $(-1)^k$. Next, the exponents are all even, beginning with 0, so instead of x^k, we use x^{2k}. The factorial on the bottom of the fractions agree with the exponents, so we use $(2k)!$ rather than $k!$. Thus, the Taylor series is

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

Example 6.5.7. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

A graph of f is depicted in Figure 6.6. It can be shown that $f^{(k)}(0) = 0$

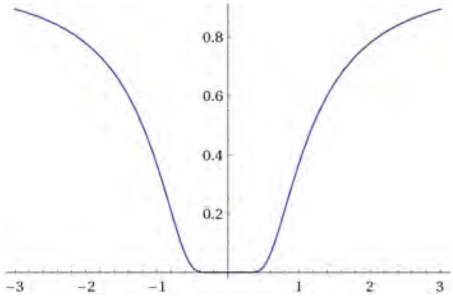

Figure 6.6: All of the derivatives of this function are equal to 0 at $x = 0$. This forces the coefficients in the Taylor series all to be 0, so the Taylor series cannot converge to f other than at $x = 0$.

for all k. Therefore, every coefficient in the Taylor series for f is 0, and the Taylor series converges to the constant 0 function. In particular, the Taylor series does not converge to f anywhere other than at $x = 0$.

Notice that for any x the question of whether or not the Taylor series for $f(x)$ actually converges to $f(x)$ is the same as the question of whether or not $\lim P_n(x) = f(x)$ (since $P_n(x)$ is the n^{th} partial sum of the Taylor series). This means that the question of whether or not the Taylor series converges to $f(x)$ is the same as whether or not the sequence of remainder terms $\langle R_n(x)\rangle$ converges to 0. To address this issue, we need a tool which will tell us how large $R_n(x)$ can be.

Theorem 6.5.8. (Taylor's Formula) *Suppose that $f : (a, b) \to \mathbb{R}$ is a function defined on an open interval containing 0 and that $f^{(n+1)}$ exists on (a, b). For every $z \neq 0$ in (a, b) there is a y between 0 and z so that $R_n(z) = \dfrac{f^{(n+1)}(y)}{(n+1)!} z^{n+1}$.*

Proof. First note that the n^{th} degree Taylor polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n$$

agrees with f and its first n derivatives at 0. That is, $f^{(k)}(0) = P_n^{(k)}(0)$ for $k = 0, 1, 2, \ldots, n$. Let

$$M = \frac{f(z) - P_n(z)}{z^{n+1}}$$

and define $F(x) = f(x) - P_n(x) - Mx^{n+1}$. Note that since f has $n+1$ derivatives on (a,b) then F also has $n+1$ derivatives on (a,b) (since $-P_n(x) - Mx^{n+1}$ is a polynomial). Also, by our choice of M we have $F^{(k)}(0) = 0$ for $k = 0, 1, 2, \ldots, n, n+1$ and that $F(z) = 0$. Since $F^{(n+1)}$ exists on (a,b), then $F^{(k)}$ is continuous on (a,b) for $k = 1, 2, \ldots, n$. We now apply Rolle's Theorem repeatedly.

- Since $F(0) = F(z) = 0$ and since F is continous and differentiable on (a,b), by Rolle's Theorem there is some c_1 between 0 and z where $F'(c_1) = 0$.

- Since $F'(0) = F'(c_1) = 0$ and since F' is continous and differentiable on (a,b), by Rolle's Theorem there is some c_2 between 0 and c_1 where $F''(c_2) = 0$.

- Since $F''(0) = F''(c_2) = 0$ and since F'' is continous and differentiable on (a,b), by Rolle's Theorem there is some c_3 between 0 and c_2 where $F'''(c_3) = 0$.

  $\vdots$

- Since $F^{(n)}(0) = F^{(n)}(c_n) = 0$ and since $F^{(n)}$ is continous and differentiable on (a,b), by Rolle's Theorem there is some y between 0 and c_n where $F^{(n+1)}(y) = 0$.

Now, if we find the $(n+1)^{st}$ derivative of F, we see that

$$F^{(n+1)}(x) = f^{(n+1)}(x) - 0 - M(n+1)!$$

We then have

$$0 = f^{(n+1)}(y) - 0 - M(n+1)! = f^{(n+1)}(y) - \frac{f(z) - P_n(z)}{z^{n+1}}(n+1)!$$

Solving for $f(z)$ now gives

$$f(z) = P_n(z) + \frac{f^{(n+1)}(y)}{(n+1)!}z^{n+1}$$

which means that

$$R_n(z) = \frac{f^{(n+1)}(y)}{(n+1)!}z^{n+1}$$

as desired. $\square$

Theorem 6.5.9. *Suppose that $f : (a, b) \to \mathbb{R}$ is a function defined on an open interval containing* 0 *and that f has derivatives of all orders on (a, b). Suppose also that there is an $M \in \mathbb{R}$ so that $|f^{(n)}(x)| < M$ for all nonnegative integers n and all $x \in (a, b)$. Then*

$$\lim R_n(z) = 0 \text{ for all } z \in (a, b).$$

In particular, the Taylor series for f around 0 *converges to f on (a, b).*

Proof. Let $z \in (a, b)$. For each n, apply Taylor's Formula to find a y_n in (a, b) so that

$$R_n(z) = \frac{f^{(n+1)}(y_n)}{(n+1)!} z^{n+1}.$$

Then

$$\begin{aligned}
\lim |R_n(z)| &= \lim \left| \frac{f^{(n+1)}(y_n)}{(n+1)!} z^{n+1} \right| \\
&\leq \lim \left| \frac{M}{(n+1)!} z^{n+1} \right| \\
&= 0
\end{aligned}$$

Since $\lim |R_n(z)| = 0$, then $\lim |f(z) - P_n(z)| = 0$ so $f(z) = \lim P_n(z)$. Thus the Taylor series for $f(x)$ at $x = z$ converges to $f(z)$. □

Example 6.5.10. Here we consider the Taylor series for e^x. We found this series in Example 6.5.5, and we saw that the series converges for all real numbers in Example 6.3.5. Suppose that $z \in \mathbb{R}$. Let (a, b) be any open interval containing z and 0. Since $f^{(n)}(x) = e^x$ for all n, we have $0 < f^{(n)}(x) = e^x < e^b$ for all x in (a, b). Applying the theorem with $M = e^b$ tells us that the Taylor series converges to e^x on (a, b). In particular, it converges at z to e^z. Therefore, the Taylor series for e^x converges to e^x for all real numbers. So

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ for all } x \in \mathbb{R}.$$

Example 6.5.11. In Example 6.5.6 we found the Taylor series for $\cos(x)$. In Exercise 6.4.6 we discovered that this series converges for all real numbers. Since the derivatives of all orders of $\cos(x)$ are bounded by 1 on every open interval containing 0, it follows that the Taylor series for $\cos(x)$ converges to $\cos(x)$ for all real numbers. That is:

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \text{ for all } x \in \mathbb{R}.$$

Exercises 6.5

6.5.1 Use the definition to find the first five nonzero terms of the Taylor series for $f(x) = \sqrt{x+1}$.

6.5.2

1. Use the definition to find the first five nonzero terms of the Taylor series for $f(x) = \sin(x)$.
2. Find a formula for the terms of the Taylor series.
3. Find the interval of convergence of the Taylor series.
4. Prove that the Taylor series converges to f on the interior of this interval.

6.5.3 Modify a power series you know to find a power series for

$$F(x) = \int_0^x e^{-t^2}\, dt.$$

Use the first five nonzero terms of this series to approximate $F(1)$.

6.5.4 Consider the integral $\displaystyle\int_0^1 \sin(t^2)dt$. There is no elementary function whose derivative is equal to $\sin(t^2)$.

1. Modify the Taylor series from 6.5.2 to find a power series for
$$F(x) = \int_0^x \sin(t^2)dt.$$
2. Use the power series from part (1) to find a series for $\displaystyle\int_0^1 \sin(t^2)dt$.
3. The series you just found is (or should be) an alternating series. Use what we know about alternating series to approximate the sum of the series with an error of less than .001.

6.5.5 Either find a power series around 0 for $f(x) = |x|$ or prove that there is not one.

Bibliography

[1] Apostol, Tom M. *Mathematical Analysis*. Reading, MA: Addison-Wesley, 1974.

[2] Gaughan, Edward. *Introduction To Analysis*. Belmont, CA: Brooks/Cole Pub. Co., 1968.

[3] Howie, John M. *Real Analysis*. London: Springer, 2001.

[4] Lang, Serge. *Analysis*. Reading (Mass.): Addison-Wesley, 1968.

[5] Lieb, Elliott H., and Michael Loss. *Analysis*. Providence, RI: American Mathematical Society, 1997.

[6] Polya, George et al. *Analysis*. Cambridge, MA: MIT Press, 1984.

[7] Protter, Murray H., and Charles Bradfield Morrey. *A First Course in Real Analysis*. New York: Springer-Verlag, 1977.

[8] Ross, Kenneth A. *Elementary Analysis: the Theory of Calculus*. New York: Springer-Verlag, 1980.

[9] Royden, H. L. *Real Analysis*. New York: Macmillan, 1968.

[10] Rudin, Walter. *Principles Of Mathematical Analysis*. New York: McGraw-Hill, 1964.

[11] Rudin, Walter. *Real And Complex Analysis*. New York: McGraw-Hill, 1966.

[12] Trench, William F. *Introduction To Real Analysis*. Upper Saddle River, NJ: Prentice Hall/Pearson Education, 2003.