Term operations in $\mathcal{V}(\mathbf{N}_5)$

John W. Snow

Abstract. Define a lattice to be order endoprimal if every order preserving operation on the lattice which is preserved by all endomorphisms is a term operation. We prove that every lattice in the variety generated by $\mathbf{N}_5$ is order endoprimal.

Mathematics Subject Classification. 08A30, 06B05.

Keywords. lattice, term operation, endoprimal, order endoprimal.

1. Introduction

In [6] Márki and Pöschel define an algebra $\mathbf{A}$ to be *endoprimal* if every operation on $\mathbf{A}$ which is preserved by every endomorphism of $\mathbf{A}$ is a term operation. They then prove that a distributive lattice $\mathbf{D}$ is endoprimal if and only $\mathbf{D}$ is not relatively complemented. In proving that if $\mathbf{D}$ is endoprimal, then $\mathbf{D}$ is not relatively complemented, Márki and Pöschel use the contrapositive. Assuming that $\mathbf{D}$ is relatively complemented, they construct a ternary term operation f on $\mathbf{D}$ which is preserved by every endomorphism of $\mathbf{D}$ but which is not a term operation. With their construction, it is clear that f is not a term operation because f does not preserve the order of $\mathbf{D}$. Since every term operation of a lattice must be order preserving, it seems natural to consider operations which are not only preserved by endomorphisms but which are also order preserving. Therefore, we define a lattice $\mathbf{L}$ to be *order endoprimal* if every order preserving operation on $\mathbf{L}$ which is preserved by all endomorphisms of $\mathbf{L}$ is a term operation. In [4] and [3], Davey, Haviar, and Priestly prove that every finite distributive lattice is dualised by its endomorphisms and order. It follows that every finite distributive lattice is order endoprimal. We prove that every lattice in the variety generated by $\mathbf{N}_5$ (including all distributive lattices) is order endoprimal.

2. Preliminaries

In this section, after we give some basic definitions, we use a result of Hegedűs and Pálfy to give conditions based on join endomorphisms for an operation p

on a finite lattice to be a term operation (Theorem 2.4). We then give some technical results about join endomorphisms which we will need within proofs in later sections. We close the section with a result (Lemma 2.11) about order preserving operations preserved by endomorphisms which will be critical to our work with $\mathbf{N}_5$ later.

For basic concepts, notation, and terminology related to universal algebra and lattice theory, we direct the reader to [7]. We will always call the least and greatest elements of a lattice (if they exist) 0 and 1, respectively. When we say that a lattice has a 0 (or 1) we mean that the lattice has a least (greatest) element which is denoted 0 (1).

Suppose that p is an operation on a finite lattice $\mathbf{L}$. For each positive n, p induces an operation $p^{\mathbf{L}^n}$ on $\mathbf{L}^n$ which is p applied coordinatewise. We will usually abuse notation and refer to these operations in direct powers also as p. If a subset R of $\mathbf{L}$ is closed under p, then we will say that p *preserves* R. The familiar Galois connection between relations and operations on a finite set declares that p is a term operation of $\mathbf{L}$ if and only if p preserves every subuniverse of every finite direct power of $\mathbf{L}$ [8]. Moreover, since $\mathbf{L}$ has a majority operation, these subuniverses of direct powers are all generated by the subuniverses of $\mathbf{L}^2$ [2, 1]. It follows that:

Lemma 2.1. *Suppose that $\mathbf{L}$ is a finite lattice. An operation p on $\mathbf{L}$ is a term operation of $\mathbf{L}$ if and only if p preserves every subuniverse of $\mathbf{L}^2$.*

Suppose that p is an idempotent, order preserving n-ary operation on a lattice $\mathbf{L}$ and that $a < b \in \mathbf{L}$. If $x_1, \ldots, x_n$ are in the interval between a and b, then

$$a = p(a, a, \ldots, a) \leq p(x_1, x_2, \ldots, x_n) \leq p(b, b, \ldots, b) = b$$

This proves that:

Lemma 2.2. *Suppose that $\mathbf{L}$ is a lattice and that p is an idempotent, order preserving operation on $\mathbf{L}$. Every bounded interval in $\mathbf{L}$ is closed under p.*

Hegedűs and Pálfy in [5] give the following characterization of sublattices of the square of a lattice as intersections of special sublattices.

Lemma 2.3. ([5] Lemma 4.7) *Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be arbitrary lattices and let $\mathbf{L}$ be a sublattice of $\mathbf{L}_1 \times \mathbf{L}_2$. Define*

$$\begin{aligned}
\mathbf{L}_1' &= \{x \in \mathbf{L}_1 : (\exists b \in \mathbf{L}_2)\langle x, b\rangle \in \mathbf{L}\} \\
\mathbf{L}_2' &= \{y \in \mathbf{L}_2 : (\exists a \in \mathbf{L}_1)\langle a, y\rangle \in \mathbf{L}\} \\
\mathbf{L}_1^* &= \{\langle x, y\rangle \in \mathbf{L}_1 \times \mathbf{L}_2 : (\exists \langle a, b\rangle \in \mathbf{L})(x \leq a \text{ and } b \leq y)\} \\
\mathbf{L}_2^* &= \{\langle x, y\rangle \in \mathbf{L}_1 \times \mathbf{L}_2 : (\exists \langle a, b\rangle \in \mathbf{L})(x \geq a \text{ and } b \geq y)\}.
\end{aligned}$$

Then $\mathbf{L} = (\mathbf{L}_1' \times \mathbf{L}_2) \cap (\mathbf{L}_1 \times \mathbf{L}_2') \cap \mathbf{L}_1^* \cap \mathbf{L}_2^*$. Moreover, if $\mathbf{L}$ is a 0-1 sublattice, then $\mathbf{L}_1^*$ and $\mathbf{L}_2^*$ are subdirect products in $\mathbf{L}_1 \times \mathbf{L}_2$ with $\langle 0, 1\rangle \in \mathbf{L}_1^*$ and $\langle 1, 0\rangle \in \mathbf{L}_2^*$.

According to Lemma 2.1, to know whether or not an operation p on a finite lattice $\mathbf{M}$ is a term operation, we must know whether or not the sublattices of $\mathbf{M}^2$ are closed under p. Corollary 2.4 shows that we actually only have to concern ourselves with certain special sublattices. We note that the proof of this theorem is identical to the proof of Theorem 4.8 of [5].

Corollary 2.4. (See [5] Theorem 4.8) *Let* $\mathbf{M}$ *be a finite lattice and let* $p : \mathbf{M}^n \to \mathbf{M}$ *be an idempotent operation on* $\mathbf{M}$. *Then* p *is a term operation of* $\mathbf{M}$ *if and only if* p *preserves every subdirect product in* $\mathbf{M}^2$ *containing* $(\{0\} \times \mathbf{M}) \cup (\mathbf{M} \times \{1\})$.

Proof. Suppose that p is an idempotent operation which preserves every subdirect product in $\mathbf{M}^2$ containing $(\{0\}\times\mathbf{M})\cup(\mathbf{M}\times\{1\})$. First, $\{\langle x, y\rangle : x \leq y\}$ is a sublattice of $\mathbf{M}^2$ containing $(\{0\}\times\mathbf{M})\cup(\mathbf{M}\times\{1\})$, so p is order preserving. According to Lemma 2.1, we need only show that every sublattice of $\mathbf{M}^2$ is closed under p. In order to do so, we will use Lemma 2.3. First we prove that that all sublattices of $\mathbf{M}$ are closed under p. Suppose that $\mathbf{L}$ is a sublattice of $\mathbf{M}$. Let $\mathbf{L}'$ be $\mathbf{L}\cup\{0, 1\}$ (it may be that $\mathbf{L} = \mathbf{L}'$). For each $a \in \mathbf{M}$, let a' be the smallest element of $\mathbf{L}'$ greater than or equal to a. Let $\mathbf{L}'' = \{\langle a, b\rangle : a' \leq b\}$. $\mathbf{L}''$ is a sublattice of $\mathbf{M}^2$ containing $(\{0\} \times \mathbf{M}) \cup (\mathbf{M} \times \{1\})$, and so is closed under p. Now, $\mathbf{L}'' \cap \{\langle a, a\rangle : a \in \mathbf{M}\} = \{\langle a, a\rangle : a \in \mathbf{L}'\}$ is closed under p (since applying p to diagonal elements must yield a diagonal element). It follows then that $\mathbf{L}'$ is closed under p. Now, $\mathbf{L}$ is an interval in $\mathbf{L}'$, and p is an idempotent, order preserving operation on $\mathbf{L}'$, so $\mathbf{L}$ is closed under p by Lemma 2.2. We have shown that every sublattice of $\mathbf{M}$ is closed under p.

Now let $\mathbf{L}$ be a sublattice of $\mathbf{M}^2$. Let $\mathbf{L}' = \mathbf{L} \cup \{\langle 0, 0\rangle, \langle 1, 1\rangle\}$. Taking $\mathbf{L}_1 = \mathbf{M}$ and $\mathbf{L}_2 = \mathbf{M}$ we can apply Lemma 2.3 to $\mathbf{L}' \subseteq \mathbf{L}_1 \times \mathbf{L}_2$. Let $\mathbf{L}_1'$, $\mathbf{L}_2'$, $\mathbf{L}_1^*$, and $\mathbf{L}_2^*$ be as in Lemma 2.3. The lattices $\mathbf{M}$, $\mathbf{L}_1'$, and $\mathbf{L}_2'$ are closed under p since they are sublattices of $\mathbf{M}$. Since $\mathbf{L}_1^*$ is a subdirect product containing $\langle 0, 1\rangle$, it follows that $\mathbf{L}_1^*$ contains $(\{0\}\times\mathbf{M})\cup(\mathbf{M}\times\{1\})$ and is closed under p. Since $\mathbf{L}_2^*$ is a subdirect product containing $\langle 1, 0\rangle$, it follows that $\mathbf{L}_2^*$ contains $(\{1\}\times\mathbf{M})\cup(\mathbf{M}\times\{0\})$. Then the converse of $\mathbf{L}_2^*$ contains $(\{0\}\times\mathbf{M})\cup(\mathbf{M}\times\{1\})$ and is closed under p. Since the converse of $\mathbf{L}_2^*$ is closed under p, so is $\mathbf{L}_2^*$. Thus $\mathbf{L}_1^*$ and $\mathbf{L}_2^*$ are closed under p. Now $\mathbf{L}' = (\mathbf{L}_1' \times \mathbf{L}_2) \cap (\mathbf{L}_1 \times \mathbf{L}_2') \cap \mathbf{L}_1^* \cap \mathbf{L}_2^*$ is closed under p. Since p is idempotent and order preserving, and since $\mathbf{L}$ is an interval in $\mathbf{L}'$, $\mathbf{L}$ is also closed under p by Lemma 2.2.

We have now proven that every sublattice of $\mathbf{L}^2$ is closed under p. Since $\mathbf{L}$ has a majority term, p must be a term operation of $\mathbf{L}$ by Lemma 2.1. $\square$

Definition 2.5. Let $\mathbf{M}$ be a finite lattice. If e is a join endomorphism of $\mathbf{M}$ fixing 0, define $e^{\uparrow} = \{\langle x, y\rangle : e(x) \leq y\}$. If $\mathbf{L}$ is a sublattice of $\mathbf{M}^2$ containing $(\{0\} \times \mathbf{M}) \cup (\mathbf{M} \times \{1\})$, then define $e_{\mathbf{L}} : \mathbf{M} \to \mathbf{M}$ so that $e_{\mathbf{L}}(x)$ is the least y with $\langle x, y\rangle \in \mathbf{L}$.

The next two lemmas demonstrate the connection between the Hegedűs and Pálfy sublattices and join endomorphisms.

Lemma 2.6. *Suppose that* $\mathbf{M}$ *is a finite lattice. If* $\mathbf{L}$ *is a sublattice of* $\mathbf{M}^2$ *containing* $(\{0\} \times \mathbf{M}) \cup (\mathbf{M} \times \{1\})$, *then* $e_{\mathbf{L}}$ *is a join endomorphism of* $\mathbf{M}$ *which fixes* 0 *and* $(e_{\mathbf{L}})^{\uparrow} = \mathbf{L}$.

Proof. Suppose that $\mathbf{M}$ is a finite lattice and that $\mathbf{L}$ is a sublattice of $\mathbf{M}^2$ containing $(\{0\} \times \mathbf{M}) \cup (\mathbf{M} \times \{1\})$. To simplify notation, we write e for $e_{\mathbf{L}}$. We will prove that e is a join endomorphism of $\mathbf{M}$. Let $x, y \in \mathbf{M}$. By the definition of $e_{\mathbf{L}}$, we know that $\langle x, e(x)\rangle$ and $\langle y, e(y)\rangle$ are in $\mathbf{L}$. Therefore, $\langle x \vee y, e(x) \vee e(y)\rangle \in \mathbf{L}$. It now follows that $e(x \vee y) \leq e(x) \vee e(y)$. On the other hand, since $\langle x, 1\rangle$ and $\langle x \vee y, e(x \vee y)\rangle$ are in $\mathbf{L}$, then also

$$\langle x, e(x \vee y)\rangle = \langle x, 1\rangle \wedge \langle x \vee y, e(x \vee y)\rangle \in \mathbf{L}.$$

Therefore, $e(x) \leq e(x \vee y)$. Similarly, $e(y) \leq e(x \vee y)$. Thus $e(x) \vee e(y) \leq e(x \vee y)$ must also be true. Since $e(x \vee y) \leq e(x) \vee e(y)$ and $e(x) \vee e(y) \leq e(x \vee y)$, then $e(x) \vee e(y) = e(x \vee y)$. This is true for all $x, y \in \mathbf{L}$, so $e = e_{\mathbf{L}}$ is actually a join homomorphism. That $e(0) = 0$ follows from the assumption that $\mathbf{L}$ contains $\{0\} \times \mathbf{M}$.

Next we prove that $e^{\uparrow} = \mathbf{L}$. Suppose first that $\langle x, y\rangle \in \mathbf{L}$. Then $e(x) \leq y$ by the definition of $e_{\mathbf{L}} = e$. Therefore, $\langle x, y\rangle \in e^{\uparrow}$. This proves $\mathbf{L} \subseteq e^{\uparrow}$. On the other hand, let $\langle a, b\rangle \in e^{\uparrow}$. This implies that $e(a) \leq b$. By our assumptions, $\langle 0, b\rangle \in \mathbf{L}$ and $\langle a, e(a)\rangle \in \mathbf{L}$. Therefore, $\mathbf{L}$ also contains $\langle 0, b\rangle \vee \langle a, e(a)\rangle = \langle a, b\rangle$. This proves that $e^{\uparrow} \subseteq \mathbf{L}$ and completes the proof that these sets are equal. □

Lemma 2.7. *Suppose that* $\mathbf{M}$ *is a finite lattice. If* e *is a join endomorphism of* $\mathbf{M}$ *which fixes* 0*, then* $e^{\uparrow}$ *is a sublattice of* $\mathbf{M}^2$ *containing* $(\{0\} \times \mathbf{M}) \cup (\mathbf{M} \times \{1\})$ *and* $e_{e^{\uparrow}} = e$*.*

Proof. Suppose that $\mathbf{M}$ is a finite lattice and that e is a join endomorphism of $\mathbf{M}$ which fixes 0. Let $x \in \mathbf{M}$. Then $e(0) = 0 \leq x$, so $\langle 0, x\rangle \in e^{\uparrow}$. This proves that $e^{\uparrow}$ contains $\{0\} \times \mathbf{M}$. On the other hand, if $x \in \mathbf{M}$, then $e(x) \leq 1$, so $\langle x, 1\rangle \in e^{\uparrow}$. This shows that $e^{\uparrow}$ contains $\mathbf{M} \times \{1\}$. We now need to prove that $e^{\uparrow}$ is a sublattice of $\mathbf{M}^2$. To that end, let $\langle x, y\rangle, \langle a, b\rangle \in e^{\uparrow}$. This means that $e(x) \leq y$ and $e(a) \leq b$. Then $e(x \vee a) = e(x) \vee e(a) \leq y \vee b$ so $\langle x \vee a, y \vee b\rangle \in e^{\uparrow}$, and $e^{\uparrow}$ is closed under joins. Also, by the order preserving nature of e, we know that $e(x \wedge a) \leq e(x)$ and $e(x \wedge a) \leq e(a)$. This implies that $e(x \wedge a) \leq e(x) \wedge e(a) \leq y \wedge b$ so $\langle x \wedge a, y \wedge b\rangle \in e^{\uparrow}$. Thus $e^{\uparrow}$ is also closed under meets and is, therefore, a sublattice of $\mathbf{M}^2$.

Now we prove that $e_{e^{\uparrow}} = e$. If $x \in \mathbf{M}$, then $e_{e^{\uparrow}}(x)$ is the least y so that $\langle x, y\rangle \in e^{\uparrow}$. From the definition of $e^{\uparrow}$, this is the least y so that $e(x) \leq y$. Clearly, the least such y is $e(x)$. Therefore, $e_{e^{\uparrow}}(x) = e(x)$. □

Corollary 2.4 along with Lemmas 2.6 and 2.7 now combine to give us this characterization of term operations on a finite lattice.

Theorem 2.8. *Let* $\mathbf{M}$ *be a finite lattice and let* p *be an idempotent operation on* $\mathbf{M}$*. Then* p *is a term operation of* $\mathbf{M}$ *if and only if* p *preserves every sublattice of* $\mathbf{M}^2$ *of the form* $e^{\uparrow}$ *where* e *is a join endomorphism of* $\mathbf{M}$ *fixing* 0*.*

The $^{\uparrow}$ operator interacts nicely with joins and intersections.

Lemma 2.9. *Suppose that* $\mathbf{M}$ *is a finite lattice and that* e *and* f *are join endomorphisms of* $\mathbf{M}$ *fixing* 0*. Then the pointwise join* $e \vee f$ *is a join endomorphism of* $\mathbf{M}$ *fixing* 0 *and* $(e \vee f)^{\uparrow} = e^{\uparrow} \cap f^{\uparrow}$*.*

Proof. That $e \vee f$ is a join homomorphism of $\mathbf{M}$ fixing 0 should be clear. Suppose that $\langle x, y\rangle \in (e \vee f)^{\uparrow}$. This means that $e(x) \vee f(x) \leq y$. Therefore, $e(x) \leq y$ and $e(x) \leq y$, so $\langle x, y\rangle$ is in $e^{\uparrow} \cap f^{\uparrow}$. Thus $(e \vee f)^{\uparrow} \subseteq e^{\uparrow} \cap f^{\uparrow}$. Now suppose that $\langle a, b\rangle \in e^{\uparrow} \cap f^{\uparrow}$. This means that $e(a) \leq b$ and $f(a) \leq b$. It follows then that $e(a) \vee f(a) \leq b$ and $\langle a, b\rangle \in (e \vee f)^{\uparrow}$. Thus $(e \vee f)^{\uparrow} = e^{\uparrow} \cap f^{\uparrow}$ as desired. $\square$

When the range of a join endomorphism e contains only 1 or 2 elements, $e^{\uparrow}$ takes on a simple form.

Lemma 2.10. *Suppose that* $\mathbf{M}$ *is a finite lattice. If* e *is a join endomorphism of* $\mathbf{M}$ *which fixes* 0 *and if the range of* e *contains at most two elements, then* $e^{\uparrow}$ *is the union of a filter and an ideal in* $\mathbf{M}^2$*.*

Proof. If the range of e contains exactly one element, then that one element must be 0 since e fixes 0. Then

$$e^{\uparrow} = \mathbf{M}^2 = \{\langle x, y\rangle \in \mathbf{M}^2 : \langle x, y\rangle \leq \langle 1, 1\rangle\} \cup \{\langle x, y\rangle \in \mathbf{M}^2 : \langle 0, 0\rangle \leq \langle x, y\rangle\}.$$

Thus, $e^{\uparrow}$ is the union of a filter and an ideal when the range of e has only one element.

Assume next that the range of e contains exactly two elements. One of these must be 0 (since e fixes 0). The other must be $e(1)$, since if $e(1) = 0$ then e would be constantly 0 by its order preserving nature. Let a be the largest element of $\mathbf{M}$ so that $e(a) = 0$, and let $b = e(1)$. Since the range of e is $\{0, b\}$, and since e is order preserving, then $e(x) = 0$ if and only if $x \leq a$. Let

$$\mathbf{L} = \{\langle x, y\rangle \in \mathbf{M}^2 : \langle x, y\rangle \leq \langle a, 1\rangle\} \cup \{\langle x, y\rangle \in \mathbf{M}^2 : \langle 0, b\rangle \leq \langle x, y\rangle\}.$$

We prove that $e^{\uparrow} = \mathbf{L}$. Suppose first that $\langle x, y\rangle \in \mathbf{L}$. If $\langle x, y\rangle \leq \langle a, 1\rangle$, then $x \leq a$. It follows that $e(x) \leq e(a) = 0$, so $e(x) = 0 \leq y$. This places $\langle x, y\rangle$ in $e^{\uparrow}$. Suppose on the other hand that $\langle 0, b\rangle \leq \langle x, y\rangle$ and that $\langle x, y\rangle \not\leq \langle a, 1\rangle$. This implies that $x \not\leq a$, so $e(x) \neq 0$. Since the range of e is $\{0, b\}$, it must be that $e(x) = b \leq y$. Thus, $\langle x, y\rangle \in e^{\uparrow}$. In either case, $\langle x, y\rangle \in e^{\uparrow}$, and $\mathbf{L} \subseteq e^{\uparrow}$.

Now suppose that $\langle x, y\rangle \in e^{\uparrow}$. If $e(x) = 0$, then $x \leq a$ and $\langle x, y\rangle \leq \langle a, 1\rangle$. This would place $\langle x, y\rangle$ in $\mathbf{L}$. On the other hand, if $e(x) \neq 0$, then $e(x) = b$. Since $\langle x, y\rangle \in e^{\uparrow}$, $b = e(x) \leq y$. This means that $\langle 0, b\rangle \leq \langle x, y\rangle$, so $\langle x, y\rangle \in \mathbf{L}$. In either case, $\langle x, y\rangle \in \mathbf{L}$, and $e^{\uparrow} \subseteq \mathbf{L}$. We have proven that $e^{\uparrow} = \mathbf{L}$, so $e^{\uparrow}$ is the union of a filter and an ideal. $\square$

Lemma 2.11. *Suppose that*

(1) $\mathbf{L}$ *is a finite lattice,*
(2) There is a homomorphism from $\mathbf{L}$ *onto the two element lattice, and*
(3) p *is an order preserving operation on* $\mathbf{L}$ *which is preserved by endomorphisms.*

Then every sublattice of $\mathbf{L}$ *or of* $\mathbf{L}^2$ *which is the union of a filter and an ideal is closed under* p.

Proof. First, since every element of $\mathbf{L}$ is the range of a constant endomorphism which must preserve p, it follows that p is idempotent. We next prove that p induces an isomorphic structure on every two element sublattice of $\mathbf{L}$ and every two element sublattice of $\mathbf{L}^2$. Since $\mathbf{L}$ has a surjective homomorphism onto the two element lattice, there is an endomorphism f of $\mathbf{L}$ whose image is $\{0,1\}$ (the bottom and top of $\mathbf{L}$). Notice that since f maps onto $\{0,1\}$ and since f must be order preserving, f must fix 0 and 1. Since p is preserved by endomorphisms, and since the range of f is $\{0,1\}$, we know that $\{0,1\}$ is closed under p. Suppose now that $a < b$ in $\mathbf{L}$. Let $e : \{0,1\} \to \{a,b\}$ be given by $e(0) = a$ and $e(1) = b$. The map $e \circ f$ is an endomorphism of $\mathbf{L}$ onto $\{a,b\}$ which must preserve p. The restriction of $e \circ f$ to $\{0,1\}$ is an isomorphism of $\langle\{0,1\},p\rangle$ and $\langle\{a,b\},p\rangle$. However, the restriction of $e \circ f$ to $\{0,1\}$ is just e since f fixes 0 and 1. Thus, every two element sublattice of $\mathbf{L}$ is closed under p, and the induced algebra on any two element sublattice is isomorphic to $\langle\{0,1\},p\rangle$ with the natural ordering.

Next, we turn our attention to two element sublattices of $\mathbf{L}^2$. Suppose that $\langle a,b\rangle < \langle c,d\rangle$ in $\mathbf{L}^2$. Then $\{a,c\}$ and $\{b,d\}$ are two element sublattices of $\mathbf{L}$. We know that these two element sets are closed under p and that p induces isomorphic structures on them via $a \mapsto b$ and $c \mapsto d$. Then $\{\langle a,b\rangle, \langle c,d\rangle\}$ is the graph of the isomorphism from $\{a,c\}$ to $\{b,d\}$ and is closed under $p^{\mathbf{L}^2}$. The projection to the first coordinate is an isomorphism of $\{\langle a,b\rangle,\langle c,d\rangle\}$ with $\{a,c\}$ under p, which is isomorphic to $\langle\{0,1\},p\rangle$. Thus every two element sublattice of $\mathbf{L}^2$ is closed under $p^{\mathbf{L}^2}$, and the structures induced on these two element sublattices are isomorphic to $\langle\{0,1\},p\rangle$.

We now know that p induces a structure isomorphic to $\langle\{0,1\},p\rangle$ on every two element sublattice of $\mathbf{L}$ and every two element sublattice of $\mathbf{L}^2$. Before we can finish with the proof, we need to describe this structure. Suppose that p is n-ary. Let A be the collection of all subsets B of $\{1,2,\ldots,n\}$ for which there exist $x_1,x_2,\ldots,x_n \in \{0,1\}$ so that $p(x_1,x_2,\ldots x_n) = 0$ and $B = \{i : x_i = 0\}$. The set A completely determines the behavior of p on 2-element sublattices. By the definition of A, if $x_1,x_2,\ldots,x_n \in \{0,1\}$ then $p(x_1,x_2,\ldots,x_n) = 0$ if and only if $\{i : x_i = 0\} \in A$. Also, since p induces an isomorphic structure on every two element sublattice, if $r < s \in \mathbf{L}$ (or $\mathbf{L}^2$), and if $x_1,x_2,\ldots,x_n \in \{r,s\}$, then $p(x_1,x_2,\ldots,x_n) = r$ if and only if $\{i : x_i = r\} \in A$.

Next, we prove that if $B \in A$, and if $B \subseteq C \subseteq \{1,2,\ldots,n\}$, then $C \in A$. Suppose that $b_1,b_2,\ldots,b_n,c_1,c_2,\ldots,c_n \in \{0,1\}$ so that $B = \{i : b_i = 0\}$ and $C = \{i : c_i = 0\}$. If some b_i is equal to 0, then $i \in B \subseteq C$ so $c_i = 0$ also. This means that $c_i \leq b_i$ for all i. Since $\{0,1\}$ is closed under p, then $p(c_1,c_2,\ldots,c_n) \in \{0,1\}$. Since p is order preserving, $p(c_1,c_2,\ldots,c_n) \leq p(b_1,b_2,\ldots,b_n)$. Since $B \in A$, $p(b_1,b_2,\ldots,b_n) = 0$. Therefore, it must be that $p(c_1,c_2,\ldots,c_n) = 0$ and $C \in A$ as desired.

We are now ready to prove the lemma. Suppose that $a, b \in \mathbf{L}$ (or $\mathbf{L}^2$) and let

$$\mathbf{M} = \{x : x \leq a\} \cup \{x : b \leq x\}.$$

We will prove that $\mathbf{M}$ is closed under p. Let $y_1, y_2, \dots, y_n \in \mathbf{M}$. For each $i \in \{1, 2, \dots, n\}$, define

$$x_i = \begin{cases} b & b \leq y_i \\ 0 & \text{else} \end{cases} \text{ and } z_i = \begin{cases} a & y_i \leq a \\ 1 & \text{else.} \end{cases}$$

Now, let $u = p(x_1, \dots, x_n)$, $v = p(y_1, \dots, y_n)$, and $w = p(z_1, \dots, z_n)$. We will prove $v \in \mathbf{M}$. Since $x_i \leq y_i \leq z_i$ for each i, we have that $u \leq v \leq w$. We will argue that either $w = a$ (in which case $v \leq a$) or $u = b$ (in which case $b \leq v$). This will force $v \in \mathbf{M}$. Suppose that $u \neq b$. Since $\{0, b\}$ is closed under p, this means that $u = 0$. Since $0 = u = p(x_1, \dots, x_n)$, it has to be that $\{i : x_i = 0\} \in A$ by the arguments above. Suppose that $x_i = 0$. This means that $b \not\leq y_i$. Since $y_i \in \mathbf{M}$, it has to be that $y_i \leq a$. In this case, $z_i = a$. This shows that $\{i : x_i = 0\} \subseteq \{i : z_i = a\}$. Since $\{i : x_i = 0\} \in A$, we know then that $\{i : z_i = a\} \in A$. By the arguments above this implies that $a = p(z_1, z_2, \dots, z_n) = w$. Thus, we have proven that either $u = b$ or $w = a$. Therefore, either $v \leq w = a$ or $b = u \leq v$, so $v \in \mathbf{M}$ as desired. Thus, every sublattice of $\mathbf{L}$ (or $\mathbf{L}^2$) which is the union of an ideal and a filter is closed under p. $\square$

3. Distributive lattices

As an example using Theorem 2.8, we prove this lemma characterizing the term operations of the two element lattice. This characterization has been known at least since Post [9].

Lemma 3.1. *The term operations on the two element lattice are precisely the idempotent, order preserving operations.*

Proof. Let $\mathbf{L}$ be the two element lattice with elements $0 < 1$. Note that every lattice term opeartion must be idempotent and order preserving, so we need only prove that every idempotent, order preserving operation is a term operation. Let p be such an operation on $\mathbf{L}$. $\mathbf{L}$ has exactly two join endomorphisms which fix 0:

$$\begin{array}{ccccccc} & e & & & & f & \\ 0 & \mapsto & 0 & \text{and} & 0 & \mapsto & 0 \\ 1 & \mapsto & 0 & & 1 & \mapsto & 1 \end{array}$$

Note that $e^{\uparrow} = \mathbf{L}^2$ is preserved by every operation on $\mathbf{L}$, so it is preserved by p. On the other hand, $f^{\uparrow} = \{\langle x, y \rangle : x \leq y\rangle$ is just the order relation on $\mathbf{L}$ and is preserved by p since we are assuming that p is order preserving. By Theorem 2.8, we can conclude that p is a term operation of $\mathbf{L}$. $\square$

We can now use this characterization of term operations in the two element lattice to prove that every distributive lattice is order endoprimal. Note that every *finite* distributive lattice is order endoprimal follows from the results on dualisability in [4] and [3].

Theorem 3.2. *Every distributive lattice is order endoprimal.*

Proof. Suppose that $\mathbf{L}$ is a distributive lattice, and suppose that p is an n-ary order preserving operation on $\mathbf{L}$ which is preserved by all endomorphisms. We need to prove that p is a term operation of $\mathbf{L}$. Let $\mathbf{M}$ be any two element sublattice of $\mathbf{L}$. Since $\mathbf{L}$ is in the variety generated by $\mathbf{M}$ and since $\mathbf{M}$ is subdirectly irreducible, $\mathbf{L}$ can be embedded into a direct power of $\mathbf{M}$. This implies that there is a family of homomorphisms $\{e_j : j \in J\}$ from $\mathbf{L}$ to $\mathbf{M}$ whose kernels intersect to the identity. Since $\mathbf{M}$ is subdirectly irreducible, and since the e_js separate points in $\mathbf{M}$, there is some e_i which is injective on $\mathbf{M}$. This means that e_i restricts to an automorphism of $\mathbf{M}$. The only automorphism of $\mathbf{M}$ is the identity, so e_i is the identity on $\mathbf{M}$.

Now we consider the operation p. Since every element of $\mathbf{L}$ is the range of a constant endomorphism of $\mathbf{L}$, and since p is preserved by endomorphisms, p must be idempotent. Every e_j is an endomorphism of $\mathbf{L}$, so every e_j preserves p. Since $\mathbf{M}$ is the range of e_i, and since e_i preserves, p, $\mathbf{M}$ is closed under p. This means that the restriction of p to $\mathbf{M}$ is an idempotent, order preserving operation on $\mathbf{M}$. Now by Lemma 3.1, there is a lattice term T so that $p^{\mathbf{M}} = T^{\mathbf{M}}$. If $x_1, x_2, \ldots, x_n \in \mathbf{L}$, then for all e_j,

$$\begin{aligned} e_j(p^{\mathbf{L}}(x_1, x_2, \ldots, x_n)) &= p^{\mathbf{M}}(e_j(x_1), e_j(x_2), \ldots, e_j(x_n)) \\ &= T^{\mathbf{M}}(e_j(x_1), e_j(x_2), \ldots, e_j(x_n)) \\ &= e_j(T^{\mathbf{L}}(x_1, x_2, \ldots, x_n)). \end{aligned}$$

Since this is true for all e_j, and since the kernels of the e_j intersect to the identity, it follows that $p^{\mathbf{L}}(x_1, x_2, \ldots, x_n) = T^{\mathbf{L}}(x_1, x_2, \ldots, x_n)$, so p is equal to a term operation on all of $\mathbf{L}$. $\square$

4. The variety generated by $\mathbf{N}_5$

We are now ready to address term operations in $\mathbf{N}_5$. Our main tool here will be to use Theorem 2.8, our closure condition involving join endomorphisms. Almost all of the join endomorphisms we encounter will decompose as joins of join endomorphisms with two element ranges, and we will be able to employ Lemmas 2.9, 2.10, and 2.11.

Theorem 4.1. $\mathbf{N}_5$ *is order endoprimal.*

Proof. Suppose that p is an n-ary order preserving operation on $\mathbf{N}_5$ which is preserved by endomorphisms. Since $\mathbf{N}_5$ has a homomorphism onto the two element lattice, we can employ Theorem 2.11. We will use Theorem 2.8 to prove that p is a term operation of $\mathbf{N}_5$. Let e be a join endomorphism of $\mathbf{N}_5$ which fixes 0. If e is injective, then e must be the identity (since the identity map is the only join automorphism of $\mathbf{N}_5$). In this case, $e^{\uparrow} = \{\langle x, y\rangle : x \leq y\}$ is closed under p because we assumed that p is order preserving. Assume, then, that e is not injective. This implies that the range $e(\mathbf{N}_5)$ of e is a distributive sublattice of $\mathbf{N}_5$. Let J be the elements of $e(\mathbf{N}_5)$ which are join prime in $e(\mathbf{N}_5)$. Since $e(\mathbf{N}_5)$ is a distributive sublattice of $\mathbf{N}_5$, every element of $e(\mathbf{N}_5)$ is a join of elements in J. Suppose $j \in J$. Define $e_j : \mathbf{N}_5 \to \mathbf{N}_5$ by

$$e_j(x) = \begin{cases} j & j \leq e(x) \\ 0 & \text{else.} \end{cases}$$

Then e_j is a join endomorphism of $\mathbf{N}_5$ which fixes 0. Since e_j has a two element range, ${e_j}^{\uparrow}$ is closed under p by Lemmas 2.10 and 2.11. Also, for any $x \in \mathbf{N}_5$, note that

$$e(x) = \bigvee_{\substack{j \in J \\ j \leq e(x)}} j = \bigvee_{\substack{j \in J \\ j \leq e(x)}} e_j(x).$$

Now, if $j \in J$ and $j \not\leq e(x)$, then $e_j(x) = 0$, so we can add this $e_j(x)$ to the join without changing the outcome to get

$$e(x) = \bigvee_{\substack{j \in J \\ j \leq e(x)}} j = \bigvee_{\substack{j \in J \\ j \leq e(x)}} e_j(x) = \bigvee_{j \in J} e_j(x).$$

It follows, then, that $e = \bigvee_{j \in J} e_j$. By Lemma 2.9,

$$e^{\uparrow} = \bigcap_{j \in J} {e_j}^{\uparrow}.$$

Since each ${e_j}^{\uparrow}$ is closed under p, we now know that $e^{\uparrow}$ is closed under p. We now have shown that if e is any join endomorphism fixing 0 then $e^{\uparrow}$ is closed under p. By Theorem 2.8, it follows that p is a term operation of $\mathbf{N}_5$. $\square$

The subdirect irreducibility of $\mathbf{N}_5$ allows us to extend Theorem 4.1 to all lattices in the variety generated by $\mathbf{N}_5$. This proof will be almost identical to the proof of Theorem 3.2.

Theorem 4.2. *Every lattice in the variety generated by* $\mathbf{N}_5$ *is order endoprimal.*

Proof. Suppose that $\mathbf{L}$ is a lattice in the variety generated by $\mathbf{N}_5$ and that p is an n-ary order preserving operation on $\mathbf{L}$ which is preserved by endomorphisms. We prove that p is a term operation of $\mathbf{L}$. If $\mathbf{L}$ is distributive, then p is a term operation of $\mathbf{L}$ by Theorem 3.2. Suppose that $\mathbf{L}$ is not distributive. Since $\mathbf{L}$ is in the variety generated by $\mathbf{N}_5$ and since $\mathbf{L}$ is not distributive, $\mathbf{L}$ contains a sublattice $\mathbf{N}$ which is isomorphic to $\mathbf{N}_5$. Since $\mathbf{L}$ is in the variety generated by $\mathbf{N}_5 \cong \mathbf{N}$ and since $\mathbf{N}$ is subdirectly irreducible, $\mathbf{L}$ can be embedded into a direct power of $\mathbf{N}$. This implies that there is a family of homomorphisms $\{e_j : j \in J\}$ from $\mathbf{L}$ to $\mathbf{N}$ whose kernels intersect to the identity. Since $\mathbf{N}$ is subdirectly irreducible, and since the e_js separate points in $\mathbf{N}$, there is some e_i which is injective on $\mathbf{N}$. This means that e_i restricts to an automorphism of **N**. The only automorphism of **N** is the identity, so e_i is the identity on **N**.

Now we consider the operation p. Every e_j is an endomorphism of **L**, so every e_j preserves p. Since **N** is the range of e_i, and since e_i preserves, p, **N** is closed under p. Suppose that g is an endomorphism of **N**. Then $g \circ e_i$ is an endomorphism of **L** and preserves p. The restriction of $g \circ e_i$ to **N** must then preserve p also. However, since e_i is the identity on **N**, the restriction of $g \circ e_i$ to **N** is just g. Thus g preserves p. This is true for all endomorphisms of **N**, so the restriction of p to **N** is an order preserving operation which is preserved by endomorphisms. By Lemma 4.1, there is a lattice term T so that $p^{\mathbf{N}} = T^{\mathbf{N}}$. If $x_1, x_2, \ldots, x_n \in \mathbf{L}$, then for all e_j,

$$\begin{aligned} e_j(p^{\mathbf{L}}(x_1, x_2, \ldots, x_n)) &= p^{\mathbf{N}}(e_j(x_1), e_j(x_2), \ldots, e_j(x_n)) \\ &= T^{\mathbf{N}}(e_j(x_1), e_j(x_2), \ldots, e_j(x_n)) \\ &= e_j(T^{\mathbf{L}}(x_1, x_2, \ldots, x_n)). \end{aligned}$$

Since this is true for all e_j, and since the kernels of the e_j intersect to the identity, it follows that $p^{\mathbf{L}}(x_1, x_2, \ldots, x_n) = T^{\mathbf{L}}(x_1, x_2, \ldots, x_n)$, so p is equal to a term operation on all of **L**. □

5. An example and a question

We now give an example to demonstrate that not all finite lattices are order endoprimal. Let **L** be a finite simple lattice with more than two elements. Define a binary operation p on **L** so that

$$p(x, y) = \begin{cases} x \vee y & 1 \in \{x, y\} \\ x \wedge y & \text{else.} \end{cases}$$

Since **L** is simple, its only nonconstant endomorphisms are automorphisms. Every automorphism of **L** fixes 1, so it is easy to see that p is preserved by every endomorphism of **L**. The operation p is also order preserving (and even idempotent). However, the only binary term operations of any lattice are meet and join and the projections, and p is none of these. The operation p is not a term operation of **L**. Therefore, we immediately have the following theorem.

Theorem 5.1. *No finite simple lattice with more than two elements is order endoprimal.*

This leads naturally to our closing question.

Question 5.2. Which (finite) lattices are order endoprimal?

References

[1] Baker, K., Pixley, A.: Polynomial interpolation and the Chinese remainder theorem for algebraic systems. Math. Zeitschr. **143**, 165–174 (1975)

[2] Bergman, G.: On the existence of subalgebras of direct products with prescribed d-fold projections. Algebra Universalis **7**, 341–356 (1977)

[3] Davey, B.A.: Dualisability in general and endodualisability in particular. Logic and algebra (Pontignano, 1994), 437–455, Lecture Notes in Pure and Appl. Math., 180, Dekker, New York, (1996)

[4] Davey, B.A., Haviar, M., Priestley, H.A.: Endoprimal distributive lattices are endodualisable. Algebra Universalis **34**(3), 444–453 (1995)

[5] Hegedűs, P., Pálfy, P.: Finite modular congruence lattices. Algebra Universalis **54**, 105–120 (2005)

[6] Márki, L., Pöschel, R.: Endoprimal distributive lattices. Algebra Universalis **30**(2), 272–274 (1993)

[7] McKenzie, R., McNulty, G., Taylor, W.: Algebras, Lattices, Varieties, Volume I. Wadsworth and Brooks/Cole, Monterey, California (1987)

[8] Pöschel, R., Kalužnin, L.A.: Funktionen und Relationenalgebren. Birkhäuser, Basel (1979)

[9] Post, E.: The two-valued iterative systems of mathematical logic. Annals of Math. Studies. Princeton University Press, N.J. (1941)

John W. Snow
Department of Mathematics
University of Mary Hardin-Baylor
Belton, Texas
USA
e-mail: jsnow@umhb.edu